RIESZ TRANSFORMS RELATED TO BESSEL OPERATORS

JORGE J. BETANCOR, DARIUSZ BURACZEWSKI, JUAN CARLOS FARIÑA, TERESA MARTÍNEZ, AND JOSÉ L. TORREA

Abstract. In this work we study the Riesz transforms $R_\mu$ related to the Bessel operators $\Delta_\mu = x^{-\mu-1/2}Dx^{2\mu+1}Dx^{-\mu-1/2}$. We develop for $R_\mu$ a theory that runs parallel to the one for the Euclidean Hilbert transform. It is proved that $R_\mu$ is actually a Calderón-Zygmund singular integral operator. Moreover, our Riesz transform can be written as a principal value and we study the speed of convergence of this limit in terms of the $L_p$ boundedness of oscillation and variation operators. Also, $R_\mu$ is seen to be the boundary value of the appropriate harmonic extension for this context. Finally we analyze weighted inequalities involving $R_\mu$.

1. Introduction.

In this paper we introduce and study the Riesz transform associated with the Bessel operator

\begin{equation}
\Delta_\mu = -\left(\frac{d^2}{dx^2} - \frac{\mu^2 - 1/4}{x^2}\right).
\end{equation}

The operator $\Delta_\mu$ is a positive self-adjoint operator in $L_2((0, \infty), dx)$ and it can be written in divergence form as

\begin{equation}
\Delta_\mu = x^{-\mu-1/2}Dx^{2\mu+1}Dx^{-\mu-1/2} = A_\mu^*A_\mu,
\end{equation}

being $A_\mu = x^{\mu+1/2}Dx^{-\mu-1/2}$ and $A_\mu^*$ the adjoint operator of $A_\mu$. The eigenfunctions of $\Delta_\mu$ are $\{\varphi_y\}_{y>0}$, where, for every $y > 0$ (see [13]),

\begin{equation}
\varphi_y(x) = (yx)^{1/2}J_\mu(yx) \quad \text{and} \quad \Delta_\mu \varphi_y(x) = y^2 \varphi_y(x),
\end{equation}

for $J_\mu(z)$ being the Bessel function of the first kind of order $\mu$. The Poisson kernel $P_\mu$ associated to $\Delta_\mu$ is

\begin{equation}
P_\mu(t, x, y) = \int_0^\infty e^{-zt} \varphi_x(z) \varphi_y(z) \, dz, \quad t, x, y \in (0, \infty).
\end{equation}
The corresponding Poisson integral was considered in [14]. By using the well known formula
\[ s^{-a} = \frac{1}{\Gamma(a)} \int_0^\infty e^{-st}t^{a-1}dt, \]
with \( s, a > 0 \), we define
\[ \Delta_{\mu}^{-1/2} f(x) = \int_0^\infty e^{-t\sqrt{\Delta_{\mu}}} f(x) dt = \int_0^\infty \int_0^\infty P_{\mu}(t, x, y)f(y) dy dt. \]

From here, following Stein’s ideas contained in the last chapter of [15], we define the Riesz transform associated to \( \Delta_{\mu} \) as
\[ (1.4) \quad R_{\mu}(f) = A_{\mu}\Delta_{\mu}^{-1/2} f. \]

The first question we treat in this paper is giving a sense to \( \Delta_{\mu}^{-1/2} \). This is done in Lemmas 2.1 and 2.2. Next, we prove the main properties of the Riesz transform \( R_{\mu} \). Namely, we see that \( R_{\mu} \) is a principal value operator when acting on smooth functions (Theorem 3.3) and also that it is the boundary value of the “conjugate harmonic extension” associated to \( \Delta_{\mu} \), as stated in Theorem 3.4. In section 4 we prove that \( R_{\mu} \) is a Calderón-Zygmund operator, and this allows us to obtain its boundedness in weighted \( L^p \)-spaces of \( R_{\mu} \) (Theorem 4.2) and of the maximal \( R_{\mu}^* \) of the truncated integrals (Theorem 4.3), for weights in Muckenhoupt class \( A_p \). Hence, by the classical theory, we deduce the existence of the principal value for every function in the same weighted \( L^p \) (Corollary 4.4). A natural way to measure the speed of this convergence is studying the oscillation and \( \rho \)-variation operators associated to \( R_{\mu} \). This is done in Theorem 5.1. Finally, we study the optimal classes of weights \( v \) such that for every \( f \in L^p(v) \), \( R_{\mu}f \) can be defined as a principal value. We characterize this class of weights by an intrinsic condition and we also show that this condition is equivalent to some boundedness properties of \( R_{\mu} \) and \( R_{\mu}^* \) (see Theorem 6.2).

Our work has been inspired by the results of Muckenhoupt and Stein in [13], where conjugations associated to the Bessel operators \( S_{\mu} = \frac{d^2}{dx^2} + \frac{2\mu+1}{x} \frac{d}{dx} \) are studied. For fixed \( \mu > -1/2 \), they considered the \( \mu \)-harmonic extension of \( f \in L_1(x^{2\mu+1}dx) \), given by
\[ \mathcal{P}_{\mu}(f)(t, x) = \int_0^\infty \mathcal{P}_{\mu}(t, x, y)f(y)y^{2\mu+1} dy, \quad t, x \in (0, \infty), \]
and satisfying
\[ \left( \frac{d^2}{dt^2} + S_{\mu,x} \right) \mathcal{P}_{\mu}(f)(t, x) = 0, \]
where \( \mathcal{P}_{\mu}(t, x, y) \) is the natural Poisson kernel associated to \( S_{\mu} \). The conjugate \( \mu \)-harmonic function of \( f \) is defined as
\[ \mathcal{Q}_{\mu}(f)(t, x) = \int_0^\infty \mathcal{Q}_{\mu}(t, x, y)f(y)y^{2\mu+1} dy, \]
where \( \mathcal{Q}_{\mu} \) is the \( \mu \)-conjugate kernel. Then, the conjugate function (analogue of the Hilbert transform) is given by (see formula (16.8) in [13])
\[ \mathcal{Q}_{\mu}(f)(x) = \lim_{t \to 0} \mathcal{Q}_{\mu}(f)(t, x). \]

\( L_\mu(x^{2\mu+1}) \)-boundedness properties of this conjugation operator were established in [13]. From our results, \( \mathcal{Q}_{\mu} \) is a principal value operator (see Remark 3.5).
Andersen and Kerman [2] investigated weighted inequalities for the operator $Q_\mu$. They characterized the weights playing in this theory the role of the $A_p$-weights of Muckenhoupt in the classical theory of the Hilbert transform.

More recently, Betancor and Stempak [4] defined and studied the operator $R_\mu f(x) = \partial_x \Delta_m^{-1/2} f(x)$ on weighted $L^p$-spaces for Muckenhoupt weights, following the ideas in [13]. The results of [4] can be obtained and improved by the ones established here (see Appendix).

Other results concerning to $Q_\mu$-conjugation can be encountered in [1], [11] and [12]. Throughout this paper we represent by $p'$ the conjugate exponent of $p$ for $1 \leq p < \infty$. We denote by $L_p(v)$ and $L_{p,\infty}(v)$ the Lebesgue spaces $L_p((0, \infty), v \, dx)$ and weak-$L_p((0, \infty), v \, dx)$. When $v \equiv 1$ we simply write $L_p$ or $L_{p,\infty}$ instead of $L_p(dx, (0, \infty))$ or weak-$L_p((0, \infty), dx)$. $A_p(0, \infty)$ stands for the class of Muckenhoupt weights on $(0, \infty)$. The parameter $\mu$ is, except otherwise stated, greater than $-1/2$. The letter $C$ denotes a suitable positive constant that may change from line to line.

2. Definition of the operator $\Delta_\mu^{-1/2}$.

The aim of this section is giving a good definition of the operator $\Delta_\mu^{-1/2}$ where $\Delta_\mu$ is defined in (1.1). It is known (see [13, (16.4)]) that the Poisson kernel associated to $\Delta_\mu$, given by (1.3), has the following expression for $x, y, t \in (0, \infty)$

$$P_\mu(t, x, y) = \frac{2\mu + 1}{\pi}(xy)^{\mu + 1/2} \int_0^{\pi} \frac{(\sin z)^{2\mu}}{(x^2 + y^2 + t^2 - 2xy \cos z)^{\mu + 3/2}} \, dz. \quad (2.1)$$

**Lemma 2.1.** For any $f \in C_c^\infty$, we define

$$\Delta_\mu^{-1/2} f = \int_0^\infty e^{-t \sqrt{\Delta_\mu}} f \, dt. \quad (2.2)$$

Then, the operator is well defined and for every $x \in (0, \infty)$ we have

$$\Delta_\mu^{-1/2} f(x) = \int_0^\infty \int_0^\infty P_\mu(t, x, y) f(y) \, dy \, dt = \int_0^\infty \left( \int_0^\infty P_\mu(t, x, y) \, dt \right) f(y) \, dy.$$

**Proof.** Let $f \in C_c^\infty$. To change the order of integration in (2.2), we will use the following inequalities ([13, p. 86]):

$$P_\mu(t, x, y) \leq C \frac{t}{|x - y|^2 + t^2}, \quad (2.3)$$

and

$$P_\mu(t, x, y) \leq C \frac{(xy)^{\mu + 1/2} t}{(|x - y|^2 + t^2)^{\mu + 3/2}}. \quad (2.4)$$

With these estimates we get that the integrand in (2.2) is uniformly integrable, since for every $y \in (0, \infty)$

$$\int_0^\infty \int_0^\infty |P_\mu(t, x, y) f(y)| \, dy \, dt \leq C \left( \int_0^\infty \int_0^1 \frac{t |f(y)|}{|x - y|^2 + t^2} \, dt \, dy \right) + \int_0^\infty \int_1^\infty \frac{(xy)^{\mu + 1/2} |f(y)|}{(|x - y|^2 + t^2)^{\mu + 3/2}} \, dt \, dy \right) \leq C \|f\|_{L^1}.$$
The rest of the section is devoted to justify our definition of $\Delta_{\mu}^{-1/2}$. The set of eigenfunctions of $\Delta_{\mu}$, $\{\varphi_y(x)\}_{y \in (0, \infty)}$ (see (1.2)), does not span a dense subset of $L^2$. Thus, we can not use the usual spectral techniques to define the classical operators associated to $\Delta_{\mu}$. Nevertheless, the natural property that $\Delta_{\mu}^{-1/2}$ should verify is that $\Delta_{\mu}^{-1/2} \varphi_y(x) = \frac{1}{y} \varphi_y(x)$, at least in a weak sense. In order to clarify this property, let us introduce some notation. From now on, we consider the domain of $\Delta_{\mu}$ to be the set of infinitely differentiable functions with compact support in $(0, \infty)$, and call it $C_\infty$. The (non modified) Hankel transform is given, for a function $f \in L^1$ as

$$H_\mu(f)(y) = \int_0^\infty (xy)^{1/2} J_\mu(xy) f(x) \, dx,$$

which plays the role of the coefficient of $f$ corresponding to the eigenvector $\varphi_y$. The integral transform $H_\mu$ has been defined in spaces of distributions of slow growth by Zemanian ([17] and [18]). He introduced the suitable “Schwartz class” $\mathcal{H}_\mu$ in this context and proved that the Hankel transform $H_\mu$ is an automorphism in the space $\mathcal{H}_\mu$ ([18, Theorem 5.4-1]) and that $H_\mu$ is extended to the dual space $\mathcal{H}_\mu'$ of $\mathcal{H}_\mu$ by transposition. That is, if $T \in \mathcal{H}_\mu'$, the Hankel transform $H_\mu(T)$ is the element of $\mathcal{H}_\mu'$ defined through

$$\langle H_\mu'(T), f \rangle = \langle T, H_\mu(f) \rangle, \quad f \in \mathcal{H}_\mu.$$

It is known ([18, Lemma 5.4-1]) that for any $f \in \mathcal{H}_\mu$,

$$H_\mu(\Delta_{\mu} f)(y) = y^2 H_\mu(f)(y).$$

According to the former discussion, the following lemma justifies our definition of $\Delta_{\mu}^{-1/2}$, given in (2.2).

