RIESZ TRANSFORMS RELATED TO BESSEL OPERATORS

JORGE J. BETANCOR, DARIUSZ BURACZEWSKI, JUAN CARLOS FARIÑA, TERESA MARTÍNEZ, AND JOSÉ L. TORREA

Abstract. In this work we study the Riesz transforms R_μ related to the Bessel operators $\Delta_\mu = x^{-\mu-1/2}Dx^{2\mu+1}Dx^{-\mu-1/2}$. We develop for R_μ a theory that runs parallel to the one for the Euclidean Hilbert transform. It is proved that R_μ is actually a Calderón-Zygmund singular integral operator. Moreover, our Riesz transform can be written as a principal value and we study the speed of convergence of this limit in terms of the L_p boundedness of oscillation and variation operators. Also, R_μ is seen to be the boundary value of the appropriate harmonic extension for this context. Finally we analyze weighted inequalities involving R_μ.

1. Introduction.

In this paper we introduce and study the Riesz transform associated with the Bessel operator

$$\Delta_\mu = -(\frac{d^2}{dx^2} - \frac{\mu^2 - 1/4}{x^2}). \tag{1.1}$$

The operator Δ_μ is a positive self-adjoint operator in $L_2((0,\infty), dx)$ and it can be written in divergence form as

$$\Delta_\mu = x^{-\mu-1/2}Dx^{2\mu+1}Dx^{-\mu-1/2} = A_\mu^* A_\mu,$$

being $A_\mu = x^{\mu+1/2}Dx^{-\mu-1/2}$, and $A_\mu^* = x^{-\mu-1/2}Dx^{\mu+1/2}$ the adjoint operator of A_μ. The eigenfunctions of Δ_μ are $\{\varphi_y\}_{y>0}$, where, for every $y > 0$ (see [13]),

$$\varphi_y(x) = (yx)^{1/2}J_\mu(yx) \qquad \text{and} \qquad \Delta_\mu\varphi_y(x) = y^2\varphi_y(x), \tag{1.2}$$

for $J_\mu(z)$ being the Bessel function of the first kind of order μ. The Poisson kernel P_μ associated to Δ_μ is

$$P_\mu(t,x,y) = \int_0^\infty e^{-zt}\varphi_x(z)\varphi_y(z)\,dz, \;\; t,x,y \in (0,\infty). \tag{1.3}$$

1991 *Mathematics Subject Classification.* Primary 42C05; Secondary 42C15.

Key words and phrases. Bessel operators, Riesz transform, Calderón-Zygmund operators.

First and third authors were partially supported by Consejería de Educación, Gobierno de Canarias PI2003/068. Second author was partially supported by KBN grant 5P03A02821. Fourth and fifth authors were partially supported by Ministerio de Ciencia y Tecnologıa BFM grant 2002-04013-C02-02. Second, fourth and fifth authors were also partially supported by European Commission HPRN-CT-2001-00273-HARP.

The corresponding Poisson integral was considered in [14]. By using the well known formula $s^{-a} = \frac{1}{\Gamma(a)} \int_0^\infty e^{-ts} t^a \frac{dt}{t}$, with $s, a > 0$, we define

$$\Delta_\mu^{-1/2} f(x) = \int_0^\infty e^{-t\sqrt{\Delta_\mu}} f(x)\, dt = \int_0^\infty \int_0^\infty P_\mu(t,x,y) f(y)\, dy\, dt.$$

From here, following Stein's ideas contained in the last chapter of [15], we define the Riesz transform associated to Δ_μ as

$$R_\mu(f) = A_\mu \Delta_\mu^{-1/2} f. \tag{1.4}$$

The first question we treat in this paper is giving a sense to $\Delta_\mu^{-1/2}$. This is done in Lemmas 2.1 and 2.2. Next, we prove the main properties of the Riesz transform R_μ. Namely, we see that R_μ is a principal value operator when acting on smooth functions (Theorem 3.3) and also that it is the boundary value of the "conjugate harmonic extension" associated to Δ_μ, as stated in Theorem 3.4. In section 4 we prove that R_μ is a Calderón-Zygmund operator, and this allows us to obtain its boundedness in weighted L^p-spaces of R_μ (Theorem 4.2) and of the maximal $R_{\mu,*}$ of the truncated integrals (Theorem 4.3), for weights in Muckenhoupt class A_p. Hence, by the classical theory, we deduce the existence of the principal value for every function in the same weighted L^p (Corollary 4.4). A natural way to measure the speed of this convergence is studying the oscillation and ρ-variation operators associated to R_μ. This is done in Theorem 5.1. Finally, we study the optimal classes of weights v such that for every $f \in L^p(v)$, $R_\mu f$ can be defined as a principal value. We characterize this class of weights by an intrinsic condition and we also show that this condition is equivalent to some boundedness properties of R_μ and $R_{\mu,*}$ (see Theorem 6.2).

