On the algebraic sum of ideals and sublattices

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$\{0\} \neq X$ is a Banach space $\Rightarrow \exists_{A,B\subset X}$ A, B closed but $A + B \neq \overline{A + B}$

Theorem

If $Y \subset X$ is a closed linear subspace of infinite dimension and infinite codimension, then there exists a closed linear subspace $G \subset X$ such that $Y \cap G = \{0\}$ and $Y + G \neq \overline{Y + G}$.

on the other hand: $Y + G$ is always closed whenever Y is closed and G is of finite dimension or G is closed and finite codimensional

- $E = (E, \|\cdot\|)$ is a Banach lattice

Let I be a closed ideal in E. Does there exist an infinite dimensional closed discrete (resp. continuous, heterogeneous) Riesz subspace F in E such that $I + F \neq \overline{I + F}$?

a linear subspace $F \subset E$ is said to be:

- a Riesz subspace (= sublattice) if $x, y \in F \;\Rightarrow\; x \vee y = \sup\{x, y\} \in F$ (equivalently, $x \wedge y = \inf\{x, y\} \in F$ or $|x| = x \vee (-x) \in F$),
- an ideal whenever F is solid, i.e., $|x| \leqslant |y|,\; y \in F \;\Rightarrow\; x \in F$

every ideal is a Riesz subspace

- Banach lattices $=$ discrete lattices $\cup$ continuous lattices $\cup$ heterogeneous lattices
- heterogeneous lattices = Banach lattices $\setminus$ (continuous lattices $\cup$ discrete lattices)

- $0 < e \in E$ is *discrete* iff $0 \leqslant x \leqslant e \;\Rightarrow\; x = te$ for some real t , equivalently:

* $[0, e]$ does not contain any two nonzero disjoint elements,
* $0 < e$ and $\mathrm{Id(e)} = \mathbb{R}\mathrm{e}$,
* $0 < e$ and $\{e\}^{\mathrm{dd}} = \mathbb{R}e$
  $(A^{\mathrm{d}} = \{x \in E : \forall_{a \in A} \; |x| \wedge |a| = 0\})$,
* $0 < e$ and $\{e\}^{\mathrm{d}}$ is a maximal ideal.

- $\mathcal{D}(E) = \{e \in E : e \text{ is discrete}\}$
  $e_1, e_2 \in \mathcal{D}(E) \;\Rightarrow\; e_1 \wedge e_2 = 0$ or $e_1 = te_2$.
- if $(e_\gamma)_{\gamma \in \Gamma_1}, (e_\gamma)_{\gamma \in \Gamma_2}$ are maximal families of discrete elements, then for every $\gamma_1 \in \Gamma_1$ there exists, uniquely determined, $\gamma_2 \in \Gamma_2$ and $t_{\gamma_1} > 0$ such that $e_{\gamma_2} = t_{\gamma_1} e_{\gamma_1}$

Examples.

- $t1_{\{\gamma\}}$ are unique discrete elements in $\ell^p(\Gamma)$, $c_0(\Gamma)$ etc.; $t1_A$, where A is an atom of a measure μ are unique discrete elements in $E(\mu)$ (= a Banach lattice of measurable functions: L^p-spaces, Orlicz spaces ...).
- $f \in C(K)$ is discrete iff $f = t1_{\{s\}}$, s is an isolated point
- E^* = the dual of E; $h \in \mathcal{D}(E^*)$ iff h is a homomorphism: $|h(x)| = h(|x|)$.
- Let F be Dedekind complete. $T \in L_{\mathrm{r}}(E, F) = \{S : E \to F : S = S_1 - S_2,\ 0 \leqslant S_i : E \to F\}$ is discrete iff $T = f \otimes e$ where e is discrete in F and f is discrete in E^*

- $I \subset E$ a closed ideal $Q : E \to E/I$ the canonical quotient map
$$Q(x) \leqslant Q(y) \ \Leftrightarrow \ \exists_{f_1, f_2 \in F} \ x + f_1 \leqslant y + f_2$$
For $0 < x \notin I$ there holds $Q(x)$ is discrete iff $I + \mathbb{R}x$ is an ideal.