**Lemma 2.2.** The operator defined as in (2.2) satisfies, for $f \in \mathcal{H}_\mu$, that

$$H_\mu'(\Delta_{\mu}^{-1/2} f)(y) = \frac{1}{y} H_\mu'(f)(y).$$

The key idea of the proof is to define, for every $0 < s < 1$,

$$G_s(f)(x) = \int_s^{1/s} \int_0^\infty P_\mu(t, x, y) f(y) \, dy \, dt, \quad x \in (0, \infty).$$

These operators verify $\lim_{s \to 0} G_s f = \Delta_{\mu}^{-1/2} f$, in the weak * topology of $\mathcal{H}_\mu$, and this gives the result. We leave the details for the interested reader.

**3. Riesz transforms $R_\mu$ associated with $\Delta_{\mu}$.**

As it was mentioned in the introduction the Bessel operator $\Delta_{\mu}$ can be written as $\Delta_{\mu} = -A_\mu^* A_\mu$, being $A_\mu = x^{\mu+1/2} D x^{-\mu-1/2}$ and denoting by $A_\mu^*$ the adjoint operator of $A_\mu$ in $L_2$. Then, this suggests to define the Riesz transform $R_\mu$ associated with $\Delta_{\mu}$ by

$$R_\mu = A_\mu \Delta_{\mu}^{-1/2},$$
where, for every \( x, y \in (0, \infty) \), \( x \neq y \),

\[
R_\mu(x, y) = \pi \mu + 1 \int_0^{\infty} \frac{d}{dx} \left( x^{-\mu-1/2} P_\mu(t, x, y) \right) dt
= \frac{2\mu + 1}{\pi} (xy)^{\mu+1/2} \int_0^{\pi} \frac{(y \cos z - x)(\sin z)^{2\mu}}{(x^2 + y^2 - 2xy \cos z)^{\mu+3/2}} dz.
\]

Due to the term \( 1 - \cos z \) in the denominator of \( R_\mu(x, y) \), the part of the kernel with \( z \in (\pi/2, \pi) \) behaves in a better way than the part with \( z \in (0, \pi/2) \). According to this, define for every \( x, y \in (0, \infty) \), \( x \neq y \),

\[
\begin{align*}
R_{\mu,1}(x, y) &= \frac{2\mu + 1}{\pi} (xy)^{\mu+1/2} \int_0^{\pi/2} \int_0^{\infty} t y^{\mu+1/2} (\sin z)^{2\mu} (x^2 + y^2 + t^2 - 2xy \cos z)^{\mu+3/2} dt dy, \\
R_{\mu,2}(x, y) &= R_\mu(x, y) - R_{\mu,1}(x, y).
\end{align*}
\]

**Lemma 3.1.** Let \( f \in C_0^\infty \). Then, for every \( x \in (0, \infty) \),

\[
\int_0^\infty R_{\mu,2}(x, y) f(y) dy = \frac{2\mu + 1}{\pi} x^{\mu+1/2} \int_0^\infty \frac{d}{dx} \left( \int_0^{\pi} \int_0^\infty \int_0^{\pi/2} \int_0^{\infty} f(y) t y^{\mu+1/2} (\sin z)^{2\mu} (x^2 + y^2 + t^2 - 2xy \cos z)^{\mu+3/2} dt dy dz dt dy \right),
\]

the first integral being absolutely convergent.

**Proof.** Let \( x \in (0, \infty) \). We can write

\[
\int_{\pi/2}^{\pi} \frac{|2x - 2y \cos z| (\sin z)^{2\mu}}{(x^2 + y^2 + t^2 - 2xy \cos z)^{\mu+5/2}} dz \leq C \frac{|x - y| + y}{(t^2 + |x - y|^2 + xy)^{\mu+5/2}}, \quad t, y > 0.
\]

Then, to justify the differentiation under the integral sign we have just to observe that

\[
\begin{align*}
\int_0^\infty |f(y)| y^{\mu+1/2} \int_0^\infty \int_0^{\pi/2} \int_0^{\infty} \frac{d}{dx} \left( \frac{(\sin z)^{2\mu}}{(x^2 + y^2 + t^2 - 2xy \cos z)^{\mu+3/2}} \right) dt dy dz
\leq C \int_0^\infty |f(y)| y^{\mu+1/2} (|x - y| + y) \int_0^\infty \frac{t}{(t^2 + |x - y|^2 + xy)^{\mu+5/2}} dt dy
\leq C \int_0^\infty |f(y)| y^{\mu+1/2} (|x - y| + y)(xy + |x - y|^2)^{-1/2} (xy + |x - y|^2)^{-\mu-1} dy
\leq C x^{-\mu-1} \int_0^\infty (y^{-1/2} + x^{-1/2}) |f(y)| dy < \infty.
\end{align*}
\]
Lemma 3.2. Let $f \in C^\infty_c$. Then, for all $x \in (0, \infty)$,

$$
\lim_{\varepsilon \to 0} \int_{|x-y| > \varepsilon} R_{\mu,1}(x, y) f(y) \, dy = \frac{2\mu + 1}{\pi} \left( \int_0^1 \int_0^\infty \int_0^{\pi/2} \frac{f(y)ty^{\mu+1/2} (\sin z)^{2\mu}}{(x^2 + y^2 + t^2 - 2xy \cos z)^{\mu+3/2}} \, dz \, dt \, dy \right).
$$

Proof. Let $x \in (0, \infty)$. Write

$$
(3.4) \quad \int_0^\infty f(y) \int_0^\infty P_{\mu,1}(t, x, y) \, dt \, dy = \int_0^\infty f(y) \left( \int_0^1 + \int_1^\infty \right) P_{\mu,1}(t, x, y) \, dt \, dy
$$

where for $y, t \in (0, \infty)$

$$
P_{\mu,1}(t, x, y) = \frac{(2\mu + 1)t}{\pi} (xy)^{\mu+1/2} \int_0^{\pi/2} \frac{(\sin z)^{2\mu}}{(x^2 + y^2 + t^2 - 2xy \cos z)^{\mu+3/2}} \, dz.
$$

Note that

$$
\int_0^\infty |f(y)|y^{\mu+1/2} \int_1^\infty \int_0^{\pi/2} \left| \frac{d}{dx} \left( \frac{(\sin z)^{2\mu}}{(x^2 + y^2 + t^2 - 2xy \cos z)^{\mu+3/2}} \right) \right| \, dz
\leq C \int_0^\infty |f(y)|y^{\mu+1/2}(x + y) \int_1^\infty t \frac{1}{(|x - y|^2 + t^2)^{\mu+5/2}} \, dt \, dy
$$

(3.5)

$$
\leq C \int_0^\infty |f(y)|y^{\mu+1/2}(x + y) \, dy < \infty.
$$

Hence we have proved that

$$
\frac{d}{dx} \left( x^{-\mu-1/2} \int_0^\infty f(y) \int_1^\infty P_{\mu,1}(t, x, y) \, dt \, dy \right)
= \frac{2\mu + 1}{\pi} \int_0^\infty \int_1^\infty \int_0^{\pi/2} \frac{d}{dx} \left( \frac{f(y)y^{\mu+1/2}t(\sin z)^{2\mu}}{(x^2 + y^2 + t^2 - 2xy \cos z)^{\mu+3/2}} \right) \, dz \, dt \, dy.
$$

(3.6)

Now we decompose the first integral in the right-hand side of (3.4) as follows

$$
\int_0^{\pi/2} \frac{(\sin z)^{2\mu}}{(x^2 + y^2 + t^2 - 2xy \cos z)^{\mu+3/2}} \, dz = \sum_{j=1}^4 I_j(x, y),
$$
where

\[ I_1(x, y) = \int_0^{\pi/2} \frac{(\sin z)^{2\mu}}{(|x - y|^2 + t^2 + 2xy(1 - \cos z))(x^2 + t^2 + xyz^2)^{\mu+3/2}} \, dz \]

\[ I_2(x, y) = \int_0^{\pi/2} \frac{(\sin z)^{2\mu} - z^{2\mu}}{(|x - y|^2 + t^2 + 2xy(1 - \cos z))(x^2 + t^2 + xyz^2)^{\mu+3/2}} \, dz \]

\[ I_3(x, y) = \int_0^{\pi/2} \frac{z^{2\mu}}{(|x - y|^2 + t^2 + 2xy(1 - \cos z))(x^2 + t^2 + xyz^2)^{\mu+3/2}} \, dz - C_\mu(xy)^{-\mu-1/2} \frac{1}{|x - y|^2 + t^2} \]

\[ I_4(x, y) = C_\mu(xy)^{-\mu-1/2} \frac{1}{|x - y|^2 + t^2}, \]

where \( C_\mu = \int_0^\infty \frac{u^{2\mu}}{(1 + u^2)^{\mu+3/2}} \, du \). If \( t, y \in (0, \infty) \), by the change of variables \( u^2 = B \frac{z^2}{A} \), with \( A = |x - y|^2 + t^2 \) and \( B = xy \), it can be easily seen that

\[ \int_0^{\pi/2} \frac{z^\alpha}{(|x - y|^2 + t^2 + xyz^2)^\beta} \, dz = A^{-(\beta-(\alpha+1)/2)}B^{-(\alpha+1)/2} \int_0^{\pi/2} \frac{\mu^\alpha}{(1 + u^2)^\beta} \, du. \]

By applying the mean value theorem we get

\[ \left| \frac{2(x - y) + 2y(1 - \cos z)}{(|x - y|^2 + t^2 + 2xy(1 - \cos z))(x^2 + t^2 + xyz^2)^{\mu+5/2}} - \frac{2(x - y) + yz^2}{(|x - y|^2 + t^2 + xyz^2)^{\mu+5/2}} \right| \]

\[ \leq Cz^4 \left( \frac{y}{(|x - y|^2 + t^2 + xyz^2)^{\mu+5/2}} + \frac{xy(|x - y| + y)}{(|x - y|^2 + t^2 + xyz^2)^{\mu+7/2}} \right), \quad z \in (0, \pi/2). \]

Moreover, we have for \( t \in (0, 1) \)

\[ \int_0^{\pi/2} \frac{y(\sin z)^{2\mu}z^4}{(|x - y|^2 + t^2 + xyz^2)^{\mu+5/2}} \, dz \leq Cy^{-\mu-3/2}x^{-\mu-5/2} \int_0^{\pi/2} \frac{u^{2\mu+4}}{(1 + u^2)^{\mu+5/2}} \, du \]

\[ \leq C \frac{x^{1/2}y^{-\mu-1}}{t^{1/2}} \int_0^\infty u^{2\mu+3} \frac{u^{2\mu+4}}{(1 + u^2)^{\mu+5/2}} \, du. \]