Our work has been inspired by the results of Muckenhoupt and Stein in [13], where conjugations associated to the Bessel operators $S_\mu = \frac{d^2}{dx^2} + \frac{2\mu+1}{x}\frac{d}{dx}$ are studied. For fixed $\mu > -1/2$, they considered the μ-harmonic extension of $f \in L_1(x^{2\mu+1}dx)$, given by

$$\mathcal{P}_\mu(f)(t,x) = \int_0^\infty \mathcal{P}_\mu(t,x,y) f(y) y^{2\mu+1}\, dy, \quad t, x \in (0,\infty),$$

and satisfying

$$\left(\frac{d^2}{dt^2} + S_{\mu,x}\right) \mathcal{P}_\mu(f)(t,x) = 0,$$

where $\mathcal{P}_\mu(t,x,y)$ is the natural Poisson kernel associated to S_μ. The conjugate μ-harmonic function of f is defined as

$$\mathcal{Q}_\mu(f)(t,x) = \int_0^\infty \mathcal{Q}_\mu(t,x,y) f(y) y^{2\mu+1}\, dy,$$

where $\mathcal{Q}_\mu$ is the μ-conjugate kernel. Then, the conjugate function (analogue of the Hilbert transform) is given by (see formula (16.8) in [13])

$$\mathcal{Q}_\mu(f)(x) = \lim_{t\to 0} \mathcal{Q}_\mu(f)(t,x).$$

$L_p(x^{2\mu+1})$-boundedness properties of this conjugation operator were established in [13]. From our results, $\mathcal{Q}_\mu$ is a principal value operator (see Remark 3.5).

Andersen and Kerman [2] investigated weighted inequalities for the operator $\mathcal{Q}_\mu$. They characterized the weights playing in this theory the role of the A_p-weights of Muckenhoupt in the classical theory of the Hilbert transform.

More recently, Betancor and Stempak [4] defined and studied the operator $\mathcal{R}_\mu f(x) = \partial_x \Delta_m^{-1/2} f(x)$ on weighted L^p-spaces for Muckenhoupt weights, following the ideas in [13]. The results of [4] can be obtained and improved by the ones established here (see Appendix).

Other results concerning to $\mathcal{Q}_\mu$-conjugation can be encountered in [1], [11] and [12].

Throughout this paper we represent by p' the conjugate exponent of p for $1 \le p < \infty$. We denote by $L_p(v)$ and $L_{p,\infty}(v)$ the Lebesgue spaces $L_p((0,\infty), v\,dx)$ and weak-$L_p((0,\infty), v\,dx)$. When $v \equiv 1$ we simply write L_p or $L_{p,\infty}$ instead of $L_p(dx,(0,\infty))$ or weak-$L_p((0,\infty), dx)$. $A_p(0,\infty)$ stands for the class of Muckenhoupt weights on $(0,\infty)$. The parameter μ is, except otherwise stated, greater than $-1/2$. The letter C denotes a suitable positive constant that may change from line to line.

2. Definition of the operator $\Delta_\mu^{-1/2}$.

The aim of this section is giving a good definition of the operator $\Delta_\mu^{-1/2}$ where Δ_μ is defined in (1.1). It is known (see [13, (16.4)]) that the Poisson kernel associated to Δ_μ, given by (1.3), has the following expression for $x, y, t \in (0,\infty)$

$$P_\mu(t,x,y) = \frac{(2\mu+1)t}{\pi}(xy)^{\mu+1/2}\int_0^\pi \frac{(\sin z)^{2\mu}}{(x^2+y^2+t^2-2xy\cos z)^{\mu+3/2}}\,dz. \tag{2.1}$$

