The Navier-Stokes equations in the weak-$L^n$ space with time-dependent external force

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Dedicated to Professor Daisuke Fujiwara on the Occasion of his Sixtieth Birthday

Abstract. We consider the Navier-Stokes equations with time-dependent external force, either in the whole time or in positive time with initial data, with domain either the whole space, the half space or an exterior domain of dimension $n \geq 3$. We give conditions on the external force sufficient for the unique existence of small solutions in the weak-$L^n$ space bounded for all time. In particular, this result gives sufficient conditions for the unique existence and the stability of small time-periodic solutions or almost periodic solutions. This result generalizes the previous result on the unique existence and the stability of small stationary solutions in the weak-$L^n$ space with time-independent external force.

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Introduction

Let $\Omega$ be the whole space $\mathbb{R}^n$, the half space $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times (0, +\infty)$, or an exterior domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, where the space dimension $n$ satisfies $n \geq 3$. We are concerned with the nonstationary Navier-Stokes equation in the whole time with the Dirichlet boundary condition in $\Omega$ as follows:

$$\frac{\partial u}{\partial t}(t, x) = \Delta_x u(t, x) - (u \cdot \nabla_x) u(t, x) - \nabla_x \pi(t, x) + f(t, x) \quad \text{in} \ \mathbb{R} \times \Omega, \quad (0.1)$$

$$\nabla_x \cdot u(t, x) = 0 \quad \text{in} \ \mathbb{R} \times \Omega, \quad (0.2)$$

$$u(t, x) = 0 \quad \text{on} \ \mathbb{R} \times \partial\Omega, \quad (0.3)$$

$$u(t, x) \to 0 \quad \text{as} \ |x| \to +\infty, \quad (0.4)$$

and the Cauchy problem for the same equations as follows:

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\[
\frac{\partial u}{\partial t}(t, x) = \Delta u(t, x) - (u \cdot \nabla) u(t, x) - \nabla \pi(t, x) + f(t, x)
\]

in \( \mathbb{R}_+ \times \Omega \), (0.5)

\[
\nabla \cdot u(t, x) = 0
\]

in \( \mathbb{R}_+ \times \Omega \), (0.6)

\[
u(t, x) = 0
\]

on \( \mathbb{R}_+ \times \partial \Omega \), (0.7)

\[
u(t, x) \to 0
\]

as \(|x| \to +\infty\), (0.8)

\[
u(0, x) = a(x)
\]

on \( \Omega \), (0.9)

where \( \mathbb{R}_+ = (0, +\infty) \), \( u(t, x) = (u_1(t, x), \ldots, u_n(t, x)) \) denotes the unknown vector function standing for the velocity at \((t, x)\), \( \pi(t, x) \) denotes the unknown scalar function standing for the pressure at \((t, x)\), and \( f(t, x) = (f_1(t, x), \ldots, f_n(t, x)) \) denotes a given function standing for the external force at \((t, x)\). We are interested in the case where the external force \( f(t, \cdot) \) does not necessarily decay as \( t \) tends to \(+\infty\), and consider the unique existence of small bounded continuous solutions of the problem (0.1)–(0.4) and the continuous dependence of the solutions on the data. We are also interested in the same problem on the Cauchy problem (0.5)–(0.9), which implies the stability of the solutions of the original problem (0.1)–(0.4) as \( t \to +\infty \). For example, we are interested in the unique existence and the stability of time-periodic and almost periodic solutions \( u(t, x) \).

In this paper we introduce the notion of mild solutions of the systems above, and show that, if \( f(t, x) \) can be written as \( P \nabla_x F(t, x) \) with a small, bounded and continuous function \( F(t, \cdot) \) with values in the space \( L^{n/2, \infty}(\Omega) \), and if \( a(x) \in L^{n, \infty}(\Omega) \) is sufficiently small, then the systems (0.1)–(0.4) and (0.5)–(0.9) admit one and only one mild solution bounded and continuous in the space \( L^{n, \infty}(\Omega) \) with norm bounded by a definite constant. Here \( P \) denotes the projection onto the space of solenoidal vector fields. We also show that the mild solution is a solution of the Navier-Stokes equation in the sense of distributions.

We moreover show that the mild solutions depend continuously on the data \( F(t, x) \) and \( a(x) \). This fact implies that, if the function \( F(t, x) \) is periodic with respect to \( t \) with period \( T \) or almost periodic with respect to \( t \), then so is the solution of the system (0.1)–(0.4). The result on (0.5)–(0.9) implies that the solution of (0.1)–(0.4) is stable as \( t \to +\infty \) under small initial perturbation at \( t = 0 \) and small perturbation on \( F(t, x) \).

We also consider the asymptotic stability of the time-global solution above with respect to the \( L^{q, 1}(\Omega) \) norm for \( q > n \) under small initial perturbation. Namely, suppose that the time-global mild solution above is sufficiently small. Then there exists another mild solution with the same external force and initial data sufficiently close to the original one, and the \( L^{q, 1}(\Omega) \)-norm of the difference between these solutions decays with definite rate. As we shall see in the next
section, we cannot expect the asymptotic stability in the space $L_{\sigma}^{n, \infty}(\Omega)$ itself, even in the trivial case $u(t, x) \equiv F(t, x) \equiv 0$.

We review previous results on the problems above. Among a number of literatures concerning time-global solutions of the Navier-Stokes equations with the external force as above, Maremonti [26], [27] was the first in which the equations in unbounded domains are treated. In fact, [26], [27] considered the case $\Omega = \mathbb{R}^3$ and $\Omega = \mathbb{R}^3_+$ respectively, and proved the unique existence of the time-global solution of problems (0.1)–(0.4) and (0.5)–(0.9) under the assumption
\[ f(t, \cdot) \text{ is small in } L^\infty \left( I : L^3(\Omega) \cap \dot{H}_p^{-1}(\Omega) \right) \] (0.10)
with some $p < 3/2$, where $I$ denotes either $\mathbb{R}$ or $\mathbb{R}_+$, and proved the stability of the solution above under a small perturbation. As a result, for $f(t, x)$ periodic in time, he showed the unique existence of time-periodic solution with the same period.

Then Kozono and Nakao [15] considered the problem (0.1)–(0.4) on $\Omega$, where $\Omega$ is the whole space $\mathbb{R}^n$ or the half space $\mathbb{R}^n_+$ for $n \geq 3$ or an exterior domain in $\mathbb{R}^n$ for $n \geq 4$, and constructed time-periodic solutions for time-periodic $f(t, x)$ satisfying the assumption
\[ f(t, \cdot) \text{ is small in } L^\infty \left( \mathbb{R} : L^r(\Omega) \cap \dot{H}_p^{-1}(\Omega) \right) \] (0.11)
with some $p < n/2$ and $r > n/3$. The condition in (0.10) corresponds to the case $r = n = 3$. Taniuchi [34] proved the stability of the periodic solutions constructed in [15] in the space $L^r(\Omega)$. These works treated solutions belonging to suitable $L^p$ spaces. Yamazaki [37] considered the problem on $\mathbb{R}^n$ for $n \geq 3$, and generalized the results of [15], [34] for Morrey spaces.

On the other hand, Salvi [32] considered the problem (0.1)–(0.4) on three-dimensional exterior domains $\Omega$, and proved the existence of a time-periodic weak solution with period $T$ for time-periodic $f(t, x)$ with period $T$ satisfying the assumption
\[ f(t, \cdot) \in L^2 \left( [0, T]; L^2(\Omega) \cap \dot{H}_2^{-1}(\Omega) \right). \] (0.12)
He also showed the existence of a time-periodic strong solution with period $T$ under the assumption that $f(t, x)$ is small in the class above. Actually he considered a more general situation; he solved the problem above on three-dimensional exterior domains with boundary moving periodically with period $T$. For the time being, this seems to be the only result for three-dimensional exterior domains. However, the uniqueness of the periodic solution is not known.

More detailed references, including results for bounded domains, are found in [26], [27], [15], [32].
If the external force \( f(t, x) \) is independent of \( t \), the problem (0.1)–(0.4) reduces to the stationary problem

\[
\Delta x u(x) - (u \cdot \nabla x) u(x) - \nabla x \pi(x) + f(x) = 0 \quad \text{in } \Omega, \tag{0.13}
\]

\[
\nabla x \cdot u(x) = 0 \quad \text{in } \Omega, \tag{0.14}
\]

\[
u(x) = 0 \quad \text{on } \partial \Omega, \tag{0.15}
\]

\[
u(x) \to 0 \quad \text{as } |x| \to +\infty, \tag{0.16}
\]

and the problem (0.5)–(0.9) with \( a(x) \) near \( u(x) \) above concerns the stability of the stationary solution \( u(x) \). This problem was studied in more detail, including 3-dimensional exterior domains. In fact, when \( \Omega \) is an \( n \)-dimensional exterior domain, Novotny and Padula [31], Galdi and Simader [10] and Borchers and Miyakawa [5] proved the following: If \( m \) is an integer such that \( 1 \leq m \leq n - 2 \) and if \( f(x) \) enjoys the condition \( |f(x)| \leq c|x|^{-m-2} \) with sufficiently small \( c \), then there exists a unique solution \( u(x) \) of (0.13)–(0.16) such that \( |u(x)| \leq C|x|^{-m} \) and that \( |\nabla x u(x)| \leq C|x|^{-m-1} \). In particular, for \( n = 3 \), they proved the unique existence of physically reasonable solutions in the sense of Finn [6], and obtained sharp estimates of the solutions and their derivatives.

On the other hand, for \( n \geq 4 \), Kozono and Sohr [17] proved the following: If \( f(x) \) can be written in the form \( P \nabla x F(x) \) such that \( F(x) \) is sufficiently small in \( L^{n/2}(\Omega) \), then there exists a unique solution \( u(x) \) of the problem (0.13)–(0.16) such that \( u(x) \in L^n(\Omega) \) and that \( \nabla x u(x) \in L^{n/2}(\Omega) \) hold with norms bounded by a certain constant.

Later on, for \( n \geq 3 \), Kozono and Yamazaki [21] unified the results above as follows: If \( f(x) \) can be written in the form \( \nabla x F(x) \) such that \( F(x) \) is sufficiently small in \( L^{n/2,\infty}(\Omega) \), then there exists a unique solution \( u(x) \) of the problem (0.13)–(0.16) such that \( u(x) \in L^{n,\infty}(\Omega) \) and that \( \nabla x u(x) \in L^{n/2,\infty}(\Omega) \) with norms bounded by a definite constant, where \( L^{p,\infty}(\Omega) \) denotes the weak-\( L^p \) space on \( \Omega \). The assumption on the external force in this result generalizes the assumption of [31], [10], [5] as well as that of [17]. Hence this result implies that, in the case \( n = 3 \), the class \( L^{3,\infty} \) is a natural generalization of the class of physically reasonable solutions satisfying \( u(x) \to 0 \) as \( |x| \to \infty \).

The stability of the stationary solutions obtained in the results above is also studied by Borchers and Miyakawa [5], Kozono and Ogawa [16] and Kozono and Yamazaki [22].

It is to be noted that, in the case \( n = 3 \), we cannot replace the spaces \( L^{3,\infty}(\Omega) \) and \( L^{3/2,\infty}(\Omega) \) by \( L^3(\Omega) \) and \( L^{3/2}(\Omega) \) respectively. In fact, Borchers and Miyakawa [5, Theorem 2.4], Kozono and Sohr [18, Theorem C] and Kozono, Sohr and Yamazaki [19, Theorem 2, (1)] showed that the solution \( u(x) \) of (0.13)–(0.16) belongs to \( L^n(\Omega) \) only under very restricted situations. More detailed references are found in [21], [22].

All the results above on the stationary solutions give the condition on \( f(x) \) in terms of norms invariant under the scaling \( (u, \pi, f) \to (\lambda u, \pi, f) \) such
Navier-Stokes equations in the weak-$L^p$ space

that \( u_\lambda(t, x) = \lambda u(\lambda^2 t, \lambda x) \), \( \pi_\lambda(t, x) = \lambda^2 \pi(\lambda^2 t, \lambda x) \), \( f_\lambda(t, x) = \lambda^3 f(\lambda^2 t, \lambda x) \);

while the conditions (0.10) and (0.11) are not in the form above and much stronger than those for the stationary solutions.

On the other hand, for the existence of weak solutions of the problem (0.13)–(0.16) in the sense of Leray [24], it suffices to assume \( f(x) = \nabla_x F(x) \) with some \( F(x) \in L^2(\Omega) \); and no smallness is necessary. The condition in Salvi [32] seems to be the composition of this one (condition for the existence of stationary weak solution) and the condition for the existence of non-stationary weak solution. (See Leray [24], [25].)

For the stationary problem (0.13)–(0.16), Galdi [9, Chapter 9, Theorem 9.4] and Miyakawa [29] showed that, if \( u(x) \) is a weak solution and if \( \sup_{x \in \Omega} (|x| + 1)|u(x)| \) is sufficiently small, then \( u(x) \) enjoys the energy identity, and every weak solution enjoying the energy inequality coincides with \( u(x) \) above. Kozono and Yamazaki [23] proved the same result under the more general assumption that \( \|u\|_{p, \infty} \) is sufficiently small. In other words, the uniqueness of weak solutions is proved only for small physically reasonable solutions, or small solutions in the class generalizing physically reasonable solutions. Hence it seems to be very difficult to prove the uniqueness of the solutions given by Salvi [32] without assuming conditions as above.

In order to describe the idea of the proof of this paper, let us recall the outline of the proof of Kozono and Nakao [15]. They modified the method of Fujita and Kato [7], in which they transformed the original equation into an integral equation on the interval \([0, t)\) for every \( t \in \mathbb{R} \). Kozono and Nakao [15] showed the unique solvability of the integral equation on the infinite interval \((-\infty, t)\) for every \( t \in \mathbb{R} \) under appropriate assumptions. If \( f(x) \) is independent of \( t \), it suffices to consider the linearization of this system around the stationary solution \( u(x) \) in the study of (0.5)–(0.9). However, if \( f(t, x) \) depends on \( t \), the linearization of this system around the solution of (0.1)–(0.4) depends on \( t \), and hence the linearization as above becomes difficult to handle. Instead, they solved the integral equation by regarding the Stokes operator as the principal part and everything else as the perturbation. However, for the integral on the infinite interval should converge so that the iteration scheme associated with the viewpoint above should work, the external force must enjoy decay property and regularity stronger than those in the case (0.13)–(0.16). Namely, under our weaker assumption, the convergence is difficult to prove.

Moreover, for three-dimensional exterior domains, the integral in question does not converge in \( L^3(\Omega) \) in general even under the stronger condition in [15], as is understood from the results of [5], [18], [19]. Hence we must work on the space \( L^{3, \infty}(\Omega) \) instead, as is stated in [5], [23]. But the weak-$L^p$ spaces contain nontrivial homogeneous functions, and the integral in question fails to converge.
in the strong topology in any of the weak-$L^p$ spaces when the integrand contains such homogeneous functions, as we shall see in Remark 1.2.

Our method is similar to the one in [15] in spirit, but in order to get around the difficulty above, we show that the integral in question does converge in the weak-$\ast$ topology of certain weak-$L^p$ spaces. For this purpose we employ duality argument, which leads naturally to the notion of mild solutions. Roughly speaking, a mild solution is a function, bounded in an appropriate function space, which solves the integral equation associated with the Navier-Stokes equation in the sense of distributions. As in Kozono and Yamazaki [21], [22], the duality between the Lorentz spaces $L^{n/(n-1),1}_p(\Omega)$ and $L^{n/2,\infty}_p(\Omega)$ plays the most important role. In order to employ the duality argument above, we prove a sharp version of the $L^p-L^q$ estimates of the Stokes semigroup formulated in the Lorentz spaces. This estimate itself seems to be of interest.