Hence,

\[ y^{\mu+1/2} \int_0^1 \int_0^{\pi/2} \frac{y(\sin z)^{2\mu}z^4}{(|x - y|^2 + t^2 + xyz^2)^{\mu+5/2}} \, dz \, dt \leq Cx^{-\mu-2}y^{-1/2}(|x - y|^2 + 1)^{1/2}. \]

Thus, we conclude that

\[ \int_0^\infty |f(y)|y^{\mu+1/2} \int_0^1 \int_0^{\pi/2} \frac{y}{(|x - y|^2 + t^2 + xyz^2)^{\mu+5/2}} \, dz \, dt \, dy < \infty. \]

On the other hand, we have

\[ \int_0^{\pi/2} \frac{xy|x - y|(\sin z)^{2\mu}z^4}{(|x - y|^2 + t^2 + xyz^2)^{\mu+7/2}} \, dz \leq C \frac{|x - y|(xy)^{-\mu-3/2}}{|x - y|^2 + t^2} \int_0^{\pi/2} \frac{u^{2\mu+4}}{(1 + u^2)^{\mu+7/2}} \, du. \]
By proceeding as above we obtain
\[ y^{\mu+1/2} \int_0^1 t \int_0^{\pi/2} \frac{y|x-y|(\sin z)^{2\mu}z^4}{(|x-y|^2 + t^2 + xyz^2)^{\mu+7/2}} \, dz \, dt \leq C x^{-\mu-1/2} y^{-1/2}. \]

Then, it follows that
\[ \int_0^\infty |f(y)| y^{\mu+1/2} \int_0^1 t \int_0^{\pi/2} \frac{y|x-y|(\sin z)^{2\mu}z^4}{(|x-y|^2 + t^2 + xyz^2)^{\mu+7/2}} \, dz \, dt \, dy < \infty. \tag{3.9} \]

Note also that
\[ \int_0^{\pi/2} \frac{xy^2(\sin z)^{2\mu}z^4}{(|x-y|^2 + t^2 + xyz^2)^{\mu+7/2}} \, dz \leq C(xy)^{-\mu-5/2} \frac{xy^2}{|x-y|^2 + t^2} \int_0^{\pi/2} \frac{u^{2\mu+4}}{(1+u^2)^{\mu+7/2}} \, du. \]

By similar arguments, we can see that
\[ y^{\mu+1/2} \int_0^1 t \int_0^{\pi/2} \frac{xy^2(\sin z)^{2\mu}z^4}{(|x-y|^2 + t^2 + xyz^2)^{\mu+7/2}} \, dz \, dt \leq C x^{-\mu-3/2} (xy)^{-1/2} (|x-y|^2 + 1)^{1/2}, \]
and from here,
\[ \int_0^\infty |f(y)| y^{\mu+1/2} \int_0^1 t \int_0^{\pi/2} \frac{y^2(\sin z)^{2\mu}z^4}{(|x-y|^2 + t^2 + xyz^2)^{\mu+7/2}} \, dz \, dt \, dy < \infty. \tag{3.10} \]

Then, (3.8), (3.9) and (3.10) implies that
\[ \int_0^\infty |f(y)| y^{\mu+1/2} \int_0^1 t \int_0^{\pi/2} (\sin z)^{2\mu} \left| \frac{1}{(|x-y|^2 + t^2 + 2xy(1-\cos z))^{\mu+3/2}} - \frac{1}{(|x-y|^2 + t^2 + xyz^2)^{\mu+3/2}} \right| \, dz 
\]
\[ \int_0^\infty |f(y)| y^{\mu+1/2} \int_0^1 t \int_0^{\pi/2} \left| \frac{d}{dx} \left( \frac{(\sin z)^{2\mu} - z^{2\mu}}{|x-y|^2 + t^2 + xyz^2)^{\mu+3/2}} \right) \right| \, dz \, dt \, dy < \infty. \tag{3.11} \]

By similar calculations, we can justify that
\[ \int_0^\infty |f(y)| y^{\mu+1/2} \int_0^1 t \int_0^{\pi/2} \left| \frac{d}{dx} \left( \frac{(\sin z)^{2\mu} - z^{2\mu}}{|x-y|^2 + t^2 + xyz^2)^{\mu+3/2}} \right) \right| \, dz \, dt \, dy < \infty, \tag{3.12} \]
and
\[ \int_0^\infty |f(y)| y^{\mu+1/2} \int_0^1 t \left| \frac{d}{dx} \left( \frac{x^{-\mu-1/2}(xy)^{\mu+1/2}}{|x-y|^2 + t^2 + xyz^2)^{\mu+3/2}} \right) \right| \, dz \, dt \, dy < \infty, \tag{3.13} \]

The inequalities (3.6), (3.11), (3.12), (3.13) and well known properties of the Hilbert transform lead to the desired result.

The following important result is an easy consequence of Lemmas 3.1 and 3.2.

**Theorem 3.3.** Let \( f \in C^\infty_c \) Then, for all \( x \in (0, \infty) \),
\[ R_\mu(f)(x) = \lim_{\varepsilon \to 0} \int_{|x-y| > \varepsilon} R_\mu(x, y) f(y) \, dy. \]
Next, we establish other form for the Riesz transform $R_\mu$ acting on $C_c^\infty$ that also can be seen as a version in our setting of the corresponding result for the Hilbert transform.

**Theorem 3.4.** Let $f \in C_c^\infty$ Then, for all $x \in (0, \infty)$,

$$R_\mu(f)(x) = \lim_{t \to 0} \int_0^\infty R_\mu(t, x, y) f(y) \, dy,$$

where, for $t, x, y \in (0, \infty)$,

$$R_\mu(t, x, y) = \frac{2\mu + 1}{\pi} (xy)^{\mu+1/2} \int_0^{\pi/2} \frac{(y \cos z - x)(\sin z)^{2\mu}}{(x^2 + y^2 + t^2 - 2xy \cos z)^{\mu+3/2}} \, dz.$$

**Proof.** To see this property we can argue as in the proof of Lemmas 3.1 and 3.2. Consider, for every $t, x, y \in (0, \infty)$,

$$R_{\mu,1}(t, x, y) = \frac{2\mu + 1}{\pi} (xy)^{\mu+1/2} \int_0^{\pi/2} \frac{(y \cos z - x)(\sin z)^{2\mu}}{(x^2 + y^2 + t^2 - 2xy \cos z)^{\mu+3/2}} \, dz,$$

$$R_{\mu,2}(t, x, y) = R_\mu(t, x, y) - R_{\mu,1}(t, x, y).$$

A similar reasoning to the one used in the proof of Lemma 3.1 allows us, by using the dominated convergence theorem, to obtain for each $x \in (0, \infty)$

$$\lim_{t \to 0} \int_0^\infty R_{\mu,2}(t, x, y) f(y) \, dy = \int_0^\infty R_{\mu,2}(x, y) f(y) \, dy.$$

Also, by proceeding as in the proof of Lemma 3.2, we can see that for some constant $C_\mu$ and a function $g \geq 0$ defined on $(0, \infty) \times (0, \infty)$ such that, for every $x \in (0, \infty)$, $g(x, \cdot) \in L_1$, we have that

$$\left| R_{\mu,1}(t, x, y) - C_\mu \frac{x - y}{|x - y|^2 + t^2} \right| \leq g(x, y), \quad x, y \in (0, \infty).$$

Then, by invoking again the dominated convergence theorem and a well known property of the Hilbert transform, we conclude that, for every $x \in (0, \infty)$ the limit

$$\lim_{t \to 0} \int_0^\infty R_\mu(t, x, y) f(y) \, dy$$

exists. Moreover, a careful study of the proof of Lemma 3.2 allows us to conclude that, for every $x \in (0, \infty)$,

$$\lim_{t \to 0} \int_0^\infty R_{\mu,1}(t, x, y) f(y) \, dy = \lim_{\varepsilon \to 0} \int_{|x - y| > \varepsilon} R_{\mu,1}(x, y) f(y) \, dy,$$

and the proof finishes.

Note that finer estimates in the proof of the above results lead to establish that Theorems 3.3 and 3.4 also hold for every $f \in \mathcal{H}_\mu$.

**Remark 3.5.** According to Theorem 3.4, if $Q_\mu$ denotes, as in the introduction, the conjugated defined by Muckenhoupt and Stein in [13], we can write, for every $f \in C_c^\infty$, $g(y) = y^{\mu+1/2} f(y)$ and

$$Q_\mu(f)(x) = x^{-\mu-1/2} R_\mu(g)(x).$$
Hence, from Theorem 3.3, we infer that the conjugated defined in [13] is also a principal value over $C_c^\infty$, that is, for every $f \in C_c^\infty$,

$$Q_\mu(f)(x) = \lim_{\varepsilon \to 0} \int_{|x-y| > \varepsilon} Q_\mu(0, x, y) f(y) \, dy, \quad x \in (0, \infty).$$

It also can be seen that $R_\mu(x,y) = (xy)^{\mu+1/2} Q(0, x, y)$, $x, y \in (0, \infty)$. Then, from [2, p. 17 and p. 20] and [12, p. 151-152] (see as well [13, p. 87]), we can deduce the following estimates for the function $R_\mu(x,y)$ that will be very useful in the sequel. There exists $C, D > 0$ for which

$$|R_\mu(x,y)| \leq C x^{-\mu-3/2} y^{\mu+1/2}, \quad 0 < y < x/2 \quad (3.15)$$

$$|R_\mu(x,y)| \leq C x^{\mu+3/2} y^{-\mu-5/2}, \quad y > 2x \quad (3.16)$$

$$R_\mu(x,y) = D \frac{1}{x-y} + O\left(\frac{1}{x} \left(1 + \log_+ \frac{\sqrt{xy}}{|x-y|}\right)\right), \quad x/2 < y < 2x \quad (3.17)$$

and, for certain $k, s > 0$,

$$R_\mu(x,y) \leq - \frac{1}{C} x^{-\mu-3/2} y^{\mu+1/2}, \quad 0 < y < x/k \quad (3.18)$$

$$R_\mu(x,y) \geq \frac{1}{C} x^{\mu+3/2} y^{-\mu-5/2}, \quad y > kx \quad (3.19)$$

$$\text{sign} (x-y) R_\mu(x,y) \geq \frac{1}{C} \frac{1}{|x-y|}, \quad x/s < y < sx \quad (3.20)$$

4. Boundedness properties of the Riesz transform

In this section we study $L^p$-boundedness properties for the Riesz transform $R_\mu$. We will prove that $R_\mu$ is actually a Calderón-Zygmund singular integral operator. Then, $L^p$-boundedness properties can be deduced from the general theory. Our procedure is different than the ones employed in [4] and [13] to establish the $L^p$ boundedness of Hankel conjugates.

First note that, according to (3.14), since $Q_\mu$ is a bounded operator from $L^2( x^{2\mu+1} \, dx)$ into itself, the Riesz transform $R_\mu$ is bounded from $L^2$ into itself. With the next proposition, it is proved that the Riesz transform $R_\mu$ is a Calderón-Zygmund operator.