Lemma 2.1. *For any $f \in \mathcal{C}_c^\infty$, we define*

$$\Delta_\mu^{-1/2} f = \int_0^\infty e^{-t\sqrt{\Delta_\mu}} f\,dt. \tag{2.2}$$

Then, the operator is well defined and for every $x \in (0,\infty)$ we have

$$\Delta_\mu^{-1/2} f(x) = \int_0^\infty\int_0^\infty P_\mu(t,x,y) f(y)\,dy\,dt = \int_0^\infty \left(\int_0^\infty P_\mu(t,x,y)\,dt\right) f(y)\,dy.$$

Proof. Let $f \in \mathcal{C}_c^\infty$. To change the order of integration in (2.2), we will use the following inequalities ([13, p. 86]):

$$P_\mu(t,x,y) \le C\frac{t}{|x-y|^2+t^2}, \tag{2.3}$$

and

$$P_\mu(t,x,y) \le C\frac{(xy)^{\mu+1/2}t}{(|x-y|^2+t^2)^{\mu+3/2}}. \tag{2.4}$$

With these estimates we get that the integrand in (2.2) is uniformly integrable, since for every $y \in (0,\infty)$

$$\begin{aligned}\int_0^\infty\int_0^\infty |P_\mu(t,x,y)f(y)|\,dy\,dt \;&\le\; C\Bigg(\int_0^\infty\int_0^1 \frac{t|f(y)|}{|x-y|^2+t^2}\,dt\,dy \\ &\quad + \int_0^\infty\int_1^\infty \frac{(xy)^{\mu+1/2}t|f(y)|}{(|x-y|^2+t^2)^{\mu+3/2}}\,dt\,dy\Bigg) \le C\|f\|_{L^1}.\end{aligned}$$

■

The rest of the section is devoted to justify our definition of $\Delta_\mu^{-1/2}$. The set of eigenfunctions of Δ_μ, $\{\varphi_y(x)\}_{y\in(0,\infty)}$ (see (1.2)), does not span a dense subset of L^2. Thus, we can not use the usual spectral techniques to define the classical operators associated to Δ_μ. Nevertheless, the natural property that $\Delta_\mu^{-1/2}$ should verify is that $\Delta_\mu^{-1/2}\varphi_y(x) = \frac{1}{y}\varphi_y(x)$, at least in a weak sense. In order to clarify this property, let us introduce some notation. From now on, we consider the domain of Δ_μ to be the set of infinitely differentiable functions with compact support in $(0,\infty)$, and call it $\mathcal{C}_c^\infty$. The (non modified) Hankel transform is given, for a function f in L_1 as

$$H_\mu(f)(y) = \int_0^\infty (xy)^{1/2} J_\mu(xy) f(x)\, dx,$$

which plays the role of the coefficient of f corresponding to the eigenvector φ_y. The integral transform H_μ has been defined in spaces of distributions of slow growth by Zemanian ([17] and [18]). He introduced the suitable "Schwartz class" $\mathcal{H}_\mu$ in this context and proved that the Hankel transform H_μ is an automorphism in the space $\mathcal{H}_\mu$ ([18, Theorem 5.4-1]) and that H_μ is extended to the dual space $\mathcal{H}'_\mu$ of $\mathcal{H}_\mu$ by transposition. That is, if $T \in \mathcal{H}'_\mu$ the Hankel transform $H'_\mu(T)$ is the element of $\mathcal{H}'_\mu$ defined through

$$\langle H'_\mu(T), f\rangle = \langle T, H_\mu(f)\rangle, \qquad f \in \mathcal{H}_\mu.$$

It is known ([18, Lemma 5.4-1]) that for any $f \in \mathcal{H}_\mu$,

$$H_\mu(\Delta_\mu f)(y) = y^2 H_\mu(f)(y).$$

According to the former discussion, the following lemma justifies our definition of $\Delta_\mu^{-1/2}$, given in (2.2).