As a result, for three-dimensional exterior domains as well, we can construct bounded solutions in the whole time, including time-periodic and almost periodic solutions under an appropriate assumptions on $f(t,x)$, which is unique in a small ball in $L^{n,\infty}(\Omega)$ and depending continuously on $f(t,x)$. We can also show their stability under small initial perturbation in the same class $L^{n,\infty}(\Omega)$, which is exactly the same as the unique existence and the stability class of stationary solutions. Our class of time-dependent solutions is equipped with a norm invariant under the scaling above, and is a natural generalization of the class of stationary solutions introduced in [21], [22], and hence of the class of reasonable stationary solutions satisfying $u(x) \to 0$ as $|x| \to \infty$.

As is seen above, our assumption is more general than those in [15] possibly except the smallness. On the other hand, neither of our assumption or the assumption of [32] implies the other. In particular, we need not assume the square summability of $f(t,x)$.

The outline of this paper is as follows. In Sect. 1 we introduce necessary notations, introduce the notion of mild solutions and state main results. In Sect. 2 we derive a Lorentz space version of the $L^p-L^q$ estimates of the Stokes semigroup. In Sect. 3 we construct an operator which gives the solution of the integral equations in question. We then construct a mild solution in Sect. 4, and verify the necessary properties in Sect. 5. We study the asymptotic behavior of the difference between two mild solutions associated with the same external force in Sect. 6. Finally, in Sect. 7, we study the regularity of the mild solutions.

1. Main results

Before stating our result, we introduce some function spaces. For $1 < p < \infty$ and $1 \leq q \leq \infty$, let $L^{p,q}(\Omega)$ denote the Lorentz space on $\Omega$ defined by

$$L^{p,q}(\Omega) = \left\{ u(x) \in L^1_{\text{loc}}(\Omega) \mid \|u\|_{p,q} < +\infty \right\},$$

where

$$\|u\|_{p,q} = \left( \int_{\Omega} \left( \int_{\Omega} |u(x)|^p \, dx \right)^{q/p} \, dx \right)^{1/q}.$$
Navier-Stokes equations in the weak-$L^n$ space

where
\[ \|u\|_{p,q} = \left( \int_0^{+\infty} \left( s \mu \left( \{ x \in \Omega \mid |u(x)| > s \} \right) \right)^{1/p} ds \right)^{1/q} \]
for $1 \leq q < \infty$ and
\[ \|u\|_{p,q} = \sup_{s>0} s \mu \left( \{ x \in \Omega \mid |u(x)| > s \} \right)^{1/p}. \]

Although the function $\|u\|_{p,q}$ above does not satisfy the triangle inequality, there exists a norm equivalent to $\|u\|_{p,q}$, and with this norm the space $L^p_{p,q}(\Omega)$ becomes a Banach space. Note that the space $L^p_{p,p}(\Omega)$ is equivalent to the standard space $L^p(\Omega)$. For these spaces, real interpolation yields the equivalence
\[ (L^p_{0,0}(\Omega),L^p_{1,1}(\Omega))_{\theta,q} = L^p_{\theta,q}(\Omega), \]
where $1 < q < \infty$ and $0 < \theta < 1$ satisfy $1/p = (1 - \theta)/p_0 + \theta/p_1$ and $1 \leq q \leq \infty$. Note that this space is determined independently of the choice of $p_0$ and $p_1$ up to equivalent norms. (See Bergh and L"ofstr"om [2] or Triebel [35] for example.)

We next recall that, for every $1 < p < \infty$, we have the Helmholtz decomposition
\[ (L^p(\Omega))^n = L^p_{\sigma}(\Omega) \oplus G^p(\Omega), \]
where
\[ L^p_{\sigma}(\Omega) = \left\{ u(x) \in (L^p(\Omega))^n \mid \text{div } u(x) \equiv 0 \text{ in } \Omega \text{ and } v \cdot u(x) \equiv 0 \text{ on } \partial \Omega \right\} \]
and
\[ G^p(\Omega) = \left\{ u(x) = \text{grad } f(x) \in (L^p(\Omega))^n \text{ for some } f(x) \in L^p_{\text{loc}}(\Omega) \right\}. \]

For the proof, see Fujiwara and Morimoto [8], Miyakawa [28] and Simader and Sohr [33]. Let $P_p$ denote the projection operator from $(L^p(\Omega))^n$ onto $L^p_{\sigma}(\Omega)$ along $G^p(\Omega)$. Then the dual of the operator $P_p$ coincides with $P_{p/(p-1)}$. In particular, the operator $P_2$ is an orthogonal projection in the Hilbert space $(L^2(\Omega))^n$.

We next generalize the Helmholtz decomposition to the Lorentz spaces following Miyakawa and Yamada [30]. We have $P_p = P_q$ on $(L^p(\Omega))^n \cap (L^q(\Omega))^n$ and hence we can extend $P_p$ as a projection operator $P$ in $(\sum_{1 < p < \infty} L^p(\Omega))_n$. It follows that $P$ is also a projection in $(L^{p,q}(\Omega))^n$. Let $(L^{p,q}(\Omega))^n = L^{p,q}_\sigma(\Omega) \oplus G^{p,q}(\Omega)$ denote the associated direct sum decomposition.

Note furthermore that, for $1 \leq q < \infty$, the space $C^\infty_{0,\sigma}(\Omega)$ consisting of all the smooth solenoidal vector fields with compact support in $\Omega$ is dense in $L^{p,q}_\sigma(\Omega)$, and we can regard $L^{p,q}_{p/(p-1),q/(q-1)}(\Omega)$ as the dual space of $L^{p,q}_\sigma(\Omega)$. The dual of the closure of $C^\infty_{0,\sigma}(\Omega)$ in $L^{p,\infty}(\Omega)$ coincides with the space $L^{p/(p-1),1}_\sigma(\Omega)$.
In order to introduce the notion of mild solution, we define some function classes. Put $\mathcal{K} = BUC(\mathbb{R}, L^n_{α,∞}(\Omega))$ and $\mathcal{L} = BUC\left(\mathbb{R}, \left(\mathcal{L}^{n/2,∞}(\Omega)\right)^n\right)$, where $BUC(\mathbb{R}, X)$ denotes the set of bounded and uniformly continuous functions on $\mathbb{R}$, equipped with the norm $\|f|BUC(\mathbb{R}, X)\| = \sup_{t \in \mathbb{R}} \|f(t, \cdot)\|_X$.

Next, put $\mathcal{K}^+ = BC(\mathbb{R}^+, L^n(\mathcal{L}^{n/2,∞}(\Omega)))$ and $\mathcal{L}^+ = BC\left(\mathbb{R}^+, \left(\mathcal{L}^{n/2,∞}(\Omega)\right)^n\right)$, where $BC(\mathbb{R}^+, X)$ denotes the set of bounded continuous functions on $\mathbb{R}^+$ with values in the Banach space $X$, equipped with the norm $\|f|BC(\mathbb{R}^+, X)\| = \sup_{t \in \mathbb{R}^+} \|f(t, \cdot)\|_X$.

**Definition 1** A function $u(t, x) \in \mathcal{K}$ is said to be a mild solution of the system (0.1)–(0.4) if the identity
\[
(u(t, \cdot), \varphi) = \sum_{j,k=1}^n \int_0^{+∞} \left( u_j(t - τ, \cdot) u_k(t - τ, \cdot) - F_{jk}(t - τ, \cdot), \frac{∂}{∂x_j} \left( \exp(-τA)\varphi \right)_k \right) dτ
\]
holds for every $\varphi \in L^{n/(n-1),1}(\Omega)$ and every $t \in \mathbb{R}$.

**Definition 2** A function $u(t, x) \in \mathcal{K}^+$ is said to be a mild solution of the system (0.5)–(0.9) if the identity
\[
(u(t, \cdot), \varphi) = (a, \exp(-tA)\varphi) + \sum_{j,k=1}^n \int_0^t \left( u_j(t - τ, \cdot) u_k(t - τ, \cdot) - F_{jk}(t - τ, \cdot), \frac{∂}{∂x_j} \left( \exp(-τA)\varphi \right)_k \right) dτ
\]
holds for every $\varphi \in L^{n/(n-1),1}(\Omega)$ and every $t > 0$.

**Remark 1.1** As is explained in the introduction, the relations (1.1) and (1.2) are the weak form of the integral equations
\[
\begin{align*}
  u(t) &= \int_0^{+∞} \exp(-τA)\left[ -P[(u \cdot \nabla)u](t - τ) + P\nabla F(t - τ) \right] dτ \\
  u(t) &= \exp(-tA)a + \int_0^t \exp(-τA)\left[ -P[(u \cdot \nabla)u](t - τ) + P\nabla F(t - τ) \right] dτ
\end{align*}
\]
respectively, if we regard the term $(u \cdot \nabla)u$ as an element of the space above by way the duality pairing $(u \cdot \nabla)u, \varphi = -(u \otimes u, \nabla \varphi)$ for $\varphi \in C^∞_{0,σ}(\Omega)$. 

Then our main result is the following:

**Theorem 1.1.** There exist positive numbers \( A, \varepsilon \) and \( C_0 \) depending on \( n \) and \( \Omega \) such that the following holds:

1. For every \( F(t,x) \in L \) such that \( \| F \|_L < \varepsilon \), there exists one and only one mild solution \( u(t,x) \in K \) of the system \((0.1)-(0.4)\) with \( f(t,x) = \nabla_x F(t,x) \) such that \( \| u \|_K < A \). Moreover, for every \( \delta \in (0,\varepsilon) \), the mapping \( T \) from the closed ball in \( L \) centered at the origin with radius \( \delta \) to \( K \) defined by \( T(F) = u \) is uniformly continuous. Furthermore, the function \( u(t,x) \) is the only solution of \((0.1)-(0.4)\) in the sense of distributions in \( \mathbb{R} \times \Omega \) such that \( \| u \|_K < A \). Namely, the function \( u(t,x) \) is the only one satisfying the estimate \( \| u \|_K < A \), the identity \((1.5)\) for every \( \varphi(x) \in C_{0,\sigma}^\infty(\Omega) \) and every \( t > 0 \), and

\[
\frac{d}{dt}(u(t,\cdot),\varphi) = (u(t,\cdot),\Delta\varphi) + \sum_{j,k=1}^n \left(u_j(t,\cdot)u_k(t,\cdot) - F_{jk}(t,\cdot), \frac{\partial}{\partial x_j}\varphi_k\right) \tag{1.5}
\]

for every \( \varphi(x) \in C_{0,\sigma}^\infty(\Omega) \) and every \( t \in \mathbb{R} \).

2. For every \( F(t,x) \in L_+ \) and every \( a(x) \in L_\infty^n \) such that \( C_0\|a\|_{n,\infty} + \| F \|_{L_+} < \varepsilon \), there exists one and only one mild solution \( u(t,x) \in K_+ \) of the system \((0.5)-(0.9)\) with \( f(t,x) = \nabla_x F(t,x) \) such that \( \| u \|_{K_+} < A \). Moreover, for every \( \delta \in (0,\varepsilon) \), the mapping \( T_+ \) from the set \( \{ (F(t,x),a) \mid \| F \|_{L_+} + C_0\|a\|_{n,\infty} \leq \delta \} \) to \( K_+ \) defined by \( T_+(F,a) = u \) is uniformly continuous. Furthermore, the function \( u(t,x) \) is the only solution of the \((0.5)-(0.9)\) in the sense of distributions in \( \mathbb{R}_+ \times \Omega \) such that \( u(t,x) \in K_+ \) with \( \| u \|_{K_+} < A \). Namely, the function \( u(t,x) \) is the only one satisfying the estimate \( \| u \|_{K_+} < A \), the identity \((1.5)\) for every \( \varphi(x) \in C_{0,\sigma}^\infty(\Omega) \) and every \( t > 0 \), and

\[
(u(t,\cdot),\varphi) \to (a,\varphi) \text{ as } t \to +0 \tag{1.6}
\]

for every \( \varphi(x) \in C_{0,\sigma}^\infty(\Omega) \).

As an application to the unique existence of time periodic and almost periodic solutions, we have the following.

**Corollary 1.2.** Suppose that \( u(t,x) \in K \) is the unique mild solution such that \( \| u \|_K < A \). Then we have the following:

1. If \( F(t,\cdot) \) is time-periodic with period \( T \), then the unique mild solution \( u(t,x) \) such that \( \| u \|_K < A \) is also time-periodic with period \( T \).
2. If \( F(t,\cdot) \) is almost periodic with respect to \( t \in \mathbb{R} \), then the unique mild solution \( u(t,x) \) such that \( \| u \|_K < A \) is also almost periodic with respect to \( t \in \mathbb{R} \).
Remark 1.2 For three-dimensional exterior domains, the best spatial decay condition expected in general is \( u(t, \cdot) \in L^{3, \infty}(\Omega) \). On the other hand, if we put \( u(t, x) = U(x) \) with some homogeneous function \( U(x) \) of degree \(-1\) on \( \mathbb{R}^3 \) such that \( U(x) \in L^{3, \infty}(\mathbb{R}^3) \), the function \( V(t, \cdot) = \exp(-\tau A)\nabla(U \otimes U) \) is forward self-similar; namely, it enjoys the equality \( V(\lambda^2 \tau, \lambda x) = \lambda^{-3} V(\tau, x) \) for every \( \lambda, \tau > 0 \) and \( x \in \mathbb{R}^3 \). It follows that

\[
\| V(\tau, \cdot) \|_{q, \infty} = \left( \frac{1}{\sqrt{\tau}} \right)^3 \left\| V \left( 1, \frac{\cdot}{\sqrt{\tau}} \right) \right\|_{q, \infty} = C \tau^{3/2q - 3/2}
\]

for every \( q \in (1, \infty) \). We thus conclude that the right-hand side of (1.3) in Remark 1.1 is not Bochner integrable in \( L^{q, \infty} \) for any \( q \in (1, \infty) \).

Remark 1.3 Assertion 2 of Theorem 1.1 implies the Lyapunov stability of the solution given in Assertion 1 of Theorem 1.1. In particular, if \( F(t, x) \) is independent of \( t \), then by the same reasoning as in Corollary 1.2, the solution given in Assertion 1 becomes the stationary solution given in Kozono and Yamazaki [21], and Assertion 2 implies the stability of this stationary solution under small initial perturbation. This result removes the technical assumption \( \nabla u(x) \in L^{q, \infty}(\Omega) \) with some \( q > n \) on the stationary solution \( u(x) \) posed in Kozono and Yamazaki [22].

We also have the following theorem, which shows the asymptotic behavior of the difference of two solutions with the same external force.