E is *discrete*: for every $0 < x \in E \quad [0, x] \cap \mathcal{D}(E) \neq \emptyset$.
equivalently
E contains a complete disjoint system $(e_\gamma)_{\gamma \in \Gamma}$ with $e_\gamma \in \mathcal{D}(E)$.
$0 \leqslant x = \sup_\gamma x(\gamma) e_\gamma$, the numbers $x(\gamma)$ are uniquely determined,
$T : E_+ \to \mathbb{R}^\Gamma_+$ defined by $T(x) = (x(\gamma))_{\gamma \in \Gamma}$ can be extended from E into $\mathbb{R}^\Gamma$ by $T(x) = T(x^+) - T(x^-)$ – the extension is an order isomorphic embedding.

$$\mathrm{span}\{1_{\{\gamma\}} : \gamma \in \Gamma\} \subset T(E) \subset \mathbb{R}^\Gamma, \quad T(E) \approx E$$

Examples of discrete spaces

- Ideals in spaces of measurable functions $E(\mu)$ over a purely atomic measure μ–in particular $\ell^p(\Gamma)$ spaces $(0 < p \leqslant \infty)$, $c_0(\Gamma)$.
- $C(K)$ is discrete iff isolated points are dense in K.
- For a Dedekind complete Banach lattice F $L_{\mathrm{r}}(E, F)$ is discrete iff E^* and F are discrete.

E is *continuous* when E does not contain discrete elements or, equivalently,
Every $0 < x \in E$ dominates infinitely many nonzero pairwise disjoint elements ($\Rightarrow \dim E = \infty$).

Examples of continuous spaces.

- Ideals in spaces of measurable functions $E(\mu)$ over an atomless measure μ–in particular $L^p[0,1]$, Orlicz spaces $L^\varphi[0,1]$.
- $C(K)$ is continuous iff K consists of accumulation points.
- $(E, \|\cdot\|)$ a σ-Dedekind complete Banach lattice,

$$E_A = \{x \in E : |x| \geqslant x_\alpha \downarrow 0 \ \Rightarrow \ \|x_\alpha\| \to 0\}.$$

  The quotient E/E_A is always continuous (in particular ℓ^∞/c_0 is continuous).
- For a Dedekind complete Banach lattice F $L_{\mathrm{r}}(E,F)$ is continuous iff E^* or F is continuous.

Every Banach lattice E embeds isometrically and order isomorphically onto a Riesz subspace of a continuous Banach lattice

$$E \hookrightarrow \Delta \subset \ell^{\infty}(E) \hookrightarrow \ell^{\infty}(E)/c_0(E)$$
$$x \longrightarrow x1_{\mathbb{N}} \longrightarrow Q(x1_{\mathbb{N}})$$

where $\Delta = \{(x_n) \in E^{\mathbb{N}} : \forall_{n,m}\; x_n = x_m\}$

None continuous Banach lattice with an order continuous norm (i.e., $E = E_A$: $x_\alpha \downarrow 0 \;\Rightarrow\; \|x_\alpha\| \to 0$) embeds in a discrete space.

Let E be heterogeneous and let $\mathcal{D}(E)$ be set of all discrete elements in E.
$E_D = \mathcal{D}(E)^{\mathrm{dd}}$ = the discrete part of E
$E_C = \mathcal{D}(E)^{\mathrm{d}}$ = the continuous part of E

$\dim E_C = \infty$ and $E = E_D + E_C$ whenever E is Dedekind complete but $E \neq E_D + E_C$ in general

$(K = [-1, 0] \cup \{\frac{1}{n} : n \in \mathbb{N}\}$
$C(K)_D = \{f \in C(K) : f([-1, 0]) = \{0\}\}$,
$C(K)_C = \{f \in C(K) : f(\frac{1}{n}) = 0$ for every $n\}$ but
$C(K)_D + C(K)_C \subset \{f \in C(K) : f(0) = 0\} \neq C(K))$

$\dim E = \infty \quad \Rightarrow \quad \exists_{F \subset E} \quad \dim F = \infty$, $F = \overline{F}$, F is discrete

$F = \overline{\text{span}}\{x_n : n \in \mathbb{N}\}$ where $x_k \wedge x_m = 0$

$$x, y \in F,\ x = \sum_{n=1}^{\infty} t_n(x) x_n, y = \sum_{n=1}^{\infty} t_n(y) x_n$$

$$\Downarrow$$

$$x \leqslant y \ \Leftrightarrow \forall_n \ t_n(x) \leqslant t_n(y)$$

Proposition

Every infinite dimensional Banach lattice E contains a continuous Riesz subspace E_1 and a heterogeneous Riesz subspace E_2 such that $\overline{E_1}$ and $\overline{E_2}$ are discrete and E_1 is order isomorphic to $C[0,1]$.