Lemma 2.2. *The operator defined as in (2.2) satisfies, for $f \in \mathcal{H}_\mu$, that*

$$H'_\mu(\Delta_\mu^{-1/2} f)(y) = \frac{1}{y} H'_\mu(f)(y). \tag{2.5}$$

The key idea of the proof is to define, for every $0 < s < 1$,

$$G_s(f)(x) = \int_s^{1/s}\int_0^\infty P_\mu(t,x,y) f(y)\, dy\, dt, \quad x \in (0,\infty).$$

These operators verify $\lim_{s\to 0} G_s f = \Delta_\mu^{-1/2} f$, in the weak $*$ topology of $\mathcal{H}_\mu$, and this gives the result. We leave the details for the interested reader.

3. Riesz transforms R_μ associated with Δ_μ.

As it was mentioned in the introduction the Bessel operator Δ_μ can be written as $\Delta_\mu = -A_\mu^* A_\mu$, being $A_\mu = x^{\mu+1/2} D x^{-\mu-1/2}$ and denoting by A_μ^* the adjoint operator of A_μ in L_2. Then, this suggests to define the Riesz transform R_μ associated with Δ_μ by

$$R_\mu = A_\mu \Delta_\mu^{-1/2},$$

where $\Delta_\mu^{-1/2}$ is as in Lemma 2.1. Our aim is to see that the Riesz transform R_μ, acting on C_c^∞, is actually a principal value, that is, for every $f \in C_c^\infty$ one has

$$R_\mu(f)(x) = \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} R_\mu(x,y) f(y)\, dy, \quad x \in (0,\infty), \tag{3.1}$$

where, for every $x, y \in (0,\infty)$, $x \neq y$,

$$\begin{aligned} R_\mu(x,y) &= x^{\mu+1/2} \int_0^\infty \frac{d}{dx}\Big(x^{-\mu-1/2} P_\mu(t,x,y)\Big)\, dt \\ &= \frac{2\mu+1}{\pi}(xy)^{\mu+1/2} \int_0^\pi \frac{(y\cos z - x)(\sin z)^{2\mu}}{(x^2+y^2-2xy\cos z)^{\mu+3/2}}\, dz. \end{aligned}$$

Due to the term $1-\cos z$ in the denominator of $R_\mu(x,y)$, the part of the kernel with $z \in (\pi/2, \pi)$ behaves in a better way than the part with $z \in (0, \pi/2)$. According to this, define for every $x, y \in (0,\infty)$, $x \neq y$,

$$R_{\mu,1}(x,y) = \frac{2\mu+1}{\pi}(xy)^{\mu+1/2} \int_0^{\frac{\pi}{2}} \frac{(y\cos z - x)(\sin z)^{2\mu}}{(x^2+y^2-2xy\cos z)^{\mu+3/2}}\, dz, \tag{3.2}$$

$$R_{\mu,2}(x,y) = R_\mu(x,y) - R_{\mu,1}(x,y). \tag{3.3}$$

Lemma 3.1. *Let $f \in C_c^\infty$. Then, for every $x \in (0,\infty)$,*

$$\begin{aligned} &\int_0^\infty R_{\mu,2}(x,y) f(y)\, dy \\ &= \frac{2\mu+1}{\pi} x^{\mu+1/2} \frac{d}{dx}\Big(\int_0^\infty \int_0^\infty \int_{\pi/2}^\pi \frac{f(y) t y^{\mu+1/2} (\sin z)^{2\mu}}{(x^2+y^2+t^2-2xy\cos z)^{\mu+3/2}}\, dz\, dt\, dy\Big), \end{aligned}$$

the first integral being absolutely convergent.

Proof. Let $x \in (0,\infty)$. We can write

$$\int_{\pi/2}^\pi \frac{|2x-2y\cos z|(\sin z)^{2\mu}}{(x^2+y^2+t^2-2xy\cos z)^{\mu+5/2}}\, dz \le C \frac{|x-y|+y}{(t^2+|x-y|^2+xy)^{\mu+5/2}}, \quad t, y > 0.$$

Then, to justify the differentiation under the integral sign we have just to observe that

$$\begin{aligned} &\int_0^\infty |f(y)| y^{\mu+1/2} \int_0^\infty t \int_{\pi/2}^\pi \left| \frac{d}{dx}\left(\frac{(\sin z)^{2\mu}}{(x^2+y^2+t^2-2xy\cos z)^{\mu+3/2}}\right)\right| dz\, dt\, dy \\ &\le C \int_0^\infty |f(y)| y^{\mu+1/2} (|x-y|+y) \int_0^\infty \frac{t}{(t^2+|x-y|^2+xy)^{\mu+5/2}}\, dt\, dy \\ &\le C \int_0^\infty |f(y)| y^{\mu+1/2} (|x-y|+y)(xy+|x-y|^2)^{-1/2}(xy+|x-y|^2)^{-\mu-1}\, dy \\ &\le C x^{-\mu-1} \int_0^\infty (y^{-1/2}+x^{-1/2})|f(y)|\, dy < \infty. \end{aligned}$$