**Theorem 1.3.** For every \( p \in (n, \infty) \) there exists a positive number \( \varepsilon(p) \leq \varepsilon \) such that the following holds. Suppose that \( a(x) \in L^{n, \infty}(\Omega) \) and \( F(t, x) \in \mathcal{L}_+ \) satisfy \( \| F \|_{\mathcal{L}_+} + C_0 \| a \|_{n, \infty} < \varepsilon(p) \), and let \( u(t, x) = K_+ \) denote the unique mild solution of (0.5)–(0.9) such that \( \| u \|_{K_+} < A \). Then there exists a positive constant \( C = C(p) \) such that, for every \( b(x) \in L^{n, \infty}(\Omega) \) such that \( C_0 \| b(x) - a(x) \|_{n, \infty} < \varepsilon(p) \), there uniquely exists a mild solution \( v(t, x) \in K_+ \) of (0.5)–(0.9) with \( a(x) \) replaced by \( b(x) \) such that \( \| v \|_{K_+} < A \) and that

\[
\| v(t, \cdot) - u(t, \cdot) \|_{p, \infty} \leq C \tau^{n/2p - 1/2}
\]

(1.7)

holds for every \( t > 0 \). Moreover, for every \( q \in (n, p) \), there exists a positive constant \( C' = C'(p, q) \) such that the estimate

\[
\| v(t, \cdot) - u(t, \cdot) \|_{q, 1} \leq C' \tau^{n/2q - 1/2}
\]

(1.8)

holds for every \( t > 0 \).

Remark 1.4 Even in the trivial case \( F(t, x) \equiv u(t, x) \equiv 0 \), we cannot expect the asymptotic stability in the space \( L^{n, \infty} \) itself. This is observed in the following fact. Suppose that \( \Omega = \mathbb{R}^3 \), and put \( b(x) = (0, 0, \log |x|) \) and

\[
a(x) = c \text{rot } b(x) = c \left( \frac{x_2}{|x|^2}, -\frac{x_1}{|x|^2}, 0 \right).
\]


Then \( a(x) \in L^3_\sigma(\mathbb{R}^3) \). Hence Kozono and Yamazaki [20] implies that, if \(|c|\) is sufficiently small, there exists a solution \( u(t, x) \in BC((0, +\infty), L^3_\sigma(\mathbb{R}^3)) \) of the evolution equation

\[
\frac{du}{dt}(t, x) = -Au(t, x) - P \left( (u(t, \cdot) \cdot \nabla)u(t, \cdot) \right)(x) + f(t, x)
\]

with \( f(t, x) \equiv 0 \) on \((0, +\infty)\), satisfying a kind of boundedness property and the initial condition \( u(0, x) = a \) in a suitable sense. Since the initial data \( a(x) \) is homogeneous of order \(-1\), it follows that the solution \( u(t, x) \) is forward self-similar; namely, \( u(t, x) \) enjoys the scaling property

\[
u(\lambda^{2} t, \lambda x) = \lambda^{-1} u(t, x).
\]

for every \( \lambda, t > 0 \) and \( x \in \mathbb{R}^n \). From this fact we see that \( \|u(t, \cdot)\|_{3, \infty} \) is independent of \( t > 0 \). This implies that even the trivial solution 0 is not asymptotically stable in the space \( L^3_\sigma(\mathbb{R}^3) \), in contrast to the space \( L^3(\mathbb{R}^3) \).

As for the regularity of the mild solutions, we have only modest results under our general assumptions.

**Theorem 1.4.** Suppose that \( n/2 < p < n \).

1. Let \( u(t, x) \in K \) be a mild solution of (0.1)–(0.4). Then, for every \( s, t \in \mathbb{R} \) such that \( s < t \), we have \( u(t, \cdot) - u(s, \cdot) \in L^{p,1}(\mathbb{R}^3) \) and \( \|u(t, \cdot) - u(s, \cdot)\|_{p,1} \leq C(t - s)^{n/2p - 1/2} \) with a constant \( C \) independent of \( s \) and \( t \).

2. Suppose that \( a(x) \) enjoys the condition \( A^{n/2p - 1/2}a(x) \in L^{p,\infty}(\Omega) \) as well, and let \( u(t, x) \in K_+ \) be a mild solution of (0.5)–(0.9). Then, for every \( s, t \in \mathbb{R} \) such that \( 0 < s < t \), we have the same conclusion as in Assertion 1.

**Remark 1.5** Theorem 1.4 implies that the function \( u(t, \cdot) \) is Hölder continuous with respect to \( t \) with values in the space \( L^3_{\sigma,\infty}(\Omega) + L^{p,1}_\sigma(\Omega) \). On the other hand, as we have seen in Remark 1.1, mild solutions \( u(t, x) \) of (0.1)–(0.4) and (0.5)–(0.9) can be regarded as formal solutions of the integral equations (1.3) and (1.4) respectively. Hence, if the external force \( f(t) \) enjoys some suitable Hölder continuity, it may be possible to deduce that the equality

\[
\frac{\partial u}{\partial t} = Au - P((u \cdot \nabla)u) + f
\]

holds as functions with values in \( \tilde{H}^{-1/n+p}((n+p), \infty) + \tilde{H}^{-1}_n(\Omega) \), if we regard the Stokes operator as a densely defined closed linear operator in the space \( \tilde{H}^{-1/n+p}((n+p), \infty) + \tilde{H}^{-1}_n(\Omega) \) with some \( q' < np/(n+p) \) and \( r' > n/2 \), which is strictly larger than \( \tilde{H}^{-1/n}_n(\Omega) \). However we do not enter this problem in detail. Amann [1] extensively studied such relationship between mild solutions and weak solutions, but his study seems to be limited to the solutions strongly continuous at \( t = 0 \) and not to be applicable to our solutions.
For the time being we can prove better results except for three-dimensional exterior domain. Namely, we have the following:

**Theorem 1.5.** Suppose either that \( n \geq 4 \), \( \Omega = \mathbb{R}^3 \) or \( \Omega = \mathbb{R}^3_+ \), and let \( q \) be a number such that \( n/3 < q < n/2 \).

1. Suppose that \( F(t, x) \in \mathcal{L} \) enjoys the condition

\[
\|F(t, \cdot) - F(s, \cdot)\|_{q, \infty} \leq C|t - s|^{n/2q - 1} \tag{1.10}
\]

for every \( s, t \in \mathbb{R} \) such that \( s < t \) with some positive constant \( C \), and let \( u(t,x) \) be a mild solution of (0.1)–(0.4). Then \( \{\nabla u(t, \cdot)\}_{t \in \mathbb{R}} \) is bounded in \( L^{n/2, \infty}(\Omega) \), and we also have

\[
\|\nabla u(t, \cdot) - \nabla u(s, \cdot)\|_{q, \infty} \leq C|t - s|^{n/2q - 1} \tag{1.11}
\]

for every \( s, t \in \mathbb{R} \) such that \( s < t \).

2. Suppose that \( F(t, x) \) enjoys the condition (1.10) for every \( s, t \in \mathbb{R} \) such that \( 0 < s < t \) with some positive constant \( C \), and that \( a \in L^{n/\infty}(\Omega) \) and \( \nabla a \in L^{n/2, \infty}(\Omega) \). Let \( u(t,x) \) be a mild solution of (0.5)–(0.9). Then \( \{\nabla u(t, \cdot)\}_{t > 0} \) is bounded in \( L^{n/2, \infty}(\Omega) \). Moreover, if we assume \( A^{n/2q - 1/2} a \in L^{q, \infty}(\Omega) \), we also have (1.11) for every \( s, t \in \mathbb{R} \) such that \( 0 < s < t \).

From the theorem above we can show that our mild solutions coincide with strong solutions under additional assumptions when the dimension is 4 or higher.

**Corollary 1.6.** Suppose that \( n \geq 4 \), and let \( r \) be a number such that \( n/4 < r < n/3 \).

1. Suppose that \( f(t, \cdot) \) is a bounded uniformly continuous function with respect to \( t \in \mathbb{R} \) with values in \( L^{n/3, \infty}(\Omega) \) satisfying the inequality

\[
\|f(t, \cdot) - f(s, \cdot)\|_{r, \infty} \leq C(t - s)^{n/2r - 3/2} \tag{1.12}
\]

for every \( s, r \in \mathbb{R} \) such that \( s < t \) with some positive constant \( C \). Let \( u(t,x) \) be a mild solution of (0.1)–(0.4) with \( F(t, x) \) such that \( f(t, x) = P \nabla F(t, x) \). Then \( u(t,x) \) is a strong solution of the evolution equation (1.9) on \( \mathbb{R} \).

2. Suppose that \( f(t, \cdot) \) is a bounded continuous function with respect to \( t > 0 \) with values in \( L^{n/3, \infty}(\Omega) \) satisfying the inequality (1.12) for every \( s, r \in \mathbb{R} \) such that \( 0 < s < t \) with some positive constant \( C \). Suppose moreover that \( a(x) \in L^{n, \infty}(\Omega) \) and that \( A^{n/2r - 1} a(x) \in L^{n/(n-r), \infty}(\Omega) \). Let \( u(t,x) \) be a mild solution of (0.5)–(0.8) with \( F(t, x) \) such that \( f(t, x) = P \nabla F(t, x) \). Then \( u(t,x) \) is a strong solution of the evolution equation (1.9) on \( (0, +\infty) \) and enjoys the condition \( u(t, \cdot) \to a \) in the sense of distributions as \( t \to +0 \).
2. \( L^p - L^q \) estimates

In this section we prove a sharp version formulated in the Lorentz spaces of the \( L^p - L^q \) estimates of the Stokes semigroup. Standard \( L^p - L^q \) estimates are proved by Kato [13], Ukai [36], Iwashita [12], Giga and Sohr [11] and Borchers and Miyakawa [3], [4]. We start with a fact which corresponds to [4, Theorem 4.4].

**Theorem 2.1.** Suppose that \( \Omega \) is an exterior domain in \( \mathbb{R}^n \). Then we have the following:

1. For every \( p \in (1, \infty) \), there exists a positive constant \( C \) such that the inequalities \( \| A^{1/2}u \|_{p,1} \leq C \| \nabla u \|_{p,1} \) and \( \| A^{1/2}u \|_{p,\infty} \leq C \| \nabla u \|_{p,\infty} \) always hold.
2. For every \( p \in (1, n) \), there exists a positive constant \( C \) such that the inequality \( \| \nabla u \|_{p,\infty} \leq C \| A^{1/2}u \|_{p,\infty} \) always holds.
3. For every \( p \in (1, n] \), there exists a positive constant \( C \) such that the inequality
   \[
   \| \nabla u \|_{p,1} \leq C \| A^{1/2}u \|_{p,1} \tag{2.1}
   \]
   always holds.

**Proof.** By real interpolation, we can derive Assertion 1 from [4, Theorem 4.4, (i)], and Assertion 2 from [4, Theorem 4.4, (ii)].

It remains to prove Assertion 3. Suppose that \( u(x) \in C^\infty_{0,\sigma}(\Omega) \), and put \( f(x) = -\Delta u(x) \). Then Kozono and Yamazaki [21, Theorem 2.1, (i)] implies that \( u(x) \) is the only solution of the equation

\[
-\Delta u(x) = f(x) \quad \text{and} \quad \text{div } u(x) = 0 \quad \text{on } \Omega
\]

such that \( \nabla u(x) \in L^{p,1}(\Omega) \), and that \( u(x) \) is subject to the estimate

\[
\| \nabla u \|_{p,1} \leq C \left\| f \right\| \hat{H}^{1}_{p/(p-1),\infty}(\Omega)^* \tag{2.2}
\]

Here \( \hat{H}^{1}_{p/(p-1),\infty}(\Omega) \) denotes the completion of the space \( C^\infty_{0,\sigma}(\Omega) \) with respect to the norm \( \| \nabla \varphi \|_{p/(p-1),\infty} \).

In view of (2.2), it suffices to show the estimate

\[
\left\| f \right\| \hat{H}^{1}_{p/(p-1),\infty}(\Omega)^* \leq C \| A^{1/2}u \|_{p,1}. \tag{2.3}
\]

Let \( \varphi(x) \) be an arbitrary element of \( C^\infty_{0,\sigma}(\Omega) \) such that \( \| \nabla \varphi \|_{p/(p-1),\infty} \leq 1 \). Then we have

\[
| (f, \varphi) | = | (Au, \varphi) | = | (A^{1/2}u, A^{1/2} \varphi) | \leq C \| A^{1/2}u \|_{p,1} \| A^{1/2} \varphi \|_{p/(p-1),\infty} \leq C \| A^{1/2}u \|_{p,1} \| \nabla \varphi \|_{p/(p-1),\infty},
\]

where the last inequality follows from Assertion 1. Now (2.3) follows immediately from this inequality.
From the theorem above we can prove the following version in the Lorentz spaces of the $L^p$-$L^q$ inequality. □

**Theorem 2.2.** For every $p \in (1, \infty)$, the operator $-A$ generates a bounded analytic semigroup $\exp(-tA)$ on $L^{p,1}(\Omega)$, and this semigroup enjoys the following estimates for $p, q$ such that $1 < p \leq q < \infty$:

1. There exists a positive constant $C$ such that the estimates
   \[ \| \exp(-tA)u \|_{q,1} \leq Ct^{n/2q-n/2p} \| u \|_{p,1} \]  
   (2.4)

   and
   \[ \| A^{1/2} \exp(-tA)u \|_{q,1} \leq Ct^{n/2q-n/2p-1/2} \| u \|_{p,1} \]  
   (2.5)

   hold for every $u \in L^{p,1}_\sigma(\Omega)$ and every $t > 0$.

2. Suppose that $q \leq n$ in the case where $\Omega$ is an exterior domain. Then there exists a positive constant $C$ such that the estimates
   \[ \| \nabla \exp(-tA)u \|_{q,1} \leq Ct^{n/2q-n/2p-1/2} \| u \|_{p,1} \]  
   (2.6)

   and
   \[ \| \nabla A^{1/2} \exp(-tA)u \|_{q,1} \leq Ct^{n/2q-n/2p-1} \| u \|_{p,1} \]  
   (2.7)

   hold for every $u \in L^{p,1}_\sigma(\Omega)$ and every $t > 0$.

**Proof.** Applying real interpolation to the results of Iwashita [12] or Borchers and Miyakawa [4, Corollary 4.6, (i)], we see that the operator $-A$ generates a bounded analytic semigroup $\exp(-tA)$ on $L^{p,1}(\Omega)$ and that the estimates (2.4) and (2.5) hold. Next, since $p \leq q \leq n$, Theorem 2.1 and the estimates (2.4) and (2.5) imply that

   \[ \| \nabla \exp(-tA)u \|_{q,1} \leq C A^{1/2} \exp\left( -\frac{t}{2} A \right) \exp\left( -\frac{t}{2} A \right) u \|_{q,1} \]
   \[ \leq C t^{-1/2} \exp\left( -\frac{t}{2} A \right) u \|_{q,1} \]
   \[ \leq C t^{-1/2+n/2q-n/2p} \| u \|_{p,1} \]

   and

   \[ \| \nabla A^{1/2} \exp(-tA)u \|_{q,1} \leq C A \exp\left( -\frac{t}{2} A \right) \exp\left( -\frac{t}{2} A \right) u \|_{q,1} \]
   \[ \leq C t^{-1} \exp\left( -\frac{t}{2} A \right) u \|_{q,1} \]
   \[ \leq C t^{-1+n/2q-n/2p} \| u \|_{p,1} \]

   which imply (2.6) and (2.7) respectively. □
Remark 2.1. The estimate (2.6) for $1 < p \leq q < n$ follows immediately from the result of Iwashita [12] and real interpolation. In order to treat the case $q = n$ we need Theorem 2.1.

The next corollary gives a sharp version of the $L^p$-$L^q$ estimates formulated in the Lorentz spaces.