Choose $(x_n) \subset E_+$, $x_k \wedge x_m = 0$, $\sum_{n=1}^{\infty} \|x_n\| < \infty$.
Let (q_n) be a sequence of all rational numbers in $[0,1]$.
Define $T : C[0,1] \to E$ by $Tf = \sum_{n=1}^{\infty} f(q_n) x_n$. T is an order isomorphism, and so $T(C[0,1])$ is a continuous Riesz subspace in E.
Fix $\varepsilon > 0$ and $k \in \mathbb{N}$. There exist $N > k$ and $f \in C[0,1]$ such that $\sum_{n=N+1}^{\infty} \|x_n\| \leqslant \varepsilon$, $f(q_k) = \|f\|_\infty = 1$, $f(\{q_m : m \in \{1,\dots,N\} \setminus \{k\}\}) = \{0\}$. We obtain $\|x_k - Tf\| < \varepsilon$, i.e, $x_k \in \overline{T(C[0,1])} = \overline{\text{span}}\{x_n : n \in \mathbb{N}\}$, and so $\overline{T(C[0,1])}$ is discrete.
It is sufficient to put $E_1 = T(C[0,1])$ and $E_2 = \overline{\text{span}}\{x_{2n-1} : n \in \mathbb{N}\} + \{\sum_{k=1}^{\infty} f(q_k) x_{2k} : f \in C[0,1]\}$.

There exists a discrete Riesz subspace $F \subset \ell^\infty \times \ell^\infty$ such that $\overline{F}$ is heterogeneous (and separable).

$\mathcal{A}$ an algebra of subsets in $\mathbb{N}$ such that

- $C \notin \mathcal{A}$ whenever $\emptyset \neq C$ is finite,
- $\forall_{n \neq m} \exists_{C \in \mathcal{A}}\ n \in C$ and $m \notin C$.

($\mathbb{Q}$ = rational numbers, $\varphi : \mathbb{Q} \to \mathbb{N}$ a bijection $\mathcal{A} = \varphi(\mathcal{A}_\mathbb{Q})$ where $\mathcal{A}_\mathbb{Q}$ = the algebra generated by the family $\{\mathbb{Q} \cap [a, b) : a, b$ dyadic numbers$\}$)

$G = \overline{\text{span}}\{1_C : C \in \mathcal{A}\} \subset \ell^\infty$ is continuous (1_C = the characteristic function of C)

Let e_k be the k-th unit vector, $x = (x_n) \in c_0$ with $x_n > 0$ for all n.

$F = \text{span}\{(x1_C, 1_C), (e_k, 0) : C \in \mathcal{A}, k \in \mathbb{N}\} \subset \ell^\infty \times \ell^\infty$ is discrete but $\overline{F} = c_0 \times G$ is heterogeneous.

e discrete in $F \quad \Rightarrow \quad e$ discrete in $\overline{F}$, i.e, the closure of a discrete Riesz subspace is never continuous.
Location of continuous Riesz subspaces in discrete Banach lattices.

Proposition

Let E be a discrete Banach lattice. If $F \subset E$ is a closed continuous Riesz subspace, then $F \cap E_A = \{0\}$.

Theorem

For a discrete σ-Dedekind complete Banach lattice E the following are equivalent.

(a) *Every closed Riesz subspace of E is discrete.*
(b) *The closure of every discrete Riesz subspace of E is discrete.*
(c) *The norm on E is order continuous.*

The assumptions in the previous theorem cannot be avoided:

- the discreteness of E is essential because modifying the construction (presented above) in $\ell^\infty \times \ell^\infty$ we obtain a discrete Riesz subspace in $L^1[0,1]$ whose closure is heterogeneous,
- σ-Dedekind completeness is also important–all closed Riesz subspaces in the lattice c of real convergent sequences are discrete.