■

Lemma 3.2. *Let $f \in C_c^\infty$. Then, for all $x \in (0,\infty)$,*

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon} R_{\mu,1}(x,y)f(y)\,dy\\ &= \frac{2\mu+1}{\pi}x^{\mu+1/2}\frac{d}{dx}\Big(\int_0^\infty\int_0^\infty\int_0^{\pi/2}\frac{f(y)ty^{\mu+1/2}(\sin z)^{2\mu}}{(x^2+y^2+t^2-2xy\cos z)^{\mu+3/2}}\,dz\,dt\,dy\Big).\end{aligned}$$

Proof. Let $x \in (0,\infty)$. Write

$$\int_0^\infty f(y)\int_0^\infty P_{\mu,1}(t,x,y)\,dt\;dy = \int_0^\infty f(y)\Big(\int_0^1+\int_1^\infty\Big)P_{\mu,1}(t,x,y)\,dt\,dy \tag{3.4}$$

where for $y,t \in (0,\infty)$

$$P_{\mu,1}(t,x,y) = \frac{(2\mu+1)t}{\pi}(xy)^{\mu+1/2}\int_0^{\pi/2}\frac{(\sin z)^{2\mu}}{(x^2+y^2+t^2-2xy\cos z)^{\mu+3/2}}\;dz.$$

Note that

$$\begin{aligned}&\int_0^\infty |f(y)|y^{\mu+1/2}\int_1^\infty\int_0^{\pi/2}\left|\frac{d}{dx}\left(\frac{(\sin z)^{2\mu}}{(x^2+y^2+t^2-2xy\cos z)^{\mu+3/2}}\right)\right|dz\\ &\le C\int_0^\infty |f(y)|y^{\mu+1/2}(x+y)\int_1^\infty\frac{t}{(|x-y|^2+t^2)^{\mu+5/2}}\,dt\,dy\\ &\le C\int_0^\infty |f(y)|y^{\mu+1/2}(x+y)\,dy<\infty.\end{aligned}\tag{3.5}$$

Hence we have proved that

$$\begin{aligned}&\frac{d}{dx}\left(x^{-\mu-1/2}\int_0^\infty f(y)\int_1^\infty P_{\mu,1}(t,x,y)\,dt\,dy\right)\\ &= \frac{2\mu+1}{\pi}\int_0^\infty\int_1^\infty\int_0^{\pi/2}\frac{d}{dx}\left(\frac{f(y)y^{\mu+1/2}t(\sin z)^{2\mu}}{(x^2+y^2+t^2-2xy\cos z)^{\mu+3/2}}\right)dz\,dt\,dy.\end{aligned}\tag{3.6}$$

Now we decompose the first integral in the right-hand side of (3.4) as follows

$$\int_0^{\pi/2}\frac{(\sin z)^{2\mu}}{(x^2+y^2+t^2-2xy\cos z)^{\mu+3/2}}\,dz = \sum_{j=1}^4 I_j(x,y),$$

where

$$\begin{aligned}
I_1(x,y) &= \int_0^{\pi/2} (\sin z)^{2\mu}\Bigg(\frac{1}{(|x-y|^2+t^2+2xy(1-\cos z))^{\mu+3/2}} \\
&\qquad - \frac{1}{(|x-y|^2+t^2+xyz^2)^{\mu+3/2}}\Bigg)\,dz \\
I_2(x,y) &= \int_0^{\pi/2} \frac{(\sin z)^{2\mu}-z^{2\mu}}{(|x-y|^2+t^2+xyz^2)^{\mu+3/2}}\,dz \\
I_3(x,y) &= \int_0^{\pi/2} \frac{z^{2\mu}}{(|x-y|^2+t^2+xyz^2)^{\mu+3/2}}\,dz - C_\mu (xy)^{-\mu-1/2}\frac{1}{|x-y|^2+t^2} \\
I_4(x,y) &= C_\mu (xy)^{-\mu-1/2}\frac{1}{|x-y|^2+t^2},
\end{aligned}$$