**Corollary 2.3.** Suppose that $1 < p < q < \infty$.

1. There exists a constant $C$ such that we have the estimates
   \[ t^{n/2} p^{-n/2q} \| \exp(-tA)u \|_{q,1} \leq C \| u \|_{p,\infty} \] (2.8)
   and
   \[ t^{n/2} p^{-n/2q + 1/2} \| A^{1/2} \exp(-tA)u \|_{q,1} \leq C \| u \|_{p,\infty} \] (2.9)
   for every $u \in L^p_{p,\infty}(\Omega)$ and every $t > 0$, and
   \[ \int_0^{+\infty} t^{n/2} p^{-n/2q} - 1 \| \exp(-tA)u \|_{q,1} \, dt \leq C \| u \|_{p,1} \] (2.10)
   and
   \[ \int_0^{+\infty} t^{n/2} p^{-n/2q - 1/2} \| A^{1/2} \exp(-tA)u \|_{q,1} \, dt \leq C \| u \|_{p,1} \] (2.11)
   for every $u \in L^p_{p,1}(\Omega)$.

2. Suppose that $q \leq n$ in the case that $\Omega$ is an exterior domain. Then the constant $C$ can be taken so that we also have the estimates
   \[ t^{n/2} p^{-n/2q + 1/2} \| \nabla \exp(-tA)u \|_{q,1} \leq C \| u \|_{p,\infty} \] (2.12)
   and
   \[ t^{n/2} p^{-n/2q + 1} \| \nabla A^{1/2} \exp(-tA)u \|_{q,1} \leq C \| u \|_{p,\infty} \] (2.13)
   for every $u \in L^p_{p,\infty}(\Omega)$ and every $t > 0$, and
   \[ \int_0^{+\infty} t^{n/2} p^{-n/2q - 1} \| \nabla \exp(-tA)u \|_{q,1} \, dt \leq C \| u \|_{p,1} \] (2.14)
   and
   \[ \int_0^{+\infty} t^{n/2} p^{-n/2q} \| \nabla A^{1/2} \exp(-tA)u \|_{q,1} \, dt \leq C \| u \|_{p,1} \] (2.15)
   for every $u \in L^p_{p,1}(\Omega)$. 
Proof. Fix \( q \) and apply real interpolation with the parameter \( p \in (1, q) \) to the inequalities (2.4), (2.5), (2.6) and (2.7), taking into account of the fact 
\[
(L_{p_1}^{p_1}, L_{p_2}^{p_2})_{\theta, \infty} = L^{p, \infty}
\]
with \( 1/p = (1-\theta)/p_1 + \theta/p_2 \). Then we see that the estimates (2.8) and (2.9) hold for every \( t > 0 \), and that the estimate (2.12) and (2.13) hold for every \( t > 0 \) as well if \( q \leq n \) in the case that \( \Omega \) is an exterior domain.

In order to prove (2.10), fix \( p \) and \( q \) and choose \( p_1 \) and \( p_2 \) such that \( 1 < p_1 < p < p_2 < q \) and \( 1/p - 1/p_2 < 2/n \). Then we put \( \rho = n/2p - n/2q - 1 \), and consider the sublinear operator \( T \) which maps a function \( u(x) \in L_{p_1}^{p_1} \sigma (\Omega), L_{p_2}^{p_2} \sigma (\Omega) \) to a function \( v(t) \) on \((0, +\infty)\) defined by the formula

\[
v(t) = t^\rho \| \exp(-tA)u \|_{q,1}.
\]

Then (2.8) implies that, for \( j = 1, 2 \), we have \( v(t) \in L^{q_1}(0, +\infty) \) and \( \| v \|_{q_1} \leq C \| u \|_{p_j, \infty} \). Furthermore, since the constant \( \theta \in (0, 1) \) such that \( 1/p = (1-\theta)/p_1 + \theta/p_2 \) enjoys the equality \( 1 = (1-\theta)/s_1 + \theta/s_2 \), we have the following real interpolation relations among quasi-Banach spaces as follows:

\[
(L_{\sigma}^{p_1, \infty}(\Omega), L_{\sigma}^{p_2, \infty}(\Omega))_{\theta, 1} = L_{\sigma}^{p, \infty}(\Omega)
\]

and

\[
(L^{q_1, \infty}(0, +\infty), L^{q_2, \infty}(0, +\infty))_{\theta, 1} = L^{q, \infty}(0, +\infty).
\]

Hence, by applying real interpolation for the operator \( T \), (see Bergh and L"ofstr"om [2, Sect. 5.3] or Komatsu [14],) we conclude that \( v(t) \in L^1((0, +\infty)) \) for \( u(x) \in L_{p_1}^{p_1}(\Omega) \) and that the estimate

\[
\int_0^{+\infty} v(t) \, dt \leq C \| u \|_{p, 1}
\]

holds with a constant \( C \) independent of \( u(x) \). Hence we have (2.10).

The estimates (2.11), (2.14) and (2.15) can be proved in the same way. \( \square \)

At the end of this section we prove the following fact.

**Corollary 2.4.** Suppose that \( p \leq n \). Then there exists a positive constant \( C \) such that, for every \( L_{p_1}^{p_1}(\Omega) \) and every \( T \in (0, +\infty] \), we have the estimate

\[
\| \nabla A^{1/2} \int_0^T \exp(-tA)\varphi \, dt \|_{p, 1} \leq C \| \varphi \|_{p, 1}.
\]

**Proof.** Since \( p \leq n \), Theorem 2.1, 3 implies that the left-hand side of (2.16) is dominated from above by

\[
C \| A \int_0^T \exp(-\tau A)\nabla \varphi \, d\tau \|_{p, 1} = C \| \varphi - \exp(-TA)\varphi \|_{p, 1}.
\]

This completes the proof. \( \square \)
Navier-Stokes equations in the weak-$L^n$ space

3. Integral equations

In this section we show a theorem on the solvability of the integral equations equivalent to the original problems. For $u(t, \cdot), v(t, \cdot) \in K$ and $F(t, \cdot) \in L$, we define $\Phi[u, v, F](t, x)$ so that the formula

$$(\Phi[u, v, F](t, \cdot), \varphi) = \sum_{j,k=1}^{n} \int_{0}^{+\infty} \left( u_j(t - \tau, \cdot)v_k(t - \tau, \cdot) - F_{jk}(t - \tau, \cdot), \frac{\partial}{\partial x_j} \left( \exp(-\tau A) \varphi \right) \right) d\tau$$

holds for every $\varphi \in L^n/(n-1), 1_{\sigma}(\Omega)$, and that

$$(\Phi[u, v, F](t, \cdot), \nabla_x \psi) = 0$$

for every scalar function $\psi$ such that $\nabla_x \psi \in \left(L^n/(n-1), 1_{\sigma}(\Omega)\right)^n$. On the other hand, for $u(t, \cdot), v(t, \cdot) \in K_+ F(t, \cdot) \in L_+$ and $a(x) \in L^{n,\infty}(\Omega)$, we define $\Psi[u, v, F, a](t, x)$ so that the formula

$$(\Psi[u, v, F, a](t, \cdot), \varphi) = (a, \exp(-tA)\varphi) + \sum_{j,k=1}^{n} \int_{0}^{t} \left( u_j(t - \tau, \cdot)v_k(t - \tau, \cdot) - F_{jk}(t - \tau, \cdot), \frac{\partial}{\partial x_j} \left( \exp(-\tau A) \varphi \right) \right) d\tau$$

holds for every $\varphi \in L^n/(n-1), 1_{\sigma}(\Omega)$, and that

$$(\Psi[u, v, F, a](t, \cdot), \nabla_x \psi) = 0$$

for every scalar function $\psi$ such that $\nabla_x \psi \in \left(L^n/(n-1), 1_{\sigma}(\Omega)\right)^n$. Then, by virtue of Corollary 2.3, we can prove the following theorem.

**Theorem 3.1.** There exist positive constants $C_1, C_2$ and $C_3$ such that the following holds:

1. For every $u(t, x), v(t, x) \in K$ and every $F(t, x) \in L$, the function $\Phi[u, v, F](t, x)$ is well-defined in $K$, and satisfies the estimate

$$\| \Phi[u, v, F] \|_{K} \leq C_1 \| F \|_{L} + C_2 \| u \|_{K} \| v \|_{K}$$

2. For every $u(t, x), v(t, x) \in K_+, F(t, x) \in L_+$ and every $a(x) \in L^{n,\infty}(\Omega)$, the function $\Psi[u, v, F, a](t, x)$ is well-defined in $K_+$, and satisfies the estimate

$$\| \Psi[u, v, F, a] \|_{K_+} \leq C_1 \| F \|_{L_+} + C_2 \| u \|_{K_+} \| v \|_{K_+} + C_3 \| a \|_{n,\infty}.$$
Proof. For every fixed \( t \) and \( \varphi \in L^{n/(n-1)}_\rho(\Omega) \), Corollary 2.3, 1 implies that the modulus of the first term of the right-hand side of (3.3) is dominated by

\[
C \|a\|_{n,\infty} \|\exp(-tA)\varphi\|_{n/(n-1),1} \leq C \|a\|_{n,\infty} \|\varphi\|_{n/(n-1),1}. \tag{3.5}
\]

On the other hand, since \( n \geq 3 \), we have \( n/(n-2) \leq n \). Hence, for every fixed \( t \) and \( \varphi \in L^{n/(n-1)}_\rho(\Omega) \), we can apply Corollary 2.3, 2 with \( p = n/(n-1) \) and \( q = n/(n-2) \) to see that the modulus of the right-hand side of (3.1) and that of the second term of the right-hand side of (3.3) is dominated by

\[
\sum_{j,k=1}^n \int_0^{+\infty} \left| \left( u_j(t - \tau, \cdot) v_k(t - \tau, \cdot) - F_{jk}(t - \tau, \cdot) \right) \frac{\partial}{\partial x_j} \left( \exp(-\tau A)\varphi \right)_k \right| d\tau \\
\leq C \sum_{j,k=1}^n \int_0^{+\infty} \|\nabla_x \exp(-\tau A)\varphi\|_{n/(n-2),1} \\
\left( \|u_j(t - \tau, \cdot)\|_{n,\infty} \|v_k(t - \tau, \cdot)\|_{n,\infty} + \|F_{jk}(t - \tau, \cdot)\|_{n/2,\infty} \right) d\tau \\
\leq C \left( ||F|\mathcal{L}| + ||u|\mathcal{K}| ||v|\mathcal{K}| \right) \|\varphi\|_{n/(n-1),1}. \tag{3.6}
\]

The estimates (3.5) and (3.6), together with the facts (3.2) and (3.4), imply that \( \Phi[u, v, F](t) \in (L^{n,\infty}(\Omega))^n \) and \( \Psi[u, v, F, a](t) \in (L^{n,\infty}(\Omega))^n \) the estimates

\[
\|\Phi[u, v, F](t)\|_{n,\infty} \leq C_1 \|F|\mathcal{L}| + C_2 \|u|\mathcal{K}| \|v|\mathcal{K}| \\
\|\Phi[u, v, F, a](t)\|_{n,\infty} \leq C_1 \|F|\mathcal{L}| + C_2 \|u|\mathcal{K}| \|v|\mathcal{K}| + C_3 \|a\|_{n,\infty}
\]

hold for every \( t \). Moreover, in view of (3.2) and (3.4), we see that \( \Phi[u, v, F](t) \) and \( \Psi[u, v, F, a](t) \) belong to \( L^{n,\infty}(\Omega) \) for every \( t \). Since \( C_1 \), \( C_2 \) and \( C_3 \) are independent of \( t \), we conclude that \( \Phi[u, v, F](t) \) and \( \Psi[u, v, F, a](t) \) are bounded functions of \( t \) with values in \( L^{n,\infty}(\Omega) \), and that they enjoy the required estimates.

It remains to show the continuity. We first consider \( \Phi[u, v, F](t) \). For every positive number \( \varepsilon \), we can take a positive number \( \delta \) so that \( \sup_t \|u(t + s, \cdot) - u(t, \cdot)\|_{n,\infty} < \varepsilon \), \( \sup_t \|v(t + s, \cdot) - v(t, \cdot)\|_{n,\infty} < \varepsilon \) and \( \sup_t \|F(t + s, \cdot) - F(t, \cdot)\|_{n/2,\infty} < \varepsilon \) holds for every \( s \in (0, \delta) \). For such \( s \) we put \( \tilde{u}(t) = u(t + s) \), \( \tilde{v}(t) = v(t + s) \) and \( \tilde{F}(t) = F(t + s) \). Then we have
Navier-Stokes equations in the weak-\(L^n\) space

\[ \Phi[\tilde{u}, \tilde{v}, \tilde{F}](t) = \Phi[u, v, F](t + s). \]

It follows that

\[
\| \Phi[u, v, F](t + s) - \Phi[u, v, F](t) \|_{n, \infty} \\
= \| \Phi[\tilde{u}, \tilde{v}, \tilde{F}](t) - \Phi[u, v, F](t) \|_{n, \infty} \\
= \| \Phi[\tilde{u}, \tilde{v} - v, 0](t) + \Phi[\tilde{u} - u, v, 0](t) + \Phi[0, 0, \tilde{F} - F](t) \|_{n, \infty} \\
\leq C_2 \| u \|_{K} \| \tilde{v} - v \|_{K} + C_2 \| \tilde{u} - u \|_{K} \| v \|_{K} + C_1 \| \tilde{F} - F \|_{L^\infty} \\
< \{ C_2 (\| u \|_{K} + \| v \|_{K}) + C_1 \} \varepsilon
\]

for every \( s \) such that \( 0 < s < \delta \). Since the right-hand side can be taken arbitrarily small, this implies the required uniform continuity of \( \Phi[u, v, F](t) \) with values in \( L^n_{\infty}(\Omega) \).