**Proposition 4.1.** There exists $C > 0$ such that, for every $x, y \in (0, \infty)$, $x \neq y$,

$$\begin{align*}
(i) \quad |\partial_x R_\mu(x,y)| + |\partial_y R_\mu(x,y)| &\leq C \frac{1}{|x-y|^2}.
(ii) \quad |R_\mu(x,y)| &\leq C \frac{1}{|x-y|}.
\end{align*}$$

**Proof.** To prove (i) we write

$$R_\mu(x,y) = R_{\mu,1}(x,y) + R_{\mu,2}(x,y), \quad x, y \in (0, \infty), \quad x \neq y,$$

where $R_{\mu,1}$ and $R_{\mu,2}$ are defined in (3.2) and (3.3). We now analyze the derivatives of $R_{\mu,1}$ and $R_{\mu,2}$. Assume that $x, y \in (0, \infty), \quad x \neq y$. In this case, we can differentiate under the
integral sign and
\[ \frac{\partial}{\partial x} R_{\mu,1}(x, y) = \frac{2\mu + 1}{\pi} \left( -(xy)^{\mu+1/2} \int_0^{\pi/2} (\sin z)^{2\mu} \frac{(\sin z)^{\mu} (x^2 + y^2 - 2xy \cos z)^{\mu+3/2}}{} dz \right. \\
\left. - (\mu + \frac{3}{2})(xy)^{\mu+1/2} \int_0^{\pi/2} (\sin z)^{2\mu} (2x - 2y \cos z)^2 (x^2 + y^2 - 2xy \cos z)^{\mu+5/2} dz \right. \\
\left. + (\mu + \frac{1}{2})x^{\mu-1/2}y^{\mu+1/2} \int_0^{\pi/2} (\sin z)^{2\mu} (y \cos z - x) (x^2 + y^2 - 2xy \cos z)^{\mu+3/2} dz \right) \\
= I_1^1 + I_2^1 + I_3^1, \]

Analogously,
\[ \frac{\partial}{\partial x} R_{\mu,2}(x, y) = I_1^2 + I_2^2 + I_3^2, \]

where \( I_1^2, I_2^2 \) and \( I_3^2 \) are defined as \( I_1^1, I_2^1 \) and \( I_3^1 \) by changing the interval of integration by \((\pi/2, \pi)\). Applying the change of variables \( z = \pi - x \) in \( I_2^1 \) and then that \( 1 + \cos x \geq cx^2 \), formula (3.7) gives that
\[ |I_1^1| + |I_2^1| \leq C(xy)^{\mu+\frac{3}{2}} \int_0^{\pi/2} z^{2\mu} \frac{z^{2\mu} d\theta}{|x - y|^2 + xy(1 - \cos z)} \leq \frac{C}{|x - y|^2}. \]

For the second term, one has
\[ |I_2^1| \leq C \left( (xy)^{\mu+\frac{1}{2}} \int_0^{\pi/2} \frac{(\sin z)^{2\mu} |x - y|^2}{(|x - y|^2 + 2xy(1 - \cos z))^{\mu+5/2}} d\theta \right. \\
\left. + (xy)^{\mu+\frac{1}{2}} \int_0^{\pi/2} \frac{(\sin z)^{2\mu} (y(1 - \cos z))^2}{(|x - y|^2 + 2xy(1 - \cos z))^{\mu+5/2}} d\theta \right) = I_1^{2,1} + I_1^{2,2}. \]

By using again (3.7) we have
\[ I_1^{2,1} \leq C(xy)^{\mu+\frac{3}{2}} \int_0^{\pi/2} \frac{z^{2\mu} |x - y|^2}{(|x - y|^2 + xy(z^2))^{\mu+5/2}} d\theta \leq \frac{C}{|x - y|^2}. \]

To study \( I_1^{2,2} \) we consider two cases. Assume first that \( y < 2x \). Then (3.7) leads to
\[ I_1^{2,2} \leq C(xy)^{\mu+\frac{1}{2}} \int_0^{\pi/2} \frac{z^{2\mu} y^2}{(|x - y|^2 + 2xy(1 - \cos z))^{\mu+5/2}} d\theta \leq \frac{Cy}{|x - y|^2} \leq \frac{C}{|x - y|^2}. \]

When \( y > 2x \), \( |y - x| \sim y \) and
\[ I_1^{2,2} \leq C(xy)^{\mu+1/2} \int_0^{\pi/2} \frac{z^{2\mu} y^2}{(|x - y|^2 + xy(1 - \cos z))^{\mu+5/2}} d\theta \leq C(xy)^{\mu+1/2} \int_0^{\pi/2} \frac{z^{2\mu}}{|x - y|^2} d\theta \leq C \frac{|y|^2}{|x - y|^2}. \]
By the change of variable $z = \pi - x$ and similar arguments as the ones for $I^2_1$, we can see that $|I^2_2| \leq \frac{C}{|x - y|^2}$. We now study the third term $I^3_1$, also in the cases $y \leq 2x$ and $y > 2x$.

In the case $y \leq 2x$, since $|1 - \cos z| \leq 1$, by (3.7), we have that

$$|I^3_1| \leq C x^{\mu - 1/2} y^{\mu + 1/2} \int_0^{\pi/2} \left( \frac{(\sin z)^{2\mu}}{(x^2 + y^2 - 2xy \cos z)^{\mu + 3/2}} \right) dz \leq \frac{C}{|x - y|^2}.$$ 

Again by the change of variable $z = \pi - x$ and similar arguments, one gets $|I^3_2| \leq \frac{C}{|x - y|^2}$ in the case $y \leq 2x$.

Finally, suppose that $y > 2x$. Note that in this case $|x - y| \sim y$. By using [11, (2.8)] we have

$$I^3 = I^3_1 + I^3_2 = (\mu + \frac{1}{2}) \frac{1}{x} R_\mu(x, y) = -(\mu + \frac{1}{2}) \frac{1}{x} \left( \frac{2(\mu + 1)}{\pi} x(xy)^{\mu + 1/2} \right) \int_0^{\pi} \left( \frac{(\sin z)^{2\mu + 2}}{(x^2 + y^2 - 2xy \cos z)^{\mu + 3/2}} dz + R_{\mu + 1}(x, y) \right) = J^1 + J^2.$$ 

We write

$$J^1 = -(\mu + \frac{1}{2}) \frac{2(\mu + 1)}{\pi} (xy)^{\mu + 1/2} \left( \int_0^{\pi/2} + \int_{\pi/2}^{\pi} \right) \left( \frac{(\sin z)^{2\mu + 2}}{(x^2 + y^2 - 2xy \cos z)^{\mu + 3/2}} dz = J^1_1 + J^1_2, \right.$$ 

and

$$J^2 = -(\mu + \frac{1}{2}) \frac{1}{x} (R_{\mu + 1, 1} + R_{\mu + 1, 2}) = J^2_1 + J^2_2.$$ 

By using (3.7), as for $I^1_1$ and $I^1_2$, we conclude that

$$|J^1_1| \leq C (xy)^{\mu + 1/2} \int_0^{\pi/2} \frac{(\sin z)^{2\mu + 2}}{(x^2 + 2xy(1 - \cos z))^{\mu + 3/2}} dz \leq \frac{C}{|x - y|^2},$$ 

$$|J^1_2| \leq \frac{C}{|x - y|^2}.$$ 

Also, as for $I^2_1$ and $I^2_2$, one has

$$|J^2_1| \leq C y(xy)^{\mu + 1/2} \int_0^{\pi/2} \frac{(\sin z)^{2\mu + 2}(|x - y| + y(1 - \cos z))}{(|x - y|^2 + 2xy(1 - \cos z))^{\mu + 5/2}} dz \leq \frac{C}{|x - y|^2},$$ 

$$|J^2_2| \leq \frac{C}{|x - y|^2}.$$ 

Thus we conclude that

$$\left| \frac{\partial}{\partial x} R_\mu(x, y) \right| \leq \frac{C}{|x - y|^2}.$$
Next we estimate the derivative with respect to $y$ of $R_\mu(x, y)$. Observe that in this case,

$$
\frac{\partial}{\partial y} R_\mu(x, y) = \frac{2\mu + 1}{\pi} \left( -(xy)^{\mu+1/2} \int_0^\pi \frac{(\sin z)^{2\mu} \cos z}{(|x-y|^2 + 2xy(1-\cos z))^{\mu+3/2}} dz \right) - (\mu + \frac{3}{2})(xy)^{\mu+1/2} \int_0^\pi \frac{(\sin z)^{2\mu}(y \cos z - x)(2y - 2x \cos z)}{(|x-y|^2 + 2xy(1-\cos z))^{\mu+5/2}} dz + (\mu + \frac{1}{2})x^{\mu+1/2}y^{\mu-1/2} \int_0^\pi \frac{(\sin z)^{2\mu}(y \cos z - x)}{(|x-y|^2 + 2xy(1-\cos z))^{\mu+3/2}} dz.
$$

By proceeding as for $I_1^1$ and $I_2^1$, we get $|S^1| \leq \frac{C}{|x-y|^2}$. In order to estimate $S^2$ we note that

$$
(x-y \cos z)(y-x \cos z) = -|x-y|^2 + y(x-y)(1-\cos z) + xy(1-\cos z)^2 + x(x-y)(1-\cos z),
$$

and we write $S^2 = \sum_{j=1}^4 S_{2j}^2$, where $S_{2j}^2$, $j = 1, 2, 3, 4$, are defined in the obvious way.

$S_{2,1}^2$ can be studied like $I_1^1$ and $I_2^1$. To analyze $S_{2,2}^2$ we consider two cases. If $y > 2x$, then $y \sim |x-y|$ and we proceed as for $S_{2,1}^2$. Otherwise, we argue as for the former estimate for $J_1^2$. $S_{2,3}^2$ coincides with $S_{2,2}^2$ when $x$ and $y$ are interchanged. To estimate $S_{2,4}^2$, we consider again two regions: $y > 2x$ and $y < 2x$, and follow the same reasoning as above. Thus we conclude that $|S^2| \leq C|x-y|^{-2}$.

Finally, observe that the estimate for $S^3$ can be obtained by proceeding as in the study of $I^3$, just by interchanging the role of $x$ and $y$. With this, we obtain that

$$
\left| \frac{\partial}{\partial y} R_\mu(x, y) \right| \leq \frac{C}{|x-y|^2}.
$$

Let us now prove $(ii)$. We write

$$
|R_\mu(x, y)| \leq C(xy)^{\mu+1/2} \int_0^\pi \frac{|\sin z|^{2\mu}|x-y|}{(|x-y|^2 + 2xy(1-\cos z))^{\mu+3/2}} dz + \int_0^\pi \frac{(|\sin z|^{2\mu}y)(1-\cos z)}{(|x-y|^2 + 2xy(1-\cos z))^{\mu+3/2}} dz = \Sigma_1 + \Sigma_2.
$$

Note that $\Sigma_1 \leq C|x-y|(I_1^1 + I_2^1)$, thus by the proof of $(i)$ $\Sigma_1 \leq C|x-y|^{-1}$. On the other hand, if $y > 2x$, then $y \sim |y-x|$ and $\Sigma_2 \leq C\Sigma_1 \leq C|x-y|^{-1}$. In the case $y \leq 2x$, then $y \leq C|\frac{x}{y}|$ and we consider

$$
\Sigma_2^1 = (xy)^{\mu+1/2} \int_0^{\pi/2} \frac{(\sin z)^{2\mu}(1-\cos z)}{(|x-y|^2 + 2xy(1-\cos z))^{\mu+3/2}} dz
$$

and $\Sigma_2^2 = \Sigma_2 - \Sigma_2^1$. By using (3.7) we get

$$
\Sigma_2^1 \leq C(xy)^{\mu+1} \int_0^{\pi/2} \frac{z^{2\mu+1}}{(|x-y|^2 + xz^{2})^{\mu+3/2}} dz \leq \frac{C}{|x-y|}.
$$

Finally, since $1 - \cos z \geq 1$ for $z \in (\pi/2, \pi)$, one easily gets $\Sigma_2^2 \leq C|x-y|^{-1}$. Thus we conclude that $(ii)$ holds.