where $C_\mu = \int_0^\infty \frac{u^{2\mu}}{(1+u^2)^{\mu+3/2}}\,du$. If $t, y \in (0,\infty)$, by the change of variables $u^2 = \frac{B}{A}z^2$, with $A = |x-y|^2+t^2$ and $B = xy$, it can be easily seen that

$$\int_0^{\frac{\pi}{2}} \frac{z^\alpha}{(|x-y|^2+t^2+xyz^2)^\beta}\,dz = A^{-(\beta-(\alpha+1)/2)}B^{-(\alpha+1)/2}\int_0^{\frac{\pi}{2}\sqrt{\frac{B}{A}}} \frac{u^\alpha}{(1+u^2)^\beta}\,du. \tag{3.7}$$

By applying the mean value theorem we get

$$\begin{aligned}
&\left|\frac{2(x-y)+2y(1-\cos z)}{(|x-y|^2+t^2+2xy(1-\cos z))^{\mu+5/2}} - \frac{2(x-y)+yz^2}{(|x-y|^2+t^2+xyz^2)^{\mu+5/2}}\right| \\
&\quad\le\ Cz^4\left(\frac{y}{(|x-y|^2+t^2+xyz^2))^{\mu+5/2}} + \frac{xy(|x-y|+y)}{(|x-y|^2+t^2+xyz^2)^{\mu+7/2}}\right), \quad z\in(0,\pi/2).
\end{aligned}$$

Moreover, we have for $t \in (0,1)$

$$\begin{aligned}
\int_0^{\pi/2} \frac{y(\sin z)^{2\mu}z^4}{(|x-y|^2+t^2+xyz^2)^{\mu+5/2}}\,dz &\le Cy^{-\mu-3/2}x^{-\mu-5/2}\int_0^{\sqrt{\frac{B}{A}}\frac{\pi}{2}} \frac{u^{2\mu+4}}{(1+u^2)^{\mu+5/2}}\,du \\
&\le C\frac{x^{1/2}y^{-\mu-1}}{(|x-y|^2+t^2)^{1/2}}\int_0^\infty \frac{u^{2\mu+3}}{(1+u^2)^{\mu+5/2}}\,du.
\end{aligned}$$

Hence,

$$y^{\mu+1/2}\int_0^1 t\int_0^{\pi/2} \frac{y(\sin z)^{2\mu}z^4}{(|x-y|^2+t^2+xyz^2)^{\mu+5/2}}\,dz\,dt \le Cx^{-\mu-2}y^{-1/2}(|x-y|^2+1)^{1/2}.$$

Thus, we conclude that

$$\int_0^\infty |f(y)|y^{\mu+1/2}\int_0^1 t\int_0^{\pi/2} (\sin z)^{2\mu}z^4\frac{y}{(|x-y|^2+t^2+xyz^2)^{\mu+5/2}}\,dz\,dt\,dy < \infty. \tag{3.8}$$

On the other hand, we have

$$\int_0^{\pi/2} \frac{xy|x-y|(\sin z)^{2\mu}z^4}{(|x-y|^2+t^2+xyz^2)^{\mu+7/2}}\,dz \le C\frac{|x-y|(xy)^{-\mu-3/2}}{|x-y|^2+t^2}\int_0^{\sqrt{\frac{B}{A}}\frac{\pi}{2}} \frac{u^{2\mu+4}}{(1+u^2)^{\mu+7/2}}\,du.$$

By proceeding as above we obtain

$$y^{\mu+1/2}\int_0^1 t\int_0^{\pi/2}\frac{y|x-y|(\sin z)^{2\mu}z^4}{(|x-y|^2+t^2+xyz^2)^{\mu+7/2}}\,dz\,dt\leq Cx^{-\mu-1}y^{-1/2}.$$

Then, it follows that

$$\int_0^\infty |f(y)|y^{\mu+1/2}\int_0^1 t\int_0^{\pi/2}\frac{y|x-y|(\sin z)^{2\mu}z^4}{(|x-y|^2+t^2+xyz^2)^{\mu+7/2}}\,dz\,dt\,dy<\infty. \tag{3.9}$$