We next consider \( \Psi[u, v, F, a](t) \). Let \( t \) and \( \varepsilon \) be positive numbers. Then there exists a positive number \( \delta < \min\{\varepsilon, t\} \) such that, for every \( \psi(x) \in L^n_{\infty}(\Omega) \), we have the inequality

\[
\int_{t-\delta}^{t+\delta} \| \nabla, \exp(-\tau A)\psi \|_{n/(n-2), 1} d\tau \leq C \int_{t-\delta}^{t+\delta} \frac{1}{\tau} d\tau \| \psi \|_{n/(n-1), 1} \\
\leq \varepsilon \| \psi \|_{n/(n-1), 1}.
\]

We now take \( s \in [t - \delta/2, t + \delta/2] \). Then we have

\[
(\Psi[u, v, F, a](t) - \Psi[u, v, F, a](s), \psi) \\
= (a, \exp(-t A)\psi - \exp(-s A)\psi) \\
+ \sum_{j,k=1}^{n} \int_{0}^{t-\delta} \left( u_j(t - \tau, \cdot)v_k(t - \tau, \cdot) - F_{jk}(t - \tau, \cdot) \\
- u_j(s - \tau, \cdot)v_k(s - \tau, \cdot) + F_{jk}(s - \tau, \cdot), \\
\frac{\partial}{\partial x_j}(\exp(-\tau A)\psi)_k \right) d\tau \\
+ \sum_{j,k=1}^{n} \int_{t-\delta}^{t} \left( u_j(t - \tau, \cdot)v_k(t - \tau, \cdot) - F_{jk}(t - \tau, \cdot), \\
\frac{\partial}{\partial x_j}(\exp(-\tau A)\psi)_k \right) d\tau
\]
\[ + \sum_{j,k=1}^{n} \int_{t-\delta}^{t} \left( F_{jk}(s-\tau, \cdot) - u_j(s-\tau, \cdot) v_k(s-\tau, \cdot), \right) \frac{\partial}{\partial x_j} \left( \exp(-\tau A) \psi \right) d\tau \]
\[ = I_1 + I_2 + I_3 + I_4. \quad (3.7) \]

Then we have
\[ |I_1| \leq C \| a \|_{n,\infty} |t - s| \sup_{0 \leq \theta \leq 1} \| A \exp(-((1 - \theta)s + \theta t)A) \varphi \|_{n/(n-1),1} \]
\[ \leq C \| a \|_{n,\infty} |t - s| \| A \exp(-(t - \tau)A) \varphi \|_{n/(n-1),1} \]
\[ \leq C \| a \|_{n,\infty} \| \varphi \|_{n/(n-1),1} \frac{\varepsilon}{t - \delta}. \quad (3.8) \]

We next have
\[ |I_3|, |I_4| \leq C \left( \| u \|_{K_{\infty}} + \| v \|_{K_\infty} + \| F \|_{L_\infty} \right) \int_{t-\delta}^{t+\delta} \| \nabla \psi \|_{n/(n-2),1} d\tau \]
\[ \leq C \left( \| u \|_{K_{\infty}} + \| v \|_{K_\infty} + \| F \|_{L_\infty} \right) \varepsilon \| \varphi \|_{n/(n-1),1}. \quad (3.9) \]

We finally estimate \( I_2 \). Since \( t - \delta/2 \leq s \leq t + \delta/2 \) and since \( 0 \leq \tau \leq t - \delta \), we have \( s - \tau, t - \tau \in \left[ \delta/2, t + \delta/2 \right] \). Since \( u(t, \cdot), v(t, \cdot) \) and \( F(t, \cdot) \) are uniformly continuous on the interval \( \left[ \delta/2, t + \delta/2 \right] \), we can take \( \gamma \in (0, \delta/2) \) so small that \( |t - s| < \gamma \) implies \( \| u(t, \cdot) - u(s, \cdot) \|_{n,\infty} < \varepsilon \), \( \| v(t, \cdot) - v(s, \cdot) \|_{n,\infty} < \varepsilon \) and \( \| F(t, \cdot) - F(s, \cdot) \|_{n/2,\infty} < \varepsilon \). It follows that
\[ |I_2| \leq C \left( \| u \|_{K_{\infty}} + \| v \|_{K_\infty} + 1 \right) \varepsilon \int_{0}^{\gamma} \| \nabla \psi \|_{n/(n-2),1} d\tau \]
\[ \leq C \left( \| u \|_{K_{\infty}} + \| v \|_{K_\infty} + 1 \right) \varepsilon \| \varphi \|_{n/(n-1),1}. \quad (3.10) \]

Combining (3.7), (3.8), (3.9) and (3.10) we conclude that \( |t - s| < \gamma \) implies
\[ \left| (\Psi[u, v, F, a](t) - \Psi[u, v, F, a](s), \varphi) \right| \leq C \| \varphi \|_{n/(n-1),1} \frac{\varepsilon}{t - \delta} \]
for every \( \varphi(x) \in L^{n/(n-1),1}_\sigma(\Omega) \). Since the dual space of \( L^{n/(n-1),1}_\sigma(\Omega) \) is identical with \( L^{\infty}_\sigma(\Omega) \), this implies
\[ \| \Psi[u, v, F, a](t) - \Psi[u, v, F, a](s) \|_{n,\infty} \leq C \frac{\varepsilon}{t - \delta}. \]

Since \( \varepsilon > 0 \) is arbitrary, this implies that \( \Psi[u, v, F, a](t) \) is continuous at \( t > 0 \) as an \( L^{\infty}_\sigma \)-valued function. \( \square \)
4. Construction of mild solutions

In this section we prove the existence part of the proof of Theorem 1.1. Suppose that either $F(t, x) \in L$ [resp. $F(t, x) \in L_+$ and $a(x) \in L^\infty_\alpha (\Omega)$.] Then we construct a sequence

$$\{u^{(j)}(t, x)\}_{j=0}^{\infty} = \left\{\left\{u^{(j)}_t(t, x)\right\}_{t=1}^{n}\right\}_{j=0}^{\infty}$$

inductively by $u^{(0)}(t, x) \equiv 0$ and $u^{(j+1)}(t, x) = \Phi [u^{(j)}_t, u^{(j)}, F](t, x)$ [resp. $u^{(j+1)}(t, x) = \Psi [u^{(j)}_t, u^{(j)}, F, a](t, x)$.] Then Theorem 3.1 immediately yields the following lemma.

**Lemma 4.1.** We have $u^{(j)}(t, x) \in K$ [resp. $u^{(j)}(t, x) \in K_+$] for every $j \in \mathbb{N},$ and putting $A_j = \|u^{(j)}| \ell \|$, [resp. $A_j = \|u^{(j)}| \ell_+ \|,$] we have $A_{j+1} \leq C_1 \|F| L \| + C_2 A_j^2$. [resp. $A_{j+1} \leq C_1 \|F| L_+ \| + C_3 \|a| n, \infty + C_2 A_j^2$.] Suppose that $\|F| L \| < 1/4 C_1 C_2$, [resp. $C_1 \|F| L_+ \| + C_3 \|a| n, \infty < 1/4 C_2,$] and let $A_\infty$ denote the smaller root of the equation $4 C_2 x^2 - x + C_1 \|F| L \| = 0$; [resp. $4 C_2 x^2 - x + C_1 \|F| L_+ \| + C_3 \|a| n, \infty = 0$] namely,

$$A_\infty = \frac{1 - \sqrt{1 - 4 C_1 C_2 \|F| L \|}}{2 C_2}.$$ 

Then it is easily seen by induction on $j$ that

$$A_j < A_\infty. \quad (4.1)$$

Next, in view of Lemma 4.1, we can put $B_j = \|u^{(j+1)} - u^{(j)}| K\|.$ Then we have the following lemma.

**Lemma 4.2.** We have $B_{j+1} \leq 2 B_j C_2 A_\infty$ for every $j \in \mathbb{N}.$

**Proof.** Since

$$u^{(j+2)} - u^{(j+1)}$$

$$= \Phi [u^{(j+1)}_t, u^{(j+1)}, F] - \Phi [u^{(j)}_t, u^{(j)}, F]$$

$$= \Phi [u^{(j+1)}_t, u^{(j+1)} - u^{(j)}, 0] + \Phi [u^{(j+1)}_t - u^{(j)}_t, u^{(j)}, 0]$$

or

$$u^{(j+2)} - u^{(j+1)}$$

$$= \Psi [u^{(j+1)}_t, u^{(j+1)}, F, a] - \Psi [u^{(j)}_t, u^{(j)}, F, a]$$

$$= \Psi [u^{(j+1)}_t, u^{(j+1)} - u^{(j)}, 0, 0] + \Psi [u^{(j+1)}_t - u^{(j)}_t, u^{(j)}, 0, 0],$$
Theorem 3.1 yields the inequality

\[
B_{j+1} = \| u^{(j+2)} - u^{(j+1)} \|_K \\
\leq C_2 \| u^{(j+1)} \|_K \| u^{(j+1)} - u^{(j)} \|_K \\
+ C_2 \| u^{(j+1)} - u^{(j)} \|_K \| u^{(j)} \|_K \\
\leq C_2 B_j (A_{j+1} + A_j) \\
\leq 2B_j C_2 A_\infty
\]

or

\[
B_{j+1} = \| u^{(j+2)} - u^{(j+1)} \|_K \\
\leq C_2 \| u^{(j+1)} \|_K \| u^{(j+1)} - u^{(j)} \|_K \\
+ C_2 \| u^{(j+1)} - u^{(j)} \|_K \| u^{(j)} \|_K \\
\leq 2B_j C_2 A_\infty
\]

This completes the proof. \(\square\)

In view of Lemma 4.2, we have

\[
\| u^{(k)} - u^{(j)} \|_K \leq \sum_{\ell=0}^{k-1} B_\ell \leq \sum_{\ell=0}^{k-1} B_0 (2C_2 A_\infty)^\ell \leq \frac{B_0 (2C_2 A_\infty)^j}{1 - 2C_2 A_\infty}
\]

or

\[
\| u^{(k)} - u^{(j)} \|_K \leq \frac{B_0 (2C_2 A_\infty)^j}{1 - 2C_2 A_\infty}
\]

for every \(j, k \in \mathbb{N}\) such that \(j < k\). Since

\[
2C_2 A_\infty = 1 - \sqrt{1 - 4C_1 C_2 \| F \|_L} < 1
\]

or

\[
2C_2 A_\infty = 1 - \sqrt{1 - 4C_1 C_2 \| F \|_L - 4C_1 C_2 \| a \|_{n, \infty}} < 1,
\]

we see that \(\| u^{(k)} - u^{(j)} \|_K \to 0\) or \(\| u^{(k)} - u^{(j)} \|_K \to 0\) as \(j, k \to +\infty\). It follows that, for every \(t\) and every \(\ell = 1, \ldots, n\), the function \(u^{(j)}(t, \cdot)\) converges to an element of \(L^{p, \infty}_b(\Omega)\) as \(j \to +\infty\). Let \(u(t, \cdot) = \{u_\ell(t, x)\}_{\ell=1}^n\) denote the limit. Then the formula (4.1) implies that \(\| u(t, \cdot) \|_{n, \infty} \leq A_\infty\) for every \(t \in \mathbb{R}\). It follows that \(u(t, x) \in K\) or \(u(t, x) \in K_+\) with \(\| u \|_K \leq A_\infty\) or \(\| u \|_{K_+} \leq A_\infty\). On the other hand, putting \(u(t, x) = v(t, x) = u^{(j)}(t, x)\)
in (3.1) or (3.3) and making use of the fact that \( u^{(ℓ+1)} = Φ[u^{(ℓ)}, F] \) or \( u^{(ℓ+1)} = Ψ[u^{(ℓ)}, F, a] \), we obtain

\[
(u^{(ℓ+1)}(t, \cdot), ϕ) = \sum_{j,k=1}^{n} \int_{0}^{+∞} \left( u_j^{(ℓ)}(t - τ, \cdot)u_k^{(ℓ)}(t - τ, \cdot) - F_{jk}(t - τ, \cdot), \right. \\
\left. \frac{∂}{∂x_j} \left( \exp(-τA)ϕ \right)_k \right) dτ
\]

or

\[
(u^{(ℓ+1)}(t, \cdot), ϕ) = (a, \exp(-tA)ϕ) + \sum_{j,k=1}^{n} \int_{0}^{t} \left( u_j^{(ℓ)}(t - τ, \cdot)u_k^{(ℓ)}(t - τ, \cdot) - F_{jk}(t - τ, \cdot), \right. \\
\left. \frac{∂}{∂x_j} \left( \exp(-τA)ϕ \right)_k \right) dτ.
\]

Letting \( ℓ → +∞ \), we obtain (1.1) or (1.2) for every \( ϕ ∈ L^n/(n−1), 1, σ(Ω) \) and \( t \). It follows that \( u(t, x) \) is the desired mild solution.

Thus the proof of the existence of mild solution is complete if we put \( A = 1/2C_2, ε = 1/4C_1C_2 \) and \( C_0 = C_3/C_1 \).

5. Properties of mild solutions

In this section we first prove the following theorem.

**Theorem 5.1.**

1. For every \( F(t, x) ∈ ℒ \), there exists at most one mild solution \( u(t, x) ∈ ℋ \) of the system (0.1)–(0.4) satisfying the condition \( ∥u∥_{ℋ} < A \). Furthermore, the mapping from \( F(t, x) ∈ ℒ \) to \( u(t, x) ∈ ℋ \), where \( u(t, x) \) is the mild solution above, is uniformly continuous on the closed ball \( B(0, δ) \) in \( ℒ \) to \( ℋ \) for every \( δ < ε \).

2. For every \( a(x) ∈ L_{n,∞}^n(Ω) \) and \( F(t, x) ∈ ℒ_+ \), there exists at most one mild solution \( u(t, x) ∈ ℋ_+ \) of the system (0.5)–(0.9) satisfying the condition \( C_0∥a∥_{n,∞} + ∥u∥_{ℋ_+} < A \). Furthermore, the mapping from \( (a(x), F(t, x)) ∈ L_{n,∞}^n(Ω) × ℒ_+ \) to \( u(t, x) ∈ ℋ_+ \), where \( u(t, x) \) is the mild solution above, is uniformly continuous on the set

\[
\left\{ (a(x), F(t, x)) \mid C_0∥a∥_{n,∞} + ∥F∥_{ℒ} \leq δ \right\} ⊂ L_{n,∞}^n(Ω) × ℒ_+
\]
to \( ℋ_+ \) for every \( δ > 0 \).
Proof. We prove Assertion 1 only, since Assertion 2 can be proved exactly in the same way.

Suppose that \( u(t,x) \) and \( v(t,x) \) are mild solutions with \( F(t,x) \in \mathcal{L} \) and \( G(t,x) \in \mathcal{L} \) satisfying the estimates \( \| u | \mathcal{K} \|, \| v | \mathcal{K} \| \leq A - \gamma \) for some \( \gamma > 0 \). Then, from the equalities \( u = \Phi[u, u, F] \) and \( v = \Phi[v, v, G] \), we obtain

\[
\begin{align*}
u - v &= \Phi[u, u, F] - \Phi[v, v, G] \\
&= \Phi[u, u, 0] + \Phi[u - v, 0] + \Phi[0, 0, F - G].
\end{align*}
\]

It follows that

\[
\| u - v | \mathcal{K} \| \leq C_2 (\| u | \mathcal{K} \| + \| v | \mathcal{K} \|) \| u - v | \mathcal{K} \| + C_1 \| F - G | \mathcal{L} \|
\]

\[
\leq (1 - 2C_2 \gamma) \| u - v | \mathcal{K} \| + C_1 \| F - G | \mathcal{L} \|.
\]

This implies

\[
\| u - v | \mathcal{K} \| \leq \frac{AC_1}{\gamma} \| F - G | \mathcal{L} \|. \tag{5.1}
\]

In particular, if \( F = G \), then we have \( u = v \). Next, suppose that \( \| F | \mathcal{L} \| \leq \delta \) for some \( \delta < \varepsilon \). Then any mild solution \( u(t,x) \) satisfying \( \| u | \mathcal{K} \| < A \) coincides with the mild solution constructed in the previous section. Hence \( u(t,x) \) enjoys the estimate

\[
\| u | \mathcal{K} \| \leq \frac{1 - \sqrt{1 - 4C_1C_2\| F | \mathcal{L} \|}}{2C_2} \leq A \left( 1 - \sqrt{1 - 4C_1C_2\delta} \right).
\]

Now put \( \gamma = A\sqrt{4C_1C_2\delta} \). Then, for every \( F(t,x), G(t,x) \in \mathcal{L} \) such that \( \| F | \mathcal{L} \|, \| G | \mathcal{L} \| \leq \delta \), the corresponding mild solution \( u(t,x), v(t,x) \in \mathcal{K} \) such that \( \| u | \mathcal{K} \|, \| v | \mathcal{K} \| < A \) enjoys the estimate \( \| u | \mathcal{K} \|, \| v | \mathcal{K} \| \leq A - \gamma \). It follows that (5.1) holds, and this implies the required uniform continuity. \( \square \)

We next prove the following theorem, which implies the equivalence between mild solutions and solutions in the sense of distributions. This completes the proof of Theorem 1.1.