Now, the general theory of Calderón-Zygmund singular integrals allows us to obtain $L_\mu$-boundedness properties for the Riesz transform $R_\mu$. 


Theorem 4.2. The Riesz transform $R_\mu$ can be uniquely extended to $L_p(w)$ as a bounded operator from $L_p(w)$ into itself, for every $1 < p < \infty$ and $w \in A_p(0, \infty)$. Also, if $w \in A_1(0, \infty)$ then, $R_\mu$ uniquely extends to $L_1(w)$ as a bounded operator from $L_1(w)$ into $L_{1, \infty}(w)$.

Define the maximal operator associated to $R_\mu$ by

$$R_{\mu,*}(f) = \sup_{\varepsilon > 0} |R_{\mu,\varepsilon}(f)|,$$

where, for every $\varepsilon > 0$,

$$R_{\mu,\varepsilon}(f)(x) = \int_{|x-y| > \varepsilon} R_{\mu}(x,y)f(y)\,dy.$$

The next theorem follows from Theorem 4.2, according to [7, Corollary 7.13].

Theorem 4.3. For every $1 < p < \infty$ and $w \in A_p(0, \infty)$, $R_{\mu,*}$ is a bounded operator from $L_p(w)$ into itself. Moreover, if $w \in A_1(0, \infty)$, $R_{\mu,*}$ is a bounded operator from $L_1(w)$ into $L_{1, \infty}(w)$.

Since $C^\infty_0$ is a dense subset of $L_p(w)$, $1 \leq p < \infty$ and $w \in A_p(0, \infty)$, Theorems 3.3 and 4.3 imply the following result.

Corollary 4.4. For every $1 \leq p < \infty$ and $w \in A_p(0, \infty)$,

$$R_{\mu}(f)(x) = \lim_{\varepsilon \to 0} \int_{|x-y| > \varepsilon} R_{\mu}(x,y)f(x)\,dx,$$

for a.e. $x \in (0, \infty)$.

5. Oscillation and Variation of the Riesz Transforms $R_\mu$.

Suppose that $(X, \mathcal{F}, \mu)$ denotes a $\sigma$-finite measure space and that $\mathcal{T} = \{T_r\}_{r > 0}$ is a family of bounded operators form $L_p(X, \mathcal{F}, \mu)$ into itself for some $1 < p < \infty$. Assume also that the limit $Tf = \lim_{r \to 0} T_rf$ exists, in some sense, for each $f \in L_p(X, \mathcal{F}, \mu)$. Then a natural question appears: to measure the speed of convergence of $T_rf$ to $Tf$. Inspired by the classical method of considering square functions to measure the speed of convergence, two kinds of operators have been introduced: the oscillation and the variation operators ([5], [6] and [10]). Let $\{t_i\}$ be a fixed decreasing sequence converging to zero. We define the oscillation operator as follows

$$\mathcal{O}(Tf)(x) = \left( \sum_{i=1}^{\infty} \sup_{t_{i+1} \leq \varepsilon_{i+1} < \varepsilon_i \leq t_i} |T_{\varepsilon_{i+1}}f(x) - T_{\varepsilon_i}f(x)|^2 \right)^{1/2}.$$

It is not hard to see that $\mathcal{O}(Tf)$ is equivalent to $\mathcal{O}'(Tf)$, being

$$\mathcal{O}'(Tf)(x) = \left( \sum_{i=1}^{\infty} \sup_{\delta_i \leq \varepsilon_i \leq t_i} |T_{\varepsilon_i}f(x) - T_{\delta_i}f(x)|^2 \right)^{1/2}.$$

For every $\rho > 2$ the variation operator $\mathcal{V}_\rho$ is defined by

$$\mathcal{V}_\rho(Tf)(x) = \sup_{\varepsilon} \left( \sum_{i=1}^{\infty} |T_{\varepsilon_{i+1}}f(x) - T_{\varepsilon_i}f(x)|^\rho \right)^{1/\rho},$$
where the supremum is taken over all decreasing sequences of real numbers \( \{ \varepsilon_i \} \) converging to zero. The oscillation and variation operators can be understood as norms of linear operators acting on suitable Banach spaces. Since our results only refer to operators acting on functions defined on \((0, \infty)\), in the sequel we will restrict ourselves to this situation. We denote by \((X, \| \cdot \|_X)\) the Banach spaces that consists of all those functions \( h \) defined on \((0, \infty) \times \mathbb{N}\) such that

\[
\| h \|_X = \left( \sum_{i=1}^{\infty} \left( \sup_{s \in (0, \infty)} |h(s, i)| \right)^2 \right)^{1/2} < \infty.
\]

Let \( T = \{ T_t \}_{t > 0} \) be a family of operators defined on \( L_p \) for some \( 1 < p < \infty \). Fix \( \{ t_i \}_i \) a sequence of real numbers that is decreasing and converging to zero and define, for every \( i \in \mathbb{N} \), the interval \( I_i = (t_{i+1}, t_i] \). Then the operator \( U(T) \) is defined as follows

\[
f \rightarrow U(T)(f) = \{ (T_{t_{i+1}} f - T_{t_i} f) \chi_{I_i} \}_{i \in \mathbb{N}}.
\]

Thus, it is clear that \( O(T f)(x) = \| U(T)(f)(x) \|_X = \| \{ (T_{t_{i+1}} f(x) - T_{t_i} f(x)) \chi_{I_i} \}_{i \in \mathbb{N}} \|_X \). To prove that the oscillation operator \( O \) of \( T \) is bounded from \( L_p \) into itself, it suffices to see that the operator \( U(T) \) is bounded from \( L_p \) into \( L_p(X) \) (being \( L_p(X) \) the usual Bochner-Lebesgue space of functions with values in \( X \)).

For the \( \rho \)-variation, let

\[
\Theta = \{ \varepsilon = \{ \varepsilon_i \}_{i \in \mathbb{N}} : \varepsilon_i \in \mathbb{R}, \ i \in \mathbb{N}, \ \varepsilon_i \downarrow 0 \}.
\]

Consider, for every \( 1 \leq \rho < \infty \) the Banach space \((X_\rho, \| \cdot \|_\rho)\), whose elements are functions \( h \) defined in \( \mathbb{N} \times \Theta \) such that

\[
\| h \|_\rho = \sup_{\varepsilon \in \Theta} \left( \sum_{i=0}^{\infty} |h(i, \varepsilon)|^\rho \right)^{1/\rho} < \infty.
\]

Now we define the operator \( V(T f)(x) = \{ T_{\varepsilon_{i+1}} f - T_{\varepsilon_i} f \}_{\varepsilon \in \Theta} \). This operator is defined for functions in \( L_p \) and takes values in \( X_\rho \). Thus, the variation operator \( V_\rho \) appears as \( V_\rho(T f) = \| V(T) f \|_{X_\rho} \).

According to Corollary 4.4, for every \( f \in L_p, 1 \leq p < \infty \), \( R_\mu(f)(x) = \lim_{\varepsilon \to 0} R_{\mu, \varepsilon}(f)(x) \), for almost every \( x \in (0, \infty) \). We now analyze the speed of convergence in the above limit by studying the boundedness of the oscillation and variation operators associated with the family \( R_\mu = \{ R_{\mu, \varepsilon} \}_{\varepsilon > 0} \). The corresponding analysis for the Hilbert transform was made in [6]. More recently, Gillespie and Torrea [10, Theorem 1.5] have obtained a weighted version of the previous result in [6] by proving that the oscillation and variation operators associated with the Hilbert transform are bounded from \( L_p(v) \) into itself for every \( 1 < p < \infty \) and \( v \in A_p \).

**Theorem 5.1.** The operators \( O(R_\mu) \) and \( V_\mu(R_\mu) \) are bounded from \( L_p(v) \) into itself, provided that \( 1 < p < \infty \) and \( v \in A_p(0, \infty) \). Moreover, the operators \( O(R_\mu) \) and \( V_\mu(R_\mu) \) are bounded from \( L_1 \) into \( L_{1, \infty} \).
Assume that \( 1 < p < \infty \) and \( v \in A_p(0, \infty) \). Let \( \{t_i\}_{i \in \mathbb{N}} \) be a decreasing sequence in \((0, \infty)\) such that \( t_i \downarrow 0 \). According to (3.15), (3.16) and (3.17) we can write

\[
|R_{\mu, t_{i+1}} f(x) - R_{\mu, t_i} f(x)| = \left| \int_{\{t_{i+1} < |x-y| < s\}} R_\mu(x, y) f(y) \, dy \right|
\]

\[
\leq C \left( x^{-\mu/2} \int_{0}^{x/2} \chi_{\{t_{i+1} < |x-y| < s\}}(y)y^{\mu+1/2} |f(y)| \, dy \right.
\]

\[
+ x^{\mu+3/2} \int_{2x}^{\infty} \chi_{\{t_{i+1} < |x-y| < s\}}(y) y^{-\mu/2} |f(y)| \, dy
\]

\[
+ \left| \int_{x/2}^{2x} \frac{1}{x-y} \chi_{\{t_{i+1} < |x-y| < s\}}(y) f(y) \, dy \right|
\]

\[
+ \int_{x/2}^{2x} \frac{1}{y} \left( 1 + \log_+ \frac{\sqrt{xy}}{|x-y|} \right) \chi_{\{t_{i+1} < |x-y| < s\}}(y) f(y) \, dy \right).
\]

Then

\[
O'(R_\mu f)(x) \leq C \sum_{j=1}^{4} \| \{I_j(s, i) (x)\} \|_X,
\]

where

\[
I_1(s, i)(x) = \frac{1}{x} \int_{0}^{x} \chi_{\{t_{i+1} < |x-y| < s\}}(y) |f(y)| \, dy
\]

\[
I_2(s, i)(x) = \int_{x}^{\infty} \frac{1}{y} \chi_{\{t_{i+1} < |x-y| < s\}}(y) |f(y)| \, dy
\]

\[
I_3(s, i)(x) = \int_{0}^{\infty} \frac{1}{x-y} \chi_{\{t_{i+1} < |x-y| < s\}}(y) f(y) \, dy
\]

\[
I_4(s, i)(x) = \int_{x/2}^{2x} \frac{1}{y} \left( 1 + \log_+ \frac{\sqrt{xy}}{|x-y|} \right) \chi_{\{t_{i+1} < |x-y| < s\}}(y) f(y) \, dy.
\]