Note also that

$$\int_0^{\pi/2}\frac{xy^2(\sin z)^{2\mu}z^4}{(|x-y|^2+t^2+xyz^2)^{\mu+7/2}}\,dz\leq C(xy)^{-\mu-5/2}\frac{xy^2}{|x-y|^2+t^2}\int_0^{\sqrt{\frac{B}{A}}\frac{\pi}{2}}\frac{u^{2\mu+4}}{(1+u^2)^{\mu+7/2}}\,du.$$

By similar arguments, we can see that

$$y^{\mu+1/2}\int_0^1 t\int_0^{\pi/2}\frac{xy^2(\sin z)^{2\mu}z^4}{(|x-y|^2+t^2+xyz^2)^{\mu+7/2}}\,dz\,dt\leq Cx^{-\mu-3/2}(xy)^{-1/2}(|x-y|^2+1)^{1/2},$$

and from here,

$$\int_0^\infty |f(y)|y^{\mu+1/2}\int_0^1 t\int_0^{\pi/2}\frac{y^2(\sin z)^{2\mu}z^4}{(|x-y|^2+t^2+xyz^2)^{\mu+7/2}}\,dz\,dt\,dy<\infty. \tag{3.10}$$

Then, (3.8), (3.9) and (3.10) implies that

$$\begin{aligned}\int_0^\infty |f(y)|y^{\mu+1/2}\int_0^1 t\int_0^{\pi/2}(\sin z)^{2\mu}\Bigg|\frac{d}{dx}\Bigg(&\frac{1}{(|x-y|^2+t^2+2xy(1-\cos z))^{\mu+3/2}}\\ &-\frac{1}{(|x-y|^2+t^2+xyz^2)^{\mu+3/2}}\Bigg)\Bigg|\,dz\,dt\,dy<\infty.\end{aligned} \tag{3.11}$$

By similar calculations, we can justify that

$$\int_0^\infty |f(y)|y^{\mu+1/2}\int_0^1 t\int_0^{\pi/2}\left|\frac{d}{dx}\frac{(\sin z)^{2\mu}-z^{2\mu}}{(|x-y|^2+t^2+xyz^2)^{\mu+3/2}}\right|dz\,dt\,dy<\infty, \tag{3.12}$$

and

$$\begin{aligned}\int_0^\infty |f(y)|y^{\mu+1/2}\int_0^1 t\Bigg|\frac{d}{dx}\Bigg(x^{-\mu-1/2}(xy)^{\mu+1/2}\Bigg(&\int_0^{\pi/2}\frac{z^{2\mu}}{(|x-y|^2+t^2+xyz^2)^{\mu+3/2}}\,dz\\ &-(xy)^{-\mu-1/2}\frac{1}{|x-y|^2+t^2}\int_0^\infty\frac{u^{2\mu}}{(1+u^2)^{\mu+3/2}}\,du\Bigg)\Bigg)\Bigg|\,dt\,dy<\infty.\end{aligned} \tag{3.13}$$

The inequalities (3.6), (3.11), (3.12), (3.13) and well known properties of the Hilbert transform lead to the desired result. ∎

The following important result is an easy consequence of Lemmas 3.1 and 3.2.

Theorem 3.3. *Let $f\in C_c^\infty$ Then, for all $x\in(0,\infty)$,*

$$R_\mu(f)(x)=\lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon}R_\mu(x,y)f(y)\,dy.$$

■

Next, we establish other form for the Riesz transform R_μ acting on C_c^∞ that also can be seen as a version in our setting of the corresponding result for the Hilbert transform.

Theorem 3.4. *Let $f \in C_c^\infty$ Then, for all $x \in (0,\infty)$,*

$$R_\mu(f)(x) = \lim_{t\to 0} \int_0^\infty R_\mu(t,x,y) f(y)\, dy,$$

where, for $t,x,y \in (0,\infty)$,

$$R_\mu(t,x,y) = \frac{2\mu+1}{\pi}(xy)^{\mu+1/2} \int_0^\pi \frac{(y\cos z - x)(\sin z)^{2\mu}}{(x^2+y^2+t^2-2xy\cos z)^{\mu+3/2}}\, dz.$$