**Theorem 5.2.**

1. Suppose that \( u(t,x) \in \mathcal{K} \). Then \( u(t,x) \) is a mild solution of (0.1)–(0.4) if and only if \( u(t,x) \) is a solution of (0.1)–(0.4) in the sense of distributions in \( \mathbb{R} \times \Omega \).

2. Suppose that \( u(t,x) \in \mathcal{K}_+ \). Then \( u(t,x) \) is a mild solution of (0.5)–(0.9) if and only if \( u(t,x) \) is a solution of (0.5)–(0.9) in the sense of distributions in \( \mathbb{R}_+ \times \Omega \).
Proof. We first introduce a notation. In the sequel we write $H_{jk}(t,x) = u_j(t,x)u_k(t,x) - F_{jk}(t,x)$ for every $j, k = 1, \ldots, n$. Then (1.1) and (1.2) are equivalent to
\begin{align*}
(u(t, \cdot), \varphi) &= \sum_{j,k=1}^{n} \int_{0}^{+\infty} \left( H_{jk}(t-\tau, \cdot), \frac{\partial}{\partial x_j} \left( \exp(-\tau A)^{\cdot} \right) \varphi \right) d\tau \\
\text{and} \quad (u(t, \cdot), \varphi) &= (a, \exp(-tA)\varphi) \\
&+ \sum_{j,k=1}^{n} \int_{0}^{t} \left( H_{jk}(t-\tau, \cdot), \frac{\partial}{\partial x_j} \left( \exp(-\tau A)^{\cdot} \right) \varphi \right) d\tau
\end{align*}
respectively.

We begin with Assertion 1. We first show that a mild solution $u(t, x)$ enjoys (1.5) at $t = t_0 \in \mathbb{R}$ for every $\varphi(x) \in C_0^\infty(\Omega)$. For this purpose it suffices to show that the term
\begin{align*}
\left( \frac{u(t, \cdot) - u(s, \cdot)}{t-s}, \varphi \right) &= \frac{1}{t-s} \int_{0}^{+\infty} \left( H_{jk}(t-\tau, \cdot) - H_{jk}(s-\tau, \cdot), \frac{\partial}{\partial x_j} \left( \exp(-\tau A)^{\cdot} \right) \varphi \right) d\tau \\
&= I_1 + I_2,
\end{align*}
where
\begin{align*}
I_1 &= \frac{1}{t-s} \sum_{j,k=1}^{n} \int_{0}^{t-s} \left( H_{jk}(\tau, \cdot), \frac{\partial}{\partial x_j} \left( \exp(-(t-\tau)A)^{\cdot} \right) \varphi \right) d\tau \\
\text{and} \\
I_2 &= \sum_{j,k=1}^{n} \int_{0}^{+\infty} \left( H_{jk}(s-\tau, \cdot), \frac{\partial}{\partial x_j} \left( \exp(-(t-s+\tau)A)^{\cdot} \right) \varphi \right) - \left( \exp(-\tau A)^{\cdot} \varphi \right) \frac{d\tau}{t-s}
\end{align*}
converges as $s \to t_0 - 0$ and $t \to t_0 + 0$, and the limit coincides with
\begin{align*}
\left( H_{jk}(t_0, \cdot), \frac{\partial}{\partial x_j} \varphi \right) + (u(t_0, \cdot), \Delta \varphi).
\end{align*}
Since \(u(\tau, \cdot)\) and \(F(\tau, \cdot)\) are continuous on \(\mathbb{R}\) as an \(L^{n,\infty}(\Omega)\)-valued function and as an \(L^{n/2,\infty}(\Omega)\)-valued function respectively and since
\[
\frac{\partial}{\partial x_j} \left( \exp (-\tau A) \varphi \right)_k \to \frac{\partial}{\partial x_j} \varphi_k
\]
in \(\left( L^{n/(n-2),1}(\Omega) \right)^n \) as \(\tau \to +0\) for \(\varphi \in C^\infty_{0,\sigma}(\Omega)\), it follows that
\[
I_1 \to \left( H_{jk}(t_0, \cdot), \frac{\partial}{\partial x_j} \varphi_k \right).
\] (5.2)

On the other hand, putting
\[
\psi(x) = \frac{(\exp(- (t-s) A) - I) \varphi}{t-s},
\] (5.3)
we see that \(P \psi(x) \in L^{n/(n-1),1}(\Omega)\) and hence \(I_2 = (u(s, \cdot), P \psi)\). Letting \(s \to t_0 - 0\), we see that \(u(s, \cdot)\) approaches to \(u(t_0, \cdot)\) in \(L^{n,\infty}_\sigma(\Omega)\). On the other hand, \(P \psi(x)\) approaches to \(P \left(- A \varphi(x) \right) = \Delta \varphi(x)\). It follows that
\[
I_2 \to \left( u(t_0, \cdot), \Delta \varphi \right).
\] (5.4)

Now the conclusion follows immediately from (5.2) and (5.4).

Conversely, suppose that \(u(t, x) \in \mathcal{K}\) is a solution of (0.1)–(0.4) in the sense of distributions. Then we have
\[
\frac{d}{dt} \left( u(t, \cdot), \varphi \right) = \left( u(t, \cdot), \Delta \varphi \right) + \sum_{j,k=1}^n \left( H_{jk}(t, \cdot), \frac{\partial}{\partial x_j} \varphi_k \right)
\]
for every \(\varphi(x) \in C^\infty_\sigma(\Omega)\). Now suppose that \(\psi(x) \in C^\infty_{0,\sigma}(\Omega)\) and \(t_0 < t_1\). Then, by approximating \(\exp\left(-(t_1 - t_0) A\right) \psi\) by functions in \(C^\infty_{0,\sigma}(\Omega)\), we obtain
\[
\frac{d}{dt} \left( u(t, \cdot), \exp\left(-(t_1 - t_0) A\right) \psi \right)\]
\[
= \left( u(t, \cdot), \Delta \exp\left(-(t_1 - t_0) A\right) \psi \right)\]
\[
+ \sum_{j,k=1}^n \left( H_{jk}(t, \cdot), \frac{\partial}{\partial x_j} \left( \exp\left(-(t_1 - t_0) A\right)\psi \right)_k \right).\] (5.5)

Here we remark that
\[
\Delta \exp\left(-(t_1 - t_0) A\right) \psi = -A \exp\left(-(t_1 - t_0) A\right) \psi
\]
\[
= -\frac{d}{dt_0} \exp\left(-(t_1 - t_0) A\right) \psi.
\]
Now put $t = t_0$ in (5.5) and making use of the equality above, we obtain
\[
\frac{d}{dt}(u(t, \cdot), \exp(-(t_1 - t)A)\psi) = \sum_{j,k=1}^{n} \left( H_{jk}(t, \cdot) \cdot \frac{\partial}{\partial x_j}(\exp(-(t_1 - t)A)\psi)_k \right).
\]
Integrating this formula on the interval $[t_0, t_1]$ with respect to $t$, we conclude
\[
(u(t_1, \cdot), \psi) - (u(t_0, \cdot), \exp(-(t_1 - t_0)A)\psi) = \sum_{j,k=1}^{n} \int_{t_0}^{t_1} \left( H_{jk}(t_1 - \tau, \cdot) \cdot \frac{\partial}{\partial x_j}(\exp(-\tau A)\psi)_k \right) d\tau.
\] (5.6)
Letting $t_0 \to -\infty$ and observing the facts $\sup_{t_0 \in \mathbb{R}} \|u(t_0, \cdot)\|_{n,\infty} < +\infty$ and $\lim_{t_1 \to +\infty} \|\exp(-tA)\psi\|_{n/(n-1),1} = 0$, we conclude that
\[
(u(t_1, \cdot), \psi) = \sum_{j,k=1}^{n} \int_{0}^{t_1 - t_0} \left( H_{jk}(t_1 - \tau, \cdot) \cdot \frac{\partial}{\partial x_j}(\exp(-\tau A)\psi)_k \right) d\tau,
\] which implies that $u(t, x)$ is a mild solution of (0.1)–(0.4).

We turn to the proof of Assertion 2. Suppose that $\varphi(x) \in C^\infty_{0,\sigma}(\Omega)$ and $t_0 > 0$. In this case we have
\[
\left( \frac{u(t, \cdot) - u(s, \cdot)}{t - s}, \varphi \right) = I_1 + I_3,
\]
where
\[
I_3 = (a, \exp(-sA)(\exp(-(t - s)A) - 1)\varphi) + \sum_{j,k=1}^{n} \int_{0}^{s} \left( H_{jk}(s - \tau, \cdot), \frac{\partial}{\partial x_j} \left( \frac{\exp(-(t - s + \tau)A)\varphi}{t - s} - \exp(-\tau A)\varphi \right)_k \right) d\tau.
\]
Putting $\psi(x)$ as in (5.3), we see that $I_3 = (u(s, \cdot), P\psi)$ also in this case. Now we can prove (1.5) in the same way as Assertion 1.

Next, for every $\varphi(x) \in C^\infty_{0,\sigma}(\Omega)$ and every $t > 0$, we have
\[
(u(t, \cdot), \varphi) - (a, \varphi) = (a, \exp(-tA)\varphi - \varphi) + \sum_{j,k=1}^{n} \int_{0}^{t} \left( H_{jk}(t - \tau, \cdot), \frac{\partial}{\partial x_j} (\exp(-\tau A)\varphi)_k \right) d\tau
\]
\[
= I_4 + I_5.
\]
We can estimate these terms as follows:

\[ |I_4| \leq \|u\|_{n,\infty} \|\exp(-tA)\varphi - \varphi\|_{n/(n-1),1} \rightarrow 0 \text{ as } t \rightarrow +0 \]

and

\[ |I_5| \leq C \left( \|u\|_K^2 + \|F\|_L \right) \int_0^t \|\nabla_x \exp(-\tau A)\varphi\|_{n/(n-2),1} d\tau \rightarrow 0 \text{ as } t \rightarrow +0, \]

which follows from Corollary 2.3. We thus conclude (1.6).

Conversely, suppose that \( u(t,x) \in K_+ \) is a solution of (0.5)–(0.9) in the sense of distributions. Then, for every \( \delta > 0 \), there exists a function \( \varphi(x) \in C_0^\infty(\Omega) \) such that

\[ \|\exp(-tA)\psi - \varphi\|_{n/(n-1),1} < \delta. \]

Then we have

\[ \left| \left( a, \exp(-t_1A)\psi \right) - \left( u(t_0, \cdot), \exp(-t_1 - t_0)A\psi \right) \right| \]

\[ \leq \left| \left( a, \exp(-t_1A) - \varphi \right) + \left| \left( u, a - u(t_0, \cdot), \varphi \right) \right| \]

\[ + \left| \left( u(t_0, \cdot), \varphi - \exp(t_1A)\psi \right) \right| \]

\[ + \left| \left( u(t_0, \cdot), \exp(-t_1A) - \exp(-t_1 - t_0)A\right\{ \psi \right| \]

\[ \leq \left| a\|\n_{n,\infty} + \left| \left( a - u(t_0, \cdot), \varphi \right) \right| + \|u(t_0, \cdot)\|_{n,\infty} \delta \]

\[ + \|u(t_0, \cdot)\|_{n,\infty} \left\{ \exp(-t_1A) - \exp(-t_1 - t_0)A\right\} \psi \|_{n/(n-1),1}. \]

Since \( \delta \) can be taken arbitrarily small and since \( u(t_0, \cdot) \rightarrow a \) in the sense of distributions, together with the fact that \( \exp(-tA)\psi \) is strongly continuous in \( L^{n/(n-1),1}(\Omega) \), we see that

\[ \left( a, \exp(-t_1A)\psi \right) - \left( u(t_0, \cdot), \exp(-t_1 - t_0)A\psi \right) \rightarrow 0 \]

as \( t_0 \rightarrow +0 \). In view of this fact, we let \( t_0 \rightarrow +0 \) in (5.6) to conclude that

\[ \left( u(t_1, \cdot), \psi \right) - \left( a, \exp(-t_1A)\psi \right) \]

\[ = \sum_{j,k=1}^n \int_0^{t_1} \left( H_{jk}(t_1 - \tau, \cdot), \frac{\partial}{\partial x_j} \exp(-\tau A)\psi \right)_k, \]

which implies that \( u(t, x) \) is a mild solution of (0.5)–(0.9). \( \Box \)

At the end of this section we prove Corollary 1.2.

**Proof (of Corollary 1.2).** Assertion 1 follows immediately from the uniqueness in Theorem 5.1 and the fact that \( F(t + T) \equiv F(t) \). Hence it suffices to show Assertion 2. Suppose that \( u(t, x) \) is the unique mild solution of (0.1)–(0.4) such that \( \|u\|_K < A \). Then, for every \( \gamma > 0 \), we can take \( \delta > 0 \) so small that

\[ \|G - F\|_L < \delta \text{ implies } \|v - u\|_K < \gamma, \]

where \( v(t, x) \) is the unique mild solution satisfying \( \|v\|_K < A \) of (0.1)–(0.4) with \( F(t, x) \) replaced by \( G(t, x) \).
From the assumption we can find \( L > 0 \) such that, for every \( a \in \mathbb{R} \), we can find \( T \in [a, a + L] \) such that \( \sup_{t \in \mathbb{R}} \| F(t) - F(t + T) \|_{n/2, \infty} < \delta \). This implies that \( G(t) = F(t + T) \) enjoys \( G \in \mathcal{L} \) and \( \| G - F \|_{n/2, \infty} < \delta \). Then \( v(t) = u(t + T) \) is the unique mild solution satisfying \( \| v \|_{\mathcal{K}} = \| u \|_{\mathcal{K}} < A \) of (0.1)–(0.4) with \( F(t) \) replaced by \( F(t + T) \). It follows that

\[
\sup_{t \in \mathbb{R}} \| u(t + T) - u(t) \|_{n, \infty} = \| v - u \|_{\mathcal{K}} < \gamma,
\]

which implies the required almost periodicity of the mild solution \( u(t, x) \).

\[\square\]

6. Asymptotic behavior

In this section we fix \( p \in (n, \infty) \), and prove Theorem 1.3. We first choose that \( \epsilon(p) \leq \epsilon/2 \). Then, for every \( b(x) \in L^\infty(\Omega) \) satisfying the assumption, we have \( C_0 \| b \|_{n, \infty} + \| F \|_{\mathcal{L}_+} < \epsilon \), and hence there uniquely exists a mild solution \( v(t, x) \in \mathcal{K}_+ \) of the problem (0.5)–(0.9) with \( a(x) \) replaced by \( b(x) \) such that \( \| v \|_{\mathcal{K}_+} < A \). We next introduce the function class

\[
\mathcal{M}_+ = \{ u(t, x) \in \mathcal{K}_+ \mid \sup_{t > 0} r^{n/2p-1/2} \| u(t, \cdot) \|_{p, \infty} < \infty \}
\]

with the norm

\[
\| u \|_{\mathcal{M}_+} = \| u \|_{\mathcal{K}_+} + \sup_{t > 0} r^{n/2p-1/2} \| u(t, \cdot) \|_{p, \infty}.
\]

Then we have the following lemma.