Assume that \( f \in L_{1, \text{loc}}([0, \infty)) \). By Minkowski’s inequality, we obtain that

\[
\| \{I_1(s, i)(x)\}_i \|_X \leq \frac{1}{x} \int_{0}^{x} \sum_{i=1}^{\infty} \sup_{t_{i+1} < s \leq t_i} \chi_{\{t_{i+1} < |x-y| < s\}}(y) |f(y)| \, dy \leq \frac{1}{x} \int_{0}^{x} |f(y)| \, dy.
\]

In a similar way we obtain that

\[
\| \{I_2(s, i)(x)\}_i \|_X \leq \int_{x}^{\infty} \frac{1}{y} |f(y)| \, dy,
\]

\[
\| \{I_4(s, i)(x)\}_i \|_X \leq \int_{x/2}^{2x} \frac{1}{y} \left( 1 + \log_+ \frac{\sqrt{xy}}{|x-y|} \right) |f(y)| \, dy.
\]

As it is known the Hardy operators \( T_j, j = 1, 2 \), defined by

\[
T_1(f)(x) = \frac{1}{x} \int_{0}^{x} f(y) \, dy \quad \text{and} \quad T_2(f)(x) = \int_{x}^{\infty} \frac{1}{y} f(y) \, dy,
\]
are bounded from $L_p(v)$ into itself. Also, by proceeding as in [2, p. 18-19] we can see that the operator
\[ T_4(f)(x) = \int_{x/2}^{2x} \frac{1}{y} \left( 1 + \log_+ \frac{\sqrt{xy}}{|x - y|} \right) |f(y)| \, dy \]
is bounded from $L_p(v)$ into itself. Therefore,
\[ \|\| \{ I_j(s,i)(x) \}_i \|\|_{L_p(v)} \leq C \|f\|_{L_p(v)}, \quad \text{for } j = 1, 2, 4 \text{ and } f \in L_p(v). \]
Finally, by using [10, Theorem 1.5], we get
\[ \|\| \{ I_3(s,i)(x) \}_i \|\|_{L_p(v)} \leq C \|f\|_{L_p(v)}, \quad f \in L_p(v). \]
Thus we conclude that the oscillation operator $O(R_\mu)$ is bounded from $L_p(v)$ into itself.

In a similar way, by using again [10, Theorem 1.5], it can be proved that the variation operators $V_\rho(R_\mu)$, $\rho > 2$, are bounded from $L_p(v)$ into itself.

On the other hand, since the operators $T_j$, $j = 1, 2, 4$, are bounded from $L_1$ into $L_{1,\infty}$, [6, Theorems 1.1 and 1.2] we get that the oscillation operator $O(R_\mu)$ and the variation operators $V_\rho(R_\mu)$, $\rho > 2$, are of weak type $(1,1)$. 

6. Weighted inequalities.

In this section we study the conditions for a pair of positive measurable functions $(u, v)$ such that the following inequality holds

\[ \int_0^\infty |R_\mu(f)(x)|^p u(x) \, dx \leq C \int_0^\infty |f(x)|^p v(x) \, dx, \quad f \in L^p(v), \]
where $1 < p < \infty$.

According to Theorem 4.2, (6.1) holds whenever $u = v \in A_p(0, \infty)$. Moreover, by using [2, Theorem 1] we can obtain a complete characterization of the measurable functions $u$ for which (6.1) is satisfied when $u = v$.

In the general case of different weights we consider the following weak problem: find conditions for a weight $v$ (respectively $u$) such that (6.1) holds true for some weight $u$ (respectively $v$). Note that, since $R_\mu$ is a Calderón-Zygmund singular integral operator (Proposition 4.1), from [9, Theorem 6.4] we deduce that if $1 < p < \infty$ and
\[ \int_0^\infty v(x)^{1-p'}(1+x)^{-p'} \, dx < \infty \quad \text{(respectively } \int_0^\infty u(x)(1+x)^{-p} \, dx < \infty), \]
then there exists a positive measurable function $u$ (respectively $v$) for which (6.1) holds.

Next we characterize the positive measurable functions $v$ (respectively $u$) satisfying (6.1) for some positive measurable function $u$ (respectively $v$).

We will need the following result established in [8].

**Theorem 6.1.** Let $(Y,d\nu)$ be a measure space, $F, G$ be Banach spaces, and $\{A_k\}_{k \in \mathbb{Z}}$ be a sequence of pairwise disjoint measurable sets of $Y$ such that $Y = \cup_{k \in \mathbb{Z}} A_k$. Consider $0 < s < p < \infty$ and $T$ a sublinear operator which satisfies the following vector valued inequality
\[ \left\| \left( \sum_j \|Tf_j\|_F^p \right)^{1/p} \right\|_{L^p(A_k,d\nu)} \leq C_k \left( \sum_j \|f_j\|_F^p \right)^{1/p}, \quad k \in \mathbb{Z}, \]
where, for every $k \in \mathbb{Z}$, $C_k$ only depends of $F$, $G$, $p$ and $s$. Then there exist a positive measurable function $u$ on $Y$ such that

$$
\left( \int_Y \|Tf(x)\|_G^p u(x) d\nu(x) \right)^{1/p} \leq C \|f\|_F,
$$

where $C$ only depends of $F$, $G$, $p$ and $s$.

Let us introduce, for every $1 < p < \infty$, the set of functions $D_p$ as follows:

$$
D_p = \left\{ w \geq 0 : \text{ measurable, such that for any } a > 0 \right\}
$$

$$
\int_a^\infty x^{(\mu+1/2)p/(p-1)}w(x)^{-1/(p-1)}dx < \infty \quad \text{and} \quad \int_a^\infty x^{-(\mu+5/2)p/(p-1)}w(x)^{-1/(p-1)}dx < \infty.
$$

**Theorem 6.2.** Let $1 < p < \infty$ and let $v$ be a measurable function such that $0 < v < \infty$. The following assertions are equivalent:

(i) $v \in D_p$.

(ii) For every $f \in L_p(v)$, there exists $\lim_{z \to 0} R_{\mu,\varepsilon}(f)(x)$, for almost all $x \in (0, \infty)$.

(iii) For every $f \in L_p(v)$, $R_{\mu,\varepsilon}(f)(x) = \sup_{\varepsilon > 0} |R_{\mu,\varepsilon}(f)(x)| < \infty$, for almost all $x \in (0, \infty)$.

(iv) There exist a positive measurable function $u$ and $C > 0$ such that, for every $f \in L_p(v),$

$$
\int_0^\infty |R_{\mu,\varepsilon}(f)(x)|^p u(x) dx \leq C \int_0^\infty |f(x)|^p v(x) dx.
$$

(v) There exist a positive measurable function $u$ and $C > 0$ such that, for every $f \in L_p(v),$

$$
\int_0^\infty |R_{\mu,\varepsilon}(f)(x)|^p u(x) dx \leq C \int_0^\infty |f(x)|^p v(x) dx.
$$

**Proof.** (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (i) are clear.

(i) $\Rightarrow$ (iv). Assume that $v \in D_p$ and take $0 < s < 1$. We define $I_0 = (\frac{1}{2}, 2)$, and, for every $k \in \mathbb{N}$, $I_k = (2^{-k-1}, 2^{-k}] \cup [2^k, 2^{k+1})$ and $B_k = (2^{-k-2}, 2^{k+2})$. Let $k \in \mathbb{N}$ and $f$ be a measurable function on $(0, \infty)$. We write $f = f_1 + f_2 + f_3$, where $f_1 = f 1_{B_k}$. $f_2 = f 1_{(2^{-k-2}, 2^{k+2})}$ and $f_3 = f 1_{(2^{k+2}, \infty)}$. Observe that for $x \in I_k$ and $y \in (0, 2^{-k-2})$ then $x > 2$, and, according to (3.15), $|R_{\mu}(x, y)| \leq C x^{-\mu-3/2} y^{\mu+1/2}$. Hence, for every $\varepsilon > 0$ and $x \in I_k,$

$$
|R_{\mu,\varepsilon}(f_2)(x)| \leq C 2^{k+3/2} \int_0^{2^{-k-2}} y^{\mu+1/2} v(y)^{1/p} v(y)^{-1/p} |f(y)| dy
$$

$$
\leq C 2^{k+3/2} \left( \int_0^\infty v(y) |f(y)|^p dy \right)^{1/p} \left( \int_0^{2^{-k-2}} y^{(\mu+1/2)p'} v(y)^{-p'/p} dy \right)^{1/p'}.
$$

Thus we conclude that

$$
(6.2) \quad \left\| \left( \sum_j |R_{\mu,\varepsilon}(f_j)|^p \right)^{1/p} \right\|_{L_p(I_k, dx)} \leq C_k \left( \sum_j \|f_j\|_{L_p(v)}^p \right)^{1/p}.
$$
Also if \( x \in I_k \) and \( y > 2^{k+2} \) then \( \frac{x}{y} < \frac{1}{2} \), and, according to (3.16), \( |R_\mu(x, y)| \leq C x^{\mu+3/2} y^{-\mu-5/2} \). Hence, for every \( \varepsilon > 0 \), one can get in a similar way that

\[
(6.3) \quad \left\| \left( \sum_j |R_{\mu,*}(f_j^1)|^p \right)^{1/p} \right\|_{L^*(I_k, dx)} \leq C_k \left( \sum_j \|f_j\|_{L^p(v)}^p \right)^{1/p}.
\]

By using standard techniques, one can easily obtain a \( \ell^p \)-valued (self)-improvement of Theorem 4.3 \( \text{p} \text{oner referencia} \) that, together with Kolmogorov’s inequality ([9, p. 485]), give us

\[
\left\| \left( \sum_j |R_{\mu,*}(f_j^1)|^p \right)^{1/p} \right\|_{L^*(I_k, dx)} \leq C_k \int_{I_k} \left( \sum_j |f_j(x)|^p \right)^{1/p} \mu(x)^{1/p} \mu(x)^{-1/p} dx \leq C_k \left( \sum_j \|f_j\|_{L^p(v)}^p \right)^{1/p}.
\]

Then

\[
(6.4) \quad \left\| \left( \sum_j |R_{\mu,*}(f_j^1)|^p \right)^{1/p} \right\|_{L^*(I_k, dx)} \leq C_k \left( \sum_j \|f_j\|_{L^p(v)}^p \right)^{1/p}.
\]

By combining now (6.2), (6.3) and (6.4) we conclude that, for every \( k \in \mathbb{N} \),

\[
\left\| \left( \sum_j |R_{\mu,*}(f_j^1)|^p \right)^{1/p} \right\|_{L^*(I_k, dx)} \leq C_k \left( \sum_j \|f_j\|_{L^p(v)}^p \right)^{1/p}.
\]

Hence, from Theorem 6.1, we deduce that there exists a weight \( u \) such that (6.1) holds true.