Sometimes $F_1 + F_2$ is not a Riesz subspace for one dimensional Riesz subspaces F_1, F_2.
Indeed, let $E = ca(\Sigma)$, $\mu \leqslant \nu \Rightarrow \mu(A) \leqslant \nu(A)$ for every $A \in \Sigma$. Let $0 \leqslant \mu$ be such that $\mu(\Sigma \cap A_i) \neq \{0\}$ for disjoint A_i, $i = 1, 2, 3$. Define $\mu_i(X) = \mu(X \cap (A_1 \cup A_i))$. Clearly $\mathbb{R}\mu_2, \mathbb{R}\mu_3$ are Riesz subspaces and $\mu_1 = \mu_2 \wedge \mu_3 \notin \mathbb{R}\mu_2 + \mathbb{R}\mu_3$, i.e., the sum is not a Riesz subspace.

Let $x, y \in E_+$ be linearly independent. The sum $\mathbb{R}x + \mathbb{R}y$ is a Riesz subspace iff x, y are of the form $tp + sq$ where t, s are positive distinct numbers and
$p, q \in \{u \in E : (x + y - u) \wedge u = 0\}$ are different.

F, G are Riesz subspaces $\Rightarrow$ $F + G$ is a Riesz subspace whenever at least one component is an ideal or $F \perp G$ (i.e., $|f| \wedge |g| = 0$ for all $f \in F, g \in G$).

Theorem (E.B. Davis, H.P. Lotz '68)

If I_1, I_2 are closed ideals in a Banach lattice, then $I_1 + I_2$ is closed.

Moreover for closed Riesz subspaces $F_1, F_2 \subset E$ such that $F_1 \perp F_2$ the sum $F_1 + F_2$ is closed.

Theorem (L. Drewnowski, 2011)

Let E be a Banach lattice, and let (I_n) be a sequence of closed ideals in E. Define $I = \sum_{n=1}^{\infty} I_n$ to be the set of elements $z \in E$ that are of the form $z = \sum_{n=1}^{\infty} x_n$ where $x_n \in I_n$ for every n. Then I is the smallest closed ideal in E that contains all the ideals I_n.

Theorem

Let E be a Banach lattice and suppose that $F_n \subset E$, $n \in \mathbb{N}$, are closed Riesz subspaces satisfying the condition $F_k \perp F_j$ for distinct k, j. Then the order sum of F_n's, i.e., the space $F = (\mathrm{o})(\oplus F_n) = \{x \in E : x = (\mathrm{o}) \sum_{k=1}^{\infty} x_k \text{ where } x_k \in F_k\}$ is a closed Riesz subspace.

(B. Wiatrowski '05): There exists a normed lattice containing a closed ideal I with $I + I^{\mathrm{d}} \neq \overline{I + I^{\mathrm{d}}}$. Hence $I^{\mathrm{dd}} + I^{\mathrm{d}} \neq \overline{I^{\mathrm{dd}} + I^{\mathrm{d}}}$.

Let I be a closed ideal in E. Does there exist an infinite dimensional closed Riesz subspace F in E with "good" order properties such that $I + F \neq \overline{I + F}$?

Theorem

Let I be a closed infinite dimensional and infinite codimensional ideal in E. Then there exists a closed separable discrete Riesz subspace G such that the induced norm is order continuous, $I \cap G = \{0\}$ and $I + G$ is not closed.

Theorem

Suppose that either

(a) *E is a σ-Dedekind complete Banach lattice and $I \subset E$ is a closed infinite dimensional ideal such that the quotient norm on E/I is not order continuous,*

or

(b) *E is a continuous Banach lattice with order continuous norm and $\{0\} \neq I \subsetneqq E$ is a closed ideal.*

Then there exists a closed continuous Riesz subspace $G_1 \subset E$ and a closed heterogeneous Riesz subspace $G_2 \subset E$ such that $I \cap G_i = \{0\}$ and $I + G_i$ is not closed for $i = 1, 2$.

If E is Dedekind complete and $E \neq E_A$, then the quotient norm on E/E_A is not order continuous, and so the Theorem can be applied to $I = E_A$ whenever $\dim E_A = \infty$ (particular case: $E = \ell^\infty$, $I = E_A = c_0$).