**Lemma 6.1.** There exists a positive number \( C_p \) such that, for every \( u(t, x) \in \mathcal{K}_+ \) and \( v(t, x) \in \mathcal{M}_+ \), we have \( \Psi[u, v, 0, 0], \Psi[v, u, 0, 0] \in \mathcal{M}_+ \), and the enjoy the estimate

\[
\| \Psi[u, v, 0, 0] \|_{\mathcal{M}_+}, \| \Psi[v, u, 0, 0] \|_{\mathcal{M}_+} \leq C_p \| u \|_{\mathcal{K}_+} \| v \|_{\mathcal{M}_+}.
\]

**Proof.** We consider the function \( \Psi[u, v, 0, 0] \), since the function \( \Psi[v, u, 0, 0] \) can be treated exactly in the same way. First we observe that the inequality

\[
\| \Psi[u, v, 0, 0] \|_{\mathcal{K}_+} \leq C_2 \| u \|_{\mathcal{K}_+} \| v \|_{\mathcal{K}_+}
\]

is already proved in Theorem 3.1, 2. Hence it suffices to show that the estimate

\[
\sup_{t > 0} r^{n/2p-1/2} \| \Psi[u, v, 0, 0](t) \|_{p, 1} \leq C_p \| u \|_{\mathcal{K}_+} \| v \|_{\mathcal{M}_+}
\]

(6.1)

holds with some positive constant \( C_p \geq C_2 \).
Suppose that \( \varphi(x) \in L^{p/(p-1)}(\Omega) \) enjoys \( \| \varphi \|_{p/(p-1), 1} \leq 1 \). Then, from the definition
\[
(\Psi[u, v, 0, 0](t, \cdot), \varphi) = \int_0^t \left( u_j(t - \tau, \cdot) v_k(t - \tau, \cdot), \frac{\partial}{\partial x_k} \left( \exp(-\tau A) \varphi \right)_k \right) d\tau,
\]
we obtain
\[
\| (\Psi[u, v, 0, 0](t, \cdot), \varphi) \| \leq \int_0^t \left| \left( u_j(t - \tau, \cdot) v_k(t - \tau, \cdot), \frac{\partial}{\partial x_k} \left( \exp(-\tau A) \varphi \right)_k \right) \right| d\tau = I_1 + I_2, \tag{6.2}
\]
where
\[
I_1 = \int_0^{t/2} \left| \left( u_j(t - \tau, \cdot) v_k(t - \tau, \cdot), \frac{\partial}{\partial x_k} \left( \exp(-\tau A) \varphi \right)_k \right) \right| d\tau
\]
and
\[
I_2 = \int_{t/2}^t \left| \left( u_j(t - \tau, \cdot) v_k(t - \tau, \cdot), \frac{\partial}{\partial x_k} \left( \exp(-\tau A) \varphi \right)_k \right) \right| d\tau.
\]
On the other hand, the inequality
\[
1 > 1 - \frac{1}{p} > 1 - \frac{1}{p} - \frac{1}{n} > 1 - \frac{2}{n} \geq 1
\]
implies
\[
1 < \frac{p}{p - 1} < \frac{np}{np - n - p} < \frac{n}{n - 2} \leq n.
\]
Hence we can apply Corollary 2.3, 2 to obtain the estimates
\[
I_1 \leq C \sup_{\tau > 0} \| u(\tau, \cdot) \|_{p, \infty} \sup_{\tau > t/2} \| v(\tau, \cdot) \|_{p, \infty} \times \int_0^{t/2} \left\| \frac{\partial}{\partial x_k} \left( \exp(-\tau A) \varphi \right)_k \right\|_{np/(np-n-p), 1} d\tau
\]
\[
\leq C \| u \|_{K^+} \left( \frac{t}{2} \right)^{n/2p - 1/2} \| v \|_{\mathcal{M}^+} \| \varphi \|_{p/(p-1), 1}, \tag{6.3}
\]
Navier-Stokes equations in the weak-$L^n$ space

\[ I_2 \leq C \sup_{\tau > 0} \| u(\tau, \cdot) \|_{n,\infty} \sup_{\tau > 0} \| u(\tau, \cdot) \|_{n,\infty} \]

\[ \times \int_{t/2}^t \left\| \frac{\partial}{\partial x_k} \left( \exp(-\tau A)\phi \right) \right\|_{n/(n-2),1} \, d\tau \]

\[ \leq C \| u \|_{K_+} \| v \|_{M_+} \]

\[ \times \int_{t/2}^t \tau^{n/2 - (p-1)/p + (n-2)/n - 1/2} \| \phi \|_{p/(p-1),1} \, d\tau \]

\[ \leq C \| u \|_{K_+} \| v \|_{M_+} \tau^{n/2p - 1/2} \| \phi \|_{p/(p-1),1}. \quad (6.5) \]

Now taking the supremum with respect to $\phi(x)$ in (6.2) and making use of (6.3) and (6.5), we conclude (6.1).

**Proof (of Theorem 1.3.).** Since the estimate (1.8) follows from the fact $u(t, x), v(t, x) \in K_+$ and the estimate (1.7) by real interpolation, it suffices to prove (1.7). We return to the iteration scheme. In Sect. 3 we constructed the mild solutions $u(t, x)$ and $v(t, x)$ as the limit of the sequence $\{u^{(j)}(t, x)\}_{j=1}^{\infty}$ and that of $\{v^{(j)}(t, x)\}_{j=1}^{\infty}$ respectively, where $u^{(0)}(t, x) \equiv v^{(0)}(t, x) \equiv 0, u^{(j+1)}(t, x) = \Psi[u^{(j)}, u^{(j)}, F, a](t, x)$ and $v^{(j+1)}(t, x) = \Psi[v^{(j)}, v^{(j)}, F, b](t, x)$ for $j = 0, 1, \ldots$. Hence, putting $w^{(j)}(t, x) = v^{(j)}(t, x) - u^{(j)}(t, x)$, we have $w^{(0)}(t, x) \equiv 0$ and

\[ w^{(j+1)}(t, x) = \Psi[u^{(j)}, w^{(j)}, 0, 0](t, x) \]

\[ + \Psi[w^{(j)}, v^{(j)}, 0, 0](t, x) + \Psi[0, 0, 0, b - a](t, x) \quad (6.6) \]

for $j = 0, 1, \ldots$. Corollary 2.3, 1 implies that $w^{(1)}(t, x) \in M_+$ and that the estimate $\| w^{(1)} \|_{M_+} \leq C_1(p)\| b - a \|_{n,\infty}$ holds with some constant $C_1(p)$. On the other hand, applying Lemma 6.1 to (6.6), we obtain the estimate

\[ \left\| w^{(j+1)} \right\|_{M_+} \leq \left\| w^{(1)} \right\|_{M_+} + C_p \left( \left\| u^{(j)} \right\|_{K_+} + \left\| v^{(j)} \right\|_{K_+} \right) \left\| w^{(j)} \right\|_{M_+}. \quad (6.7) \]

From the inequalities $C_0\|a\|_{n,\infty} + \| F \|_{L_+} < \varepsilon(p)$ and $C_0\|b\|_{n,\infty} + \| F \|_{L_+} < 2\varepsilon(p)$, we deduce that the inequality

\[ \left\| u^{(j)} \right\|_{K_+}, \left\| v^{(j)} \right\|_{K_+} \leq \frac{1}{4C_p} \]

holds if we choose $\varepsilon(p) > 0$ so small. Substituting this inequality into (6.7) we can deduce

\[ \| w^{(j)} \|_{M_+} < 2 \| w^{(1)} \|_{M_+} \leq 2C_1(p)\| b - a \|_{n,\infty} \]
by induction on \( j \).
Since \( w^{(j)}(t, x) = v^{(j)}(t, x) - u^{(j)}(t, x) \) converges to \( v(t, x) - u(t, x) \) in \( K_+ \) as \( j \to \infty \), we see that \( v(t, x) - u(t, x) \in M_+ \) with the estimate \( \| v - u \|_{M_+} \leq 2C_p \), which yields (1.7).

\[ \square \]

7. Regularity

In this section we prove Theorem 1.4, Theorem 1.5 and Corollary 1.6. For technical reasons, the proof of Theorem 1.5 is divided into two subsections. In the former we prove the boundedness of \( \nabla u(t, \cdot) \) and in the latter the Hölder continuity.

7.1. Proof of Theorem 1.4

We first prove Assertion 1. It suffices to show the estimate

\[ \left| (u(t, \cdot) - u(s, \cdot), \varphi) \right| \leq C(t - s)^{n/2p - 1/2} \| \varphi \|_{p/(p - 1), \infty} \tag{7.1} \]

for \( \varphi \in C_0^\infty(\Omega) \). We have

\[ I_1 \leq \int_0^{t-s} \sum_{j,k=1}^n H_{jk}(t - \tau, \cdot, \frac{\partial}{\partial x_j} \exp(-\tau A) \varphi)_k \, d\tau \]

\[ + \int_0^{\infty} \sum_{j,k=1}^n H_{jk}(s - \tau, \cdot, \frac{\partial}{\partial x_j} \exp(-(t - s + \tau) A) \varphi)_k - \frac{\partial}{\partial x_j} (\exp(-\tau A) \varphi)_k \, d\tau \]

\[ = I_1 + I_2, \tag{7.2} \]

where the functions \( H_{jk}(t, \cdot) \) are introduced in the proof of Theorem 5.2.

Since \( n \geq 3 \), it follows that \( p/(p - 1) < n/(n - 2) \leq n \). Hence Corollary 2.3, 2 implies that the term \( I_1 \) can be estimated as follows:

\[ I_1 \leq C \sup_{\tau \in \mathbb{R}} \left( \| u(\tau, \cdot) \|_{n, \infty}^2 + \| F(\tau, \cdot) \|_{n/2, \infty} \right) \int_0^{t-s} \| \nabla_x \exp(-\tau A) \varphi \|_{n/(n-2), 1} \, d\tau \]

\[ \leq C \int_0^{t-s} \tau^{n/2p - 1/2} \| \varphi \|_{p/(p - 1), \infty} \, d\tau \]

\[ \leq C(t - s)^{n/2p - 1/2} \| \varphi \|_{p/(p - 1), \infty}. \tag{7.3} \]
On the other hand, by using Corollary 2.3, 2 again, we obtain

\[ I_2 \]
\[
\leq C \sup_{\tau \in \mathbb{R}} \left( \|u(\tau, \cdot)\|_{n, \infty}^2 + \|F(\tau, \cdot)\|_{n/2, \infty} \right) 
\int_0^{+\infty} \|\nabla_x \exp(-t - s + \tau) A) \varphi - \nabla_x \exp(-\tau A) \varphi\|_{n/(n-2), \infty} d\tau 
\leq C \int_0^{t-s} \|\nabla_x \exp(-\tau A) \left( \exp(-t - s) A) \varphi - \varphi \right)\|_{n/(n-2), \infty} d\tau 
+ C \int_{t-s}^{+\infty} \|\nabla_x \exp\left(-\frac{\tau}{2} A\right) \left( \exp(-t - s) A - 1 \right) 
\cdot \exp\left(-\frac{\tau}{2} A\right) \varphi\|_{n/(n-2), \infty} \, d\tau
\leq C \int_0^{t-s} \tau^{n/2p - 1 - 1/2} \|\exp(-t - s) A) \varphi - \varphi\|_{p/(p-1), \infty} d\tau 
+ C \int_{t-s}^{+\infty} \tau^{n/2p - 1 - 1/2} \left| \int_0^{t-s} A \exp\left(-\frac{\theta}{2} A\right) \varphi \, d\theta \right|_{p/(p-1), \infty} d\tau
\leq C(t - s)^{n/2p - 1/2} \|\varphi\|_{p/(p-1), \infty} 
+ C \int_{t-s}^{+\infty} \tau^{n/2p - 1/2} \|\exp\left(-\frac{\tau}{2} A\right) \varphi\|_{p/(p-1), \infty} d\tau
\leq C(t - s)^{n/2p - 1/2} \|\varphi\|_{p/(p-1), \infty} 
+ C(t - s) \int_{t-s}^{+\infty} \tau^{n/2p - 2 - 1/2} \|\varphi\|_{p/(p-1), \infty} d\tau
\leq C(t - s)^{n/2p - 1/2} \|\varphi\|_{p/(p-1), \infty}.
\]  
(7.4)

Substituting (7.3) and (7.4) into (7.2), we obtain Assertion 1.

We turn to Assertion 2. It suffices to show the estimate

\[ \left| (u(t, \cdot) - u(s, \cdot), \varphi) \right| \leq C(t - s)^{n/2p - 1/2} \|\varphi\|_{p/(p-1), \infty} \]
for \( \varphi \in C_{0,\sigma}^{\infty}(\Omega) \). We have

\[
\begin{align*}
|\{u(t, \cdot) - u(s, \cdot), \varphi\}| & \leq \left| \int_0^{t-s} \sum_{j,k=1}^n \left( H_{jk}(t - \tau, \cdot), -\frac{\partial}{\partial x_j}(\exp(-\tau A)\varphi)_k \right) d\tau \right| \\
& \quad + \left| \int_0^{t-s} \sum_{j,k=1}^n \left( H_{jk}(s - \tau, \cdot), -\frac{\partial}{\partial x_j}(\exp(-s A)\varphi)_k \right) d\tau \right| \\
& \quad + \left| (a, \{\exp(-(t - s)A) - 1\} \exp(-s A)\varphi) \right| \\
& = I_1 + I_2 + I_3.
\end{align*}
\]

The terms \( I_1 \) and \( I_2 \) are estimated exactly in the same way as in the proof of Assertion 1. The term \( I_3 \) can be estimated as

\[
I_3 = \left| (\{\exp(-(t - s)A) - 1\} a, \exp(-s A)\varphi) \right| \\
= \left| \int_0^{t-s} (A \exp(-\tau A)a, \exp(-s A)\varphi) d\tau \right| \\
\leq C \left( \int_0^{t-s} \| A \exp(-\tau A)a \|_{p,\infty} \| \exp(-s A)\varphi \|_{p/(p-1),1} \right) \\
\leq C \left( \int_0^{t-s} \tau^{-1 + (n/2p - 1/2)} \| A^{n/2p-1/2}a \|_{p,\infty} \| \varphi \|_{p/(p-1),1} \right) \\
= C (t - s)^{n/2p - 1/2} \| A^{n/2p-1/2}a \|_{p,\infty} \| \varphi \|_{p/(p-1),1}.
\]

This estimate completes the proof of Assertion 2. \( \square \)

7.2. Proof of Theorem 1.5: The boundedness

For technical reasons, we first show the boundedness of \( \nabla u \) first.