\( (v) \Rightarrow (i) \). Duality arguments allow us to conclude that if for a pair of weights \( (u, v) \) (6.1) holds then,

\[
(6.5) \quad \int_0^\infty |R^*_\mu(f)(x)|^p v(x)^{1-p'} dx \leq C \int_0^\infty |f(x)|^p u(x)^{1-p'} dx, \quad f \in L^{p'}(v^{1-p'}),
\]

where \( R^*_\mu \) denotes the adjoint operator of \( R_\mu \) and it is given by

\[
R^*_\mu(f)(x) = \int_0^\infty R_\mu(y, x) f(y) dy.
\]

Assume that (6.5) holds for some weight \( u \). Let \( a > 0 \) and choose a bounded set \( J \subset (ak, \infty) \), where \( k \) is a suitable positive constant given in (3.18) and (3.19), having positive Lebesgue measure and such that \( \int_J u(x)^{1-p'} dx < \infty \). According to (3.18), since \( \frac{x}{y} < \frac{1}{k} \), provided that \( x \in (0, a) \) and \( y \in J \),

\[
|R^*_\mu(\chi_J)(x)| = \left| \int_J R_\mu(y, x) dy \right| \geq C x^{\mu+1/2} \int_J y^{-\mu-3/2} dy, \quad x \in (0, a).
\]
Then
\[ \int_0^a x^{(\mu+1/2)p'} v(x)^{1-p'} dx \leq C \int_0^a |R^*_\mu(\chi J)(x)|^p v(x)^{1-p'} dx \]
\[ \leq C \int_0^\infty \chi_J(x) u(x)^{1-p'} dx \leq C \int_j u(x)^{1-p'} dx. \]

Now we take a bounded set \( J \subset (0, a/k) \) having positive Lebesgue measure and such that \( \int_j u(x)^{1-p'} dx < \infty \). According to (3.19), since \( \frac{x}{y} > k \), provided that \( x \in (a, \infty) \) and \( y \in J \),
\[ |R^*_\mu(\chi J)(x)| = \left| \int_j R_\mu(y, x) dy \right| \geq C x^{-\mu-5/2} \int_j y^{\mu+3/2} dy, \quad x \in (a, \infty). \]

Then
\[ \int_a^\infty x^{-(\mu+5/2)p'} v(x)^{1-p'} dx \leq C \int_a^\infty |R^*_\mu(\chi J)(x)|^p v(x)^{1-p'} dx \]
\[ \leq C \int_0^\infty \chi_J(x) u(x)^{1-p'} dx \leq C \int_j u(x)^{1-p'} dx. \]

And with this we prove that \( v \in D_p \).

(iv) ⇒ (ii). Since the measure \( v dx \) is σ-finite on \( (0, \infty) \), \( L_p(v) \cap L_2(dx) \) is a dense subset of \( L_p(v) \). Moreover, according to Corollary 4.4, for every \( f \in L_2(dx) \) there exists \( \lim_{\epsilon \to 0} R_{\mu, \epsilon} f \) almost everywhere. Then, for every \( f \in L_p(v) \), there exists \( \lim_{\epsilon \to 0} R_{\mu, \epsilon} f \) almost everywhere, provided that (iv) holds. To see this it is sufficient to argue as in [7, p. 27 and 28].

(iii) ⇒ (i). Assume that (iii) holds. First, we see that the sublinear operator \( R_{\mu, \epsilon} \), defined on \( L_p(v) \), is continuous in measure. Since \( R_{\mu, \epsilon} f(x) < \infty \) a.e., for every \( f \in L_p(v) \), according to [9, Proposition VI.1.4], it is sufficient to show that, for every \( \epsilon > 0 \), \( R_{\mu, \epsilon} \) is continuous in measure. In fact, we shall see that the sublinear operators
\[ S_{\mu, \epsilon} f(x) = \int_{|x-y| > \epsilon} |R_{\mu}(x, y)| \| f(x) \| dx \]
are continuous in measure. We now define, for each \( m \in \mathbb{N} \) and \( l \in \mathbb{Z} \), the set \( A_{m, l} = \{ x \in (m, m+1) : 2^l < v(x) < 2^{l+1} \} \). Thus, we can write, for every \( m, n \in \mathbb{N} \) and \( l, k \in \mathbb{Z} \),
\[ S_{\mu, \epsilon} f(x) = \sum_{m, l} \sum_{k, n} \chi_{A_{m, l}}(x) \int_{|x-y| > \epsilon} \chi_{A_{n, k}}(y) \| R_{\mu}(x, y) \| \| f(y) \| dy = \sum_{m, l} \sum_{k, n} S_{\mu, \epsilon}^{(m, l), (n, k)}(f)(x). \]

Fix \( m, n \in \mathbb{N} \) and \( l, k \in \mathbb{Z} \). By Proposition 4.1, (ii) and Holder’s inequality, we have
\[ S_{\mu, \epsilon}^{(m, l), (n, k)}(f)(x) \leq C \int_{A_{n, k}} \| f(y) \|^{1/p} \| v^{-1/p} dy \]
\[ \leq C \left( \int_{A_{n, k}} v^{-1/(p-1)} dy \right)^{1/p'} \| f \|_{L_p(v)}, \quad x \in A_{m, l}. \]

Hence, \( S_{\mu, \epsilon}^{(m, l), (n, k)} \) is bounded from \( L_p(v) \) into \( L_\infty(dx) \) and then, \( S_{\mu, \epsilon}^{(m, l), (n, k)} \) is continuous in measure. According to our hypothesis, if \( f \in L_p(v) \) then \( S_{\mu, \epsilon} f(x) < \infty \) for almost every \( x \in (0, \infty) \). From Proposition VI.1.4 in [9] it follows that \( S_{\mu, \epsilon} \) is continuous in measure (\( S_{\mu, \epsilon} f(x) \) is the supremum of finite sums of continuous-in-measure operators).
Since \(|R_{\mu,\varepsilon}f| \leq S_{\mu,\varepsilon}f, R_{\mu,\varepsilon}|,\) and by the same reasoning \(R_{\mu,\varepsilon}^*,\) are also continuous-in-measure operators. Nikishin’s theorem ([9, Corollary VI.2.7]) implies that there exists a measurable function \(u > 0,\) a.e., such that

\[
\int_{\{x \in (0,\infty): R_{\mu,\varepsilon}f(x) > \lambda\}} u(x)dx \leq \left(\frac{||f||_{L_p(v)}}{\lambda}\right)^q, \quad f \in L_p(v) \quad \text{and} \quad \lambda > 0,
\]

where \(q = \inf(p,2).\) Assume first that \(1 < p \leq 2.\) Then we have that

\[
\int_{\{x \in (0,\infty): R_{\mu,\varepsilon}f(x) > \lambda\}} u(x)dx \leq \left(\frac{||f||_{L_p(v)}}{\lambda}\right)^p, \quad f \in L_p(v) \quad \text{and} \quad \lambda > 0,
\]

\(u\) being a positive measurable function. By Corollary 4.4, \(R_{\mu}(f)(x) = \lim_{\varepsilon \to 0} R_{\mu,\varepsilon}(f)(x),\) a.e., for every \(f \in L_2.\) Then, since \(L_2 \cap L_p(v)\) is a dense subset in \(L_p(v),\) (6.7) allows us to extend \(R_{\mu}\) to \(L_p(v)\) and the extended operator satisfies

\[
\int_{\{x \in (0,\infty): R_{\mu}f(x) > \lambda\}} u(x)dx \leq \left(\frac{||f||_{L_p(v)}}{\lambda}\right)^p, \quad f \in L_p(v) \quad \text{and} \quad \lambda > 0.
\]

From (6.8) we deduce that \(u \in L_{1,\text{loc}}.\) Indeed, let \(x_0 > 0.\) We define, for every \(k \in \mathbb{N},\) the set \(C_k = B(x_0,2^{-k-1}x_0) \cap \left\{y \in (0,\infty): |y-x| < r\right\},\) \(r, x > 0.\) We choose \(k_0,l_0 \in \mathbb{N}\) such that \(\frac{1}{1-l} < \frac{1}{l}\) \(l\) being a positive constant for which

\[
R_{\mu}(x,y) \geq C \frac{\text{sign}(x-y)}{x-y}, \quad 1 \leq y < lx
\]

(see (3.20)), and we take a measurable set \(A \subset C_{k_0} \cap (0,x_0)\) such that \(|A| > 0 and \int_A v dy < \infty.\) Note that this set \(A\) exists because \(0 < v < \infty\) a.e.. Assume now that \(x \in B(x_0,2^{-k_0-l_0}x_0)\) and \(y \in A.\) Then \(y > x/l\) and

\[
R_{\mu}(x,y) \geq C \frac{\text{sign}(x-y)}{x-y} \geq \frac{C}{x_0(2^{-k_0-l_0} - 2^{-(k_0-l_0)})} = C_0.
\]

Hence

\[
R_{\mu}(x_A)(x) \geq \int_A C_0 dx = C_0|A| > 0, \quad x \in B(x_0,2^{-k_0-l_0}x_0).
\]

By taking \(0 < \lambda_0 < C_0|A|,\) from (6.8) we deduces that

\[
\int_{B(x_0,2^{-k_0-l_0}x_0)} u(z)dz \leq \int_{\{z \in (0,\infty): R_{\mu}(x_A)(z) > \lambda_0\}} u(z)dz \\
\leq C\left(\frac{||x_A||_{L_p(v)}}{\lambda_0}\right)^p \leq \frac{C}{\lambda_0^p} \int_A v dy < \infty.
\]

Thus, \(u \in L_{1,\text{loc}}.\) Since \(u \in L_{1,\text{loc}}\) the measure \(u dx\) is \(\sigma\)-finite and from Kolmogorov’s inequality ([9, Lemma V.2.8]), if \(1 < r < p,\)

\[
\sup_{E: 0 < \int_E v < \infty} \left(\int_E u dy\right)^{1/p-1/r} \left(\int_E |R_{\mu}(f)(x)|^r u(x)dx\right)^{1/r} \leq C||R_{\mu}f||_{L_{p,\infty}(u)} \leq C||f||_{L_p(v)}.
\]
Choose a measurable set $A$ such that $|A| < \infty$, $0 < \int_A u\,dy < \infty$ and $\int_A u^{1-r'}(x)\,dx < \infty$. It is not hard to see that this set $A$ can be found. Since the set $M_p(v) = L_p \cap L_p(v^{-1}) \cap L_\infty$ is dense in $L_p$, we can write

$$\|R_p^*(\chi_A)\|_{L_p(v^{-1/p})} = \|R_p^*(\chi_A)v^{-1/p}\|_{L_p(v)} = \sup_{g \in M_p(v), \|g\|_p = 1} \left| \int_0^\infty R_p^*(\chi_A)(x)v(x)^{-1/p}g(x)\,dx \right|.$$ 

Take $g \in M_p(v)$. It is clear that $\chi_A \in L_p^*$ and then $R_p^*(\chi_A) \in L_{p'}$ (Theorem 4.2). Also $gv^{-1/p} \in L_p$. Therefore

$$\left| \int_0^\infty R_p^*(\chi_A)(x)v(x)^{-1/p}g(x)\,dx \right| \leq \left( \int_A |R_p^*(v^{-1/p}g)(x)|^r u(x)\,dx \right)^{1/r} \left( \int_A u(x)^{-r'/r}\,dx \right)^{1/r'} \leq C \left( \int_A u(z)\,dz \right)^{1/r-1/p} \|v^{-1/p}g\|_{L_p(v)} \left( \int_A u(x)^{-r'/r}\,dx \right)^{1/r'} \leq C \left( \int_A u(z)\,dz \right)^{1/r-1/p} \|g\|_{L_p} \left( \int_A u(x)^{-r'/r}\,dx \right)^{1/r'}.$$ 

Hence, we obtain

$$\|R_p^*(\chi_A)\|_{L_{p'}(v^{1-1/p})} \leq \left( \int_A u(z)\,dz \right)^{1/r-1/p} \left( \int_A u(x)^{-r'/r}\,dx \right)^{1/r'}.$$ 

(6.9)

Next step is proving that $v \in D_p$. Let $a > 0$ and take measurable set $A \subset (k, \infty)$, where $k$ is a suitable positive constant given in (3.18) and (3.19), such that, as above, $|A| < \infty$, $0 < \int_A u\,dy < \infty$ and $\int_A u^{1-\tau}(x)\,dx < \infty$. By proceeding as in the proof of (v) $\Rightarrow$ (i) we get

$$|R_p^*(\chi_A)(x)| \geq Cx^{\mu+1/2} \int_A y^{-\mu-3/2}\,dy, \quad x \in (0, a).$$

Then (6.9) leads to

$$\int_0^a x^{(\mu+1/2)p'}v(x)^{1-p'}\,dx \leq C \int_0^a |R_p^*(\chi_A)(x)|^{p'} v(x)^{1-p'}\,dx \leq C\|R_p^*(\chi_A)\|_{L_{p'}(v^{1-1/p})}^{p'} < \infty.$$ 

Also by using (6.9) and by proceeding as in the proof of (v) $\Rightarrow$ (i) we obtain that

$$\int_a^\infty x^{-(\mu+5/2)p'} v(x)^{1-p'}\,dx < \infty.$$ 

Thus we have proved that $v \in D_p$ when $1 < p \leq 2$.

