GRAPHICAL MODELS 2020 – GAUSSIAN VECTORS: BASICS

CASE OF DIMENSION 1. $X \sim \mathcal{N}(m, \sigma^2)$; parameters $m \in \mathbb{R}$, $\sigma^2 > 0$,

Density with respect to the Lebesgue measure

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}, \quad m \in \mathbb{R}, \sigma^2 > 0$$

Characteristic function for $\boxed{m \in \mathbb{R}, \sigma^2 \geq 0}$ (we include $\delta_m = \mathcal{N}(m, 0)$)

$$\phi_X(t) = \mathbb{E}e^{itX} = e^{itm - \frac{1}{2}t^2\sigma^2}, \qquad t \in \mathbb{R}$$

CASE OF DIMENSION n.

Definition. Let $m = (m_1, \dots, m_n) \in \mathcal{R}^n$ and $C = (c_{kl})$ real symmetric non–negative definite $n \times n$ matrix (with eigenvalues $d_k \geq 0$). A random vector $X = {}^t(X_1, \dots, X_n)$ is called **Gaussian**, $X \sim \mathcal{N}(m, C)$ if it has the characteristic function

$$\phi_X(t) = \mathbb{E}e^{i\langle t, X\rangle} = e^{i\langle t, m\rangle - (1/2)\langle Ct, t\rangle}.$$

Definition of Gaussian density. Let K be a real symmetric **positive** definite $n \times n$ matrix (with eigenvalues $d_k > 0$).

The Gaussian density is defined by

$$f_{m,K}(x) = \frac{|K|^{1/2}}{(2\pi)^{n/2}} \exp\{-\frac{1}{2}\langle K(x-m), x-m\rangle\}, \qquad x \in \mathcal{R}^n,$$

où $|K| = \det K$.

Properties of Gaussian vectors. 1. The function $f_{m,K}$ is a probability density.

Let X be a random vector with density $f_{m,K}$. The characteristic function of X is

$$\phi_X(t) = e^{i\langle t, m\rangle - \frac{1}{2}\langle Ct, t\rangle}, \qquad C = K^{-1}.$$

2. (degenerate Gaussian vectors) When C is not inversible, the law $\mathcal{N}(m, C)$ exists. It is called *degenerate Gaussian.*

3. Let $Y \sim \mathcal{N}(0, Id)$ the standard Gaussian vector and C real symmetric non–negative definite $n \times n$ matrix (with eigenvalues $d_k \geq 0$). Let $C^{1/2}$ the square root matrix of C (if the diagonalization of C is $C = UD\,{}^tU$ then $C^{1/2} = UD^{1/2}\,{}^tU$.

Then $X = C^{1/2}Y \sim \mathcal{N}(0, C)$. In particular, if C has some zero eigenvalues, X is concentrated on the subspace $Image(C^{1/2}) \neq \mathbb{R}^n$ and X has not a density.

Let $m \in \mathbb{R}^n$. Then $Z = m + C^{1/2}Y \sim \mathcal{N}(m, C)$. if C has some zero eigenvalues, Z is concentrated on the set $m + Image(C^{1/2}) \neq \mathbb{R}^n$ and Z has not a density.

4. Let $Z \sim \mathcal{N}(m, C)$. Then $\mathbb{E}Z = m$ and $CovZ = C$.

5. Components X_k, X_l of a Gaussian vector X are non-correlated if and only if X_k, X_l are independent (it is false without hypothesis of a Gaussian vector!)

6. A random vector $X = {}^t(X_1, \dots, X_n)$ is Gaussian if and only if for all vectors $v = {}^t(v_1, \dots, v_n) \in \mathcal{R}^n$ the linear combination $\langle X, v\rangle = v_1X_1 + \dots + v_nX_n$ is a Gaussian random variable.

7. Marginal variables X_k of a Gaussian vector X are Gaussian and $X_k \sim \mathcal{N}(m_k, c_{kk})$.

Let $I \subset \{1, \dots, n\}$. Marginal subvectors X_I of a Gaussian vector X are Gaussian and $X_I \sim \mathcal{N}(m_I, C_I)$.

GRAPHICAL MODELS 2020 – GAUSSIAN VECTORS: EXERCISES

1. The characteristic function $\phi_{(X,Y)}$ of a random vector ${}^t(X,Y)$ equals for $(t_1,t_2) \in \mathbb{R}^2$

   (a) $\varphi_{(X,Y)}(t_1,t_2) = e^{-(t_1^2+t_1t_2+t_2^2)}$

   (b) $\varphi_{(X,Y)}(t_1,t_2) = e^{-\frac{1}{2}t_1^2}$

   (c) $\phi_{(X,Y)}(t_1,t_2) = e^{-\frac{1}{2}(t_1^2+2t_1t_2+t_2^2)}$

   (d) $\phi_{(X,Y)}(t_1,t_2) = e^{-\frac{1}{2}(t_1^2-2t_1t_2+t_2^2)}$

   Determine the law of the vector ${}^t(X,Y)$ and $\mathrm{Cov}(X,Y)$.

2. Let X a Gaussian random vector with density of the form

$$f(x) = c\exp\{-\frac{1}{2}\langle Kx, x\rangle\}, \quad x \in \mathbb{R}^n,$$

   for

   (a) $K = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}, \quad n = 2$

   (b) $K = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix}, \quad n = 3.$

   1. Determine c.
   2. Find $CovX$.
   3. Are there independent components of X?

   Does existance of 0 in K implies that there are independent components of X?

3. Let X and Y with law $\mathcal{N}(0,1)$ and independent. Show that $X+Y$ et $X-Y$ are independent and give their laws.

4. Let $U = {}^t(X,Y,Z)$ a random vector in $(\mathbb{R}^3, \mathcal{B}(\mathbb{R}^3))$ with density

$$f_U(x,y,z) = \frac{1}{(\sqrt{2\pi})^3} \exp\left(-\frac{1}{2}\left(x^2 + y^2 + 2z^2 + 2yz \right) \right).$$

   (a) Show that U is a Gaussian vector.

   (b) Give $CovU$, the covariance matrix.

   (c) Are variables X, Y, Z independent? Some of them are pairwise independent?

   (d) Consider $V = {}^t(X,Y)$.

      i. Give $CovV$.

      ii. Give the law of V.