We first prove Assertion 1. Since \( n/2 < n \), it suffices to show \( \| A^{1/2}u(t, \cdot) \|_{n/2,\infty} \leq C \) in view of Theorem 2.1, 2. Let \( \varphi \) be an element of \( C_{0,\sigma}^{\infty}(\Omega) \), and suppose that \( u(t, x) \) is a mild solution of (0.1)–(0.4). Then we
have
\[
\left| (A^{1/2} u(t, \cdot), \varphi) \right| \\
\leq \sum_{j,k=1}^{n} \left| \int_{0}^{+\infty} \left( H_{jk}(t - \tau, \cdot) - H_{jk}(t, \cdot) \right) \frac{\partial}{\partial x_j} \left( \exp(-\tau A) A^{1/2} \varphi \right)_k \, d\tau \right| \\
\leq \sum_{j,k=1}^{n} \left| \left( H_{jk}(t, \cdot), \int_{0}^{+\infty} \frac{\partial}{\partial x_j} \left( \exp(-\tau A) A^{1/2} \varphi \right)_k \, d\tau \right) \right| \\
+ \sum_{j,k=1}^{n} \left| \left( H_{jk}(t, \cdot), \int_{0}^{+\infty} \frac{\partial}{\partial x_j} \left( \exp(-\tau A) A^{1/2} \varphi \right)_k \, d\tau \right) \right| \\
= I_1 + I_2. \tag{7.5}
\]

Now put \( p = nq/(n - q) \). Then we have \( n/2 < p < n \). Hence Proposition 1.4 implies that
\[
\left\| u_j(t - \tau, \cdot) - u_j(t, \cdot) \right\|_{q, \infty} \\
\leq \left\| u_j(t - \tau, \cdot) - u_j(t, \cdot) \right\|_{n, \infty} \left\| u_j(t - \tau, \cdot) - u_j(t, \cdot) \right\|_{p, \infty} \\
+ \left\| u_j(t - \tau, \cdot) - u_j(t, \cdot) \right\|_{p, \infty} \left\| u_j(t, \cdot) \right\|_{n, \infty} \\
\leq C \tau^{n/2p - 1/2} = C \tau^{n/2q - 1}. \tag{7.6}
\]

From this fact and (1.10) we have
\[
\left\| H_{jk}(t - \tau, \cdot) - H_{jk}(t, \cdot) \right\|_{q, \infty} \leq \tau^{n/2q - 1}. \tag{7.7}
\]
for \( j, k = 1, \ldots, n \). From this estimate and the inequalities \( n/(n - 2) < q/(q - 1) < n/(n - 3) \leq n \) in the case where \( \Omega \) is an exterior domain, we can dominate the term \( I_1 \) from above by
\[
\sum_{j,k=1}^{n} \int_{0}^{+\infty} \left\| H_{jk}(t - \tau, \cdot) - H_{jk}(t, \cdot) \right\|_{q, \infty} \\
\times \left\| \nabla \exp(-\tau A) A^{1/2} \varphi \right\|_{q/(q-1), 1} \, d\tau \\
\leq C \int_{0}^{+\infty} \tau^{n/2q - 1} \left\| \nabla A^{1/2} \exp(-\tau A) \varphi \right\|_{q/(q-1), 1} \, d\tau \\
\leq C \left\| \varphi \right\|_{n/(n-2), 1} \tag{7.8}
\]
by using Corollary 2.3, 2.
On the other hand, since \( n/(n - 2) \leq n \), the term \( I_2 \) can be dominated from above by
\[
C \left( \|F\|_L + \|u\|_{\infty} \right) \left\| \nabla A^{1/2} \int_0^{+\infty} \exp(-\tau A) \varphi d\tau \right\|_{n/(n-2),1}.
\]
Applying Corollary 2.4 we see that the right-hand side of this formula is dominated from above by
\[
C \|\varphi\|_{n/(n-2),1}.
\]
Now we can easily see from (7.5), (7.8) and the fact above that the family \( \{\nabla u(t, \cdot)\}_{t \in \mathbb{R}} \) is bounded in \( L^{n/2,\infty}(\Omega) \).

We next prove Assertion 2. Let \( \varphi \) be an element of \( C^{0,\sigma}_0(\Omega) \) such that \( \|\varphi\|_{n/(n-2),1} \leq 1 \), and suppose that \( u(t, x) \) is a mild solution of (0.5)–(0.9). Then we have
\[
\left| \left( A^{1/2}u(t, \cdot), \varphi \right) \right| \\
\leq \sum_{j,k=1}^n \left| \int_0^t \left( H_{jk}(t-\tau, \cdot), \frac{\partial}{\partial x_j} \left( \exp(-\tau A) A^{1/2} \varphi \right)_k \right) d\tau \right| \\
\leq \sum_{j,k=1}^n \left| \int_0^t \left( H_{jk}(t-\tau, \cdot) - H_{jk}(t, \cdot), \frac{\partial}{\partial x_j} \left( \exp(-\tau A) A^{1/2} \varphi \right)_k \right) d\tau \right| \\
+ \sum_{j,k=1}^n \left| \left( H_{jk}(t, \cdot), \int_0^{+\infty} \frac{\partial}{\partial x_j} \left( \exp(-\tau A) A^{1/2} \varphi \right)_k d\tau \right) \right| \\
= I_1 + I_2 + I_3.
\]
The terms \( I_1 \) and \( I_2 \) can be dominated in the same way as in the proof of Assertion 1, and \( I_3 \) can be dominated from above by
\[
\left| \left( A^{1/2}a, \exp(-tA)\varphi \right) \right| \leq C \left\| A^{1/2}a \right\|_{n/2,\infty} \left\| \exp(-tA)\varphi \right\|_{n/(n-2),1} \\
\leq C \left\| \nabla a \right\|_{n/2,\infty} \left\| \varphi \right\|_{n/(n-2),1}
\]
by Theorem 2.1, 1. This completes the proof. \( \square \)

7.3. Proof of Theorem 1.5: The Hölder continuity

In this subsection we prove the Hölder continuity of \( \nabla u(t, \cdot) \). We first prove Assertion 1. In view of Theorem 2.1, 3, it suffices to show the estimate
\[
\left| \left( A^{1/2}u(t, \cdot) - A^{1/2}u(s, \cdot), \varphi \right) \right| \leq C(t-s)^{n/2q-1} \left\| \varphi \right\|_{q/(q-1),1} \quad (7.9)
\]

\[
\left| \left( A^{1/2}u(t, \cdot) - A^{1/2}u(s, \cdot), \varphi \right) \right| \leq C(t-s)^{n/2q-1} \left\| \varphi \right\|_{q/(q-1),1} \quad (7.9)
\]
for \( \varphi \in L^q/(q-1) (\Omega) \). We have

\[
\begin{align*}
    \left(A^{1/2} u(t, \cdot), \varphi\right) &= 
    \sum_{j,k=1}^{n} \left( \int_{t-s}^{t-s} \left( H_{jk}(t - \tau, \cdot) - H_{jk}(t, \cdot), \frac{\partial}{\partial x_j}\left(\exp(-\tau A) A^{1/2} \varphi\right)_k\right) \, d\tau \\
    &+ \left(H_{jk}(t, \cdot), \int_{t-s}^{t-s} \frac{\partial}{\partial x_j} \left(\exp(-\tau A) A^{1/2} \varphi\right)_k \, d\tau\right) \\
    &+ \int_{t-s}^{+\infty} \left( H_{jk}(t - \tau, \cdot) - H_{jk}(s, \cdot), \frac{\partial}{\partial x_j}\left(\exp(-\tau A) A^{1/2} \varphi\right)_k \, d\tau\right) \\
    &+ \left(H_{jk}(s, \cdot), \int_{t-s}^{+\infty} \frac{\partial}{\partial x_j} \left(\exp(-\tau A) A^{1/2} \varphi\right)_k \, d\tau\right) \right).
\end{align*}
\]

It follows that

\[
\begin{align*}
    \left| \left(A^{1/2} u(t, \cdot) - A^{1/2} u(s, \cdot), \varphi\right) \right| 
    &\leq \left| \left(\int_{t-s}^{t-s} \sum_{j,k=1}^{n} \left( H_{jk}(t - \tau, \cdot) - H_{jk}(t, \cdot), \frac{\partial}{\partial x_j}\left(\exp(-\tau A) A^{1/2} \varphi\right)_k\right) \, d\tau\right) \right| \\
    &+ \left| \left(\sum_{j,k=1}^{n} \left( H_{jk}(t, \cdot) - H_{jk}(s, \cdot), \int_{0}^{t-s} \frac{\partial}{\partial x_j} \left(\exp(-\tau A) A^{1/2} \varphi\right)_k \, d\tau\right) \right) \right| \\
    &+ \left| \left(\int_{0}^{+\infty} \sum_{j,k=1}^{n} \left( H_{jk}(s - \tau, \cdot) - H_{jk}(s, \cdot), \right.ight.ight. \\
    &\quad \quad \quad \quad \left. \left. \left. \left. \frac{\partial}{\partial x_j}\left(\exp(-(t-s+\tau) A) \varphi\right)_k \right) \right) \right) \right| \\
    &= I_1 + I_2 + I_3.
\end{align*}
\]

(7.10)

If \( \Omega \) is an exterior domain, we have \( q/(q-1) < n/(n-3) \leq n \). Hence, we can substitute (7.7) and apply Corollary 2.3, 2 to estimate the term \( I_1 \) as follows:

\[
\begin{align*}
    I_1 &\leq C \int_{0}^{t-s} \tau^{n/2q-1} \left\| \nabla A^{1/2} \exp(-\tau A) \varphi \right\|_{q/(q-1),1} \, d\tau \\
    &\leq C \int_{0}^{t-s} \tau^{n/2q-2} \| \varphi \|_{q/(q-1),1} \, d\tau \\
    &\leq C (t-s)^{n/2q-1} \| \varphi \|_{q/(q-1),1}.
\end{align*}
\]

(7.11)
Next, substituting (7.7) and applying Corollary 2.4, we can estimate the term $I_2$ as follows:

$$I_2 \leq C(t - s)^{n/2q - 1} \left\| \int_0^{t-s} \nabla A^{1/2} \exp(-\tau A) \varphi d\tau \right\|_{q/(q-1),1}$$

$$\leq C(t - s)^{n/2q - 1} \| \varphi \|_{q/(q-1),1}. \quad (7.12)$$

Finally, in the same way as in (7.11), the term $I_3$ is estimated as follows:

$$I_3 \leq C \int_0^{+\infty} \tau^{n/2q - 1}$$

$$\left\| \nabla A^{1/2} \left( \exp(-(t-s+\tau)A) - \exp(-\tau A) \right) \varphi \right\|_{q/(q-1),1} d\tau$$

$$\leq C \int_0^{t-s} \tau^{n/2q - 1} \left( \left\| \nabla A^{1/2} \exp(-(t-s+\tau)A)\varphi \right\|_{q/(q-1),1} + \left\| \nabla A^{1/2} \exp(-\tau A)\varphi \right\|_{q/(q-1),1} \right) d\tau$$

$$+ C \int_{t-s}^{+\infty} \tau^{n/2q - 1} \left\| \nabla A^{1/2} \exp\left( -\frac{\tau}{2} A \right) \right\|_{q/(q-1),1} \int_0^{t-s} \left\| A \exp\left( -\left( \theta + \frac{\tau}{2} \right) A \right) \varphi \right\|_{q/(q-1),1} d\theta d\tau$$

$$\leq C \int_0^{t-s} \tau^{n/2q - 1} \left( \frac{1}{\tau} + \frac{1}{t-s+\tau} \right) \| \varphi \|_{q/(q-1),1} d\tau$$

$$+ C \int_{t-s}^{+\infty} \tau^{n/2q - 1} \left( \frac{1}{\tau} \right)^{-1} \int_0^{t-s} \left\| A \exp\left( -\left( \theta + \frac{\tau}{2} \right) A \right) \varphi \right\|_{q/(q-1),1} d\theta d\tau$$

$$\leq C(t - s)^{n/2q - 1} \| \varphi \|_{q/(q-1),1}$$

$$+ C(t - s) \int_{t-s}^{+\infty} \tau^{n/2q - 2} \| \varphi \|_{q/(q-1),1} d\tau$$

$$\leq C(t - s)^{n/2q - 1} \| \varphi \|_{q/(q-1),1}. \quad (7.13)$$

Combining (7.10), (7.11), (7.12) and (7.13), we obtain (7.9). This completes the proof of Assertion 1.
We turn to Assertion 2. In the same way as (7.10), we have
\[
\left| \left( A^{1/2}u(t, \cdot) - A^{1/2}u(s, \cdot) \right) \right| \\
\leq \left| \int_0^{t-s} \sum_{j,k=1}^n \left( H_{jk}(t - \tau, \cdot) - H_{jk}(t, \cdot), \frac{\partial}{\partial x_j} \left( A^{1/2} \exp(-\tau A) \varphi \right)_k \right) d\tau \right| \\
+ \left| \int_0^{t-s} \sum_{j,k=1}^n \left( H_{jk}(t, \cdot) - H_{jk}(s, \cdot), \int_0^\tau \frac{\partial}{\partial x_j} \left( A^{1/2} \exp(-\tau A) \varphi \right)_k \right) d\tau \right| \\
+ \left| \int_0^{t-s} \sum_{j,k=1}^n \left( H_{jk}(s - \tau, \cdot) - H_{jk}(s, \cdot), \frac{\partial}{\partial x_j} \left( A^{1/2} \exp(-(t - s + \tau) A) \varphi \right)_k \right) d\tau \right| \\
+ \left| \left( a, A^{1/2} \exp(-t A) \varphi - A^{1/2} \exp(-s A) \varphi \right) \right| \\
= I_1 + I_2 + I_3 + I_4. \tag{7.14}
\]

The terms \(I_1, I_2\) and \(I_3\) can be estimated in the same way as in the proof of Assertion 1. \(I_4\) is estimated as follows:
\[
I_4 = \left| \left\{ \exp(-(t - s) A) - 1 \right\} A^{1/2}a, \exp(-s A) \varphi \right| \\
= \left| \int_0^{t-s} \left( A^{3/2} \exp(-\tau A) a, \exp(-s A) \varphi \right) d\tau \right| \\
\leq C \int_0^{t-s} \left\| A^{3/2} \exp(-\tau A) a \right\|_{q, \infty} d\tau \left\| \exp(-s A) \varphi \right\|_{q/(q-1), 1} \\
\leq C \int_0^{t-s} \tau^{-3/2 + (n/2q - 1/2)} \left\| A^{n/2q - 1/2} u \right\|_{q, \infty} \left\| \varphi \right\|_{q/(q-1), 1} \\
\leq C \left( t - s \right)^{n/2q - 1} \left\| A^{n/2q - 1/2} u \right\|_{q, \infty} \left\| \varphi \right\|_{q/(q-1), 1}.
\]
Substituting this estimate into (7.14) we obtain the desired estimate. \(\square\)

7.4. Proof of Corollary 1.6

We first consider Assertion 1. Put \(q = nr/(n-r)\). Then we have \(n/3 < q < n/2\), and the Biot-Savard law implies that there exists a function \(F(t, x) \in L\) enjoying (1.10). Hence we can apply Theorem 1.11, 1. It follows that \((u(t, \cdot) \cdot \nabla)u(t, \cdot)\) is uniformly continuous with respect to \(t\) and enjoys the estimate
\[
\left\| (u(t, \cdot) \cdot \nabla)u(t, \cdot) - (u(s, \cdot) \cdot \nabla)u(s, \cdot) \right\|_{r, \infty} \leq (t - s)^{n/2r - 3/2}
\]
for every $s, t$ such that $s < t$. It follows from this fact and (1.12) that, if we put
\[
v(t, x) = \int_{t_0}^t \exp(-(t - \tau)A) \left[ -P(u(\tau, \cdot) \cdot \nabla)u(\tau, \cdot) + f(\tau, \cdot) \right] d\tau,
\]
we see that the equality
\[
\frac{\partial v}{\partial t}(t, x) = -Av(t, x) - P(u(t, x) \cdot \nabla)u(t, x) + f(t, x)
\]
holds on $(t_0, +\infty)$. Hence the difference $w(t, x) = u(t, x) - v(t, x)$ enjoys the linear Stokes equation on $(t_0, +\infty)$. This implies that $u(t, x)$ solves (1.9) on $(t_0, +\infty)$. Since $t_0 \in \mathbb{R}$ is arbitrary, this implies that $u(t, x)$ solves (1.9) on $\mathbb{R}$.

Assertion 2 can be proved exactly in the same way.

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References

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