Assume now that $p > 2$. In this case, from (6.6) we infer that

$$\int_{\{x \in (0, \infty) : R_p f(x) > \lambda\}} u(x)\,dx \leq \left( \|f\|_{L_p(v)/\lambda} \right)^2,$$

(6.10) $f \in L_p(v)$ and $\lambda > 0$.

Define the functions

$$w(x) = \begin{cases} v^{1/(p-1)} x^{(\mu+1/2)(2-p')}, & 0 < x < 1 \\ v^{1/(p-1)} x^{-(\mu+5/2)(2-p')}, & x > 1. \end{cases}$$
and
\[ \alpha(x) = \begin{cases} x^{\mu+1/2}, & 0 < x < 1 \\ x^{-(\mu+3/2)}, & x > 1. \end{cases} \]

Then \( L_2(w) \subset L_p(v) + L_1(\alpha) \). Indeed, let \( f \in L_2(w) \). We write \( f = g + h \) where
\[ g(x) = \begin{cases} f(x), & |f(x)| \leq (w(x)/v(x))^{1/(p-2)} \\ 0, & \text{otherwise}, \end{cases} \]

and \( h = f - g \). Since \( p > 2 \) we can write
\[
\int_0^\infty |g(x)|^p v(x) dx \leq \int_{\{x: |f(x)| \leq (w(x)/v(x))^{1/(p-2)}\}} |f(x)|^2 |f(x)|^{p-2} v(x) dx \leq \int_0^\infty |f(x)|^2 w(x) dx.
\]

Hence \( g \in L_p(v) \).

On the other hand, we get
\[
\int_0^\infty |h(x)| \alpha(x) dx = \int_{\{x: |f(x)| > (w(x)/v(x))^{1/(p-2)}\}} |f(x)| \alpha(x) dx
\]
\[
\leq \int_{\{x: |f(x)| > (w(x)/v(x))^{1/(p-2)}\}} |f(x)|^2 w(x)^{-1/(p-2)} (w(x)/v(x))^{1/(p-2)} \alpha(x) dx
\]
\[
\leq \int_0^\infty |f(x)|^2 w(x) dx,
\]

that proves that \( h \in L_1(\alpha) \).

Our next aim is to show that, for every \( f \in L_2(w) \), \( R_{\mu,\varphi}(f)(x) < \infty \), for almost every \( x \in (0, \infty) \). According to our hypothesis, it is sufficient to see that \( R_{\mu,\varphi}(h)(x) < \infty \), a.e., for each \( h \in L_1(\alpha) \). Let \( h \in L_1(\alpha) \). By (3.15), (3.16) and (3.17) we can write, for every \( \varepsilon > 0 \),
\[
|R_{\mu,\varepsilon}h(x)| \leq C \left( x^{-\mu-3/2} \int_0^x y^{\mu+1/2} |h(y)| dy + \int_{\{x/2, 2x\} \cap \{|x-y| > \varepsilon\}} \frac{h(y)}{x-y} dy \right)
\]
\[
+ \int_{x/2}^x \frac{1}{y} \left( 1 + \log_+ \frac{\sqrt{xy}}{|x-y|} \right) |h(y)| dy + x^{\mu+3/2} \int_x^\infty y^{-\mu-5/2} |h(y)| dy \right)
\]
\[
\leq C \left( \sum_{j=1}^3 T_j(h)(x) + |H_{0,\varepsilon}(h)(x)| \right),
\]

where
\[
T_1(h)(x) = x^{-\mu-3/2} \int_0^x y^{\mu+1/2} |h(y)| dy
\]
\[
T_2(h)(x) = x^{\mu+3/2} \int_x^\infty y^{-\mu-5/2} |h(y)| dy
\]
\[
T_3(h)(x) = \int_{x/2}^x \frac{1}{y} \left( 1 + \log_+ \frac{\sqrt{xy}}{|x-y|} \right) |h(y)| dy
\]
\[
H_{0,\varepsilon}(f)(x) = \int_{\{x/2, 2x\} \cap \{|x-y| > \varepsilon\}} \frac{h(y)}{x-y} dy.
\]
It is not hard to see that $T_j(h)(x) < \infty$, $j = 1, 2$, for every $x \in (0, \infty)$. On the other hand, if $x \in (0, \infty)$ there exists $C(x) > 0$ such that

$$T_j(h)(x) \leq C(x) \int_{1/2}^{2} |h_1(ux)| \left(1 + \log_+ \frac{\sqrt{u}}{|1 - u|}\right) \frac{du}{u},$$

where $h_1 = h\alpha$. Moreover,

$$\int_{0}^{\infty} \int_{1/2}^{2} |h_1(ux)| \left(1 + \log_+ \frac{\sqrt{u}}{|1 - u|}\right) \frac{du}{u} dx = \int_{1/2}^{2} \int_{0}^{\infty} |h_1(ux)| dx \left(1 + \log_+ \frac{\sqrt{u}}{|1 - u|}\right) \frac{du}{u} = \int_{1/2}^{2} \left(1 + \log_+ \frac{\sqrt{u}}{|1 - u|}\right) \frac{du}{u^2} \|h_1\|_{L^1} \leq \infty.$$

Hence $T_j(h)(x) < \infty$, for almost all $x \in (0, \infty)$.

To prove that $\sup_{\varepsilon > 0} |H_{0,\varepsilon}(h)(x)| < \infty$, for almost every $x \in (0, \infty)$, we will show that $\lim_{\varepsilon \to 0} H_{0,\varepsilon}h(x)$ exists for almost all $x \in (0, \infty)$. In order to establish the last assertion it is sufficient to see that the operator $L$ defined by

$$h \to L(h) = \lim_{\varepsilon \to 0} H_{0,\varepsilon}h(x)$$

is bounded from $L_1(\alpha)$ into $L_{1,\infty}(\alpha)$. According to [3, Lemma 1] we have the boundedness of the operator $L$ provided that $\alpha$ is in $A_{1,\text{loc}}$, that is, if there exists a constant $C > 0$ for which

$$\int_{a}^{b} \alpha(x) dx \sup_{x \in (a,b)} \frac{1}{\alpha(x)} \leq C(b - a),$$

for every $0 < a < b < 2a$. This property for $\alpha$ can be established by using the mean value theorem in a suitable way. Hence we conclude that $\sup_{\varepsilon > 0} |H_{0,\varepsilon}(h)(x)| < \infty$, for almost every $x \in (0, \infty)$.

Thus it is proved that, for every $f \in L_2(w)$, $R_{\mu,*}(f)(x) < \infty$, for almost every $x \in (0, \infty)$. Then, since we already have shown that $(iii) \Rightarrow (i)$ when $p = 2$, $w \in D_2$, that is, $v \in D_0$, and the proof of this theorem is now completed.

Duality arguments would allow us to deduce from Theorem 6.2 necessary and sufficient conditions on a measurable function $u$ in order to guarantee that there exists a positive measurable function $v$ such that (6.1) holds true.

7. APPENDIX

It is clear that $\Delta_{\mu} = \Delta_{-\mu}$. Then, we also have for $\Delta_{\mu}$ the decomposition $\Delta_{\mu} = B^*_\mu B_{\mu}$, where $B_{\mu} = x^{-\mu + 1/2} D x^{\mu - 1/2}$. According to the general idea described in the introduction, we can define other Riesz transform, that will be denoted by $T_\mu$, associated with $\Delta_{\mu}$ as

$$T_\mu = B_{\mu} \Delta_{\mu}^{-1/2}.$$

In this case, by proceeding as for $R_{\mu}$, we get that for every $f \in C^\infty_c$

$$T_\mu(f)(x) = \lim_{\varepsilon \to 0} \int_{|x - y| > \varepsilon} T_\mu(x, y) f(y) dy, \quad x \in (0, \infty),$$
being, for every \( x, y \in (0, \infty), x \neq y \),

\[
T_\mu(x, y) = x^{-\mu+1/2} \frac{d}{dx} \left( x^{\mu-1/2} \int_0^\infty P_\mu(t, x, y) \ dt \right) = x^{-\mu+1/2} \frac{d}{dx} \left( x^{\mu-1/2} \int_0^\infty \int_0^\infty e^{-zt} (xz)^{1/2} J_\mu(xz)(yz)^{1/2} J_\mu(yz) \ dz \ dt \right) = \int_0^\infty \int_0^\infty e^{-zt} (xz)^{1/2} J_{\mu-1}(xz)(yz)^{1/2} J_{\mu}(yz) \ z \ dz \ dt = -R_{\mu-1}(y, x).
\]

Hence, \( T_\mu = -R_{\mu-1} \). Then, the results obtained in this paper can be transferred to the Riesz transform \( T_\mu \), provided that \( \mu > 1/2 \).

On the other hand, the Riesz transform defined in [4] can be written as

\[
\mathcal{R}_\mu = \frac{d}{dx} \Delta_{\mu}^{-1/2},
\]

and a straightforward manipulation allows us to get

\[
\mathcal{R}_\mu = \frac{1}{2\mu} \left( (\mu - 1/2)R_\mu + (\mu + 1/2)T_\mu \right).
\]

By using this relation and our results in Sections 3-6, we can obtain new results for the Riesz transforms \( \mathcal{R}_\mu \). In particular, from Theorem 4.2, it follows that, when \( \mu > 1/2 \), \( \mathcal{R}_\mu \) is a Calderón-Zygmund singular integral operator. Then, for every \( w \in A_p(0, \infty) \), \( \mathcal{R}_\mu \) defines a bounded operator from \( L_p(w) \) into itself, \( 1 < p < \infty \), and from \( L_1(w) \) into \( L_{1,\infty}(w) \). Thus we improve [4, Theorem 3.4].

References