Proof. To see this property we can argue as in the proof of Lemmas 3.1 and 3.2. Consider, for every $t,x,y \in (0,\infty)$,

$$\begin{aligned} R_{\mu,1}(t,x,y) &= \frac{2\mu+1}{\pi}(xy)^{\mu+1/2} \int_0^{\pi/2} \frac{(y\cos z - x)(\sin z)^{2\mu}}{(x^2+y^2+t^2-2xy\cos z)^{\mu+3/2}}\, dz, \\ R_{\mu,2}(t,x,y) &= R_\mu(t,x,y) - R_{\mu,1}(t,x,y). \end{aligned}$$

A similar reasoning to the one used in the proof of Lemma 3.1 allows us, by using the dominated convergence theorem, to obtain for each $x \in (0,\infty)$

$$\lim_{t\to 0} \int_0^\infty R_{\mu,2}(t,x,y) f(y)\, dy = \int_0^\infty R_{\mu,2}(x,y) f(y)\, dy.$$

Also, by proceeding as in the proof of Lemma 3.2, we can see that for some constant C_μ and a function $g \geq 0$ defined on $(0,\infty)\times(0,\infty)$ such that, for every $x \in (0,\infty)$, $g(x,.) \in L_1$, we have that

$$\left| R_{\mu,1}(t,x,y) - C_\mu \frac{x-y}{|x-y|^2+t^2} \right| \leq g(x,y), \quad x,y \in (0,\infty).$$

Then, by invoking again the dominated convergence theorem and a well known property of the Hilbert transform, we conclude that, for every $x \in (0,\infty)$ the limit

$$\lim_{t\to 0} \int_0^\infty R_\mu(t,x,y) f(y)\, dy$$

exists. Moreover, a careful study of the proof of Lemma 3.2 allows us to conclude that, for every $x \in (0,\infty)$,

$$\lim_{t\to 0} \int_0^\infty R_{\mu,1}(t,x,y) f(y)\, dy = \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} R_{\mu,1}(x,y) f(y)\, dy,$$

and the proof finishes. ■

Note that finer estimates in the proof of the above results lead to establish that Theorems 3.3 and 3.4 also hold for every $f \in \mathcal{H}_\mu$.

Remark 3.5. According to Theorem 3.4, if $\mathcal{Q}_\mu$ denotes, as in the introduction, the conjugated defined by Muckenhoupt and Stein in [13], we can write, for every $f \in C_c^\infty$, $g(y) = y^{\mu+1/2} f(y)$ and

$$\mathcal{Q}_\mu(f)(x) = x^{-\mu-1/2} R_\mu(g)(x). \tag{3.14}$$

Hence, from Theorem 3.3, we infer that the conjugated defined in [13] is also a principal value over C_c^∞, that is, for every $f \in C_c^\infty$,

$$\mathcal{Q}_\mu(f)(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \mathcal{Q}_\mu(0,x,y) f(y)\, dy, \quad x \in (0,\infty).$$

It also can be seen that $R_\mu(x,y) = (xy)^{\mu+1/2}\mathcal{Q}(0,x,y)$, $x,y \in (0,\infty)$. Then, from [2, p. 17 and p. 20] and [12, p. 151-152] (see as well [13, p. 87]), we can deduce the following estimates for the function $R_\mu(x,y)$ that will be very useful in the sequel. There exists $C, D > 0$ for which

$$|R_\mu(x,y)| \leq Cx^{-\mu-3/2}y^{\mu+1/2}, \quad 0 < y < x/2 \tag{3.15}$$

$$|R_\mu(x,y)| \leq Cx^{\mu+3/2}y^{-\mu-5/2}, \quad y > 2x \tag{3.16}$$

$$R_\mu(x,y) = D\frac{1}{x-y} + O\left(\frac{1}{x}\left(1 + \log_+ \frac{\sqrt{xy}}{|x-y|}\right)\right), \quad x/2 < y < 2x. \tag{3.17}$$

and, for certain $k, s > 0$,

$$R_\mu(x,y) \leq -\frac{1}{C}x^{-\mu-3/2}y^{\mu+1/2}, \quad 0 < y < x/k \tag{3.18}$$

$$R_\mu(x,y) \geq \frac{1}{C}x^{\mu+3/2}y^{-\mu-5/2}, \quad y > kx \tag{3.19}$$

$$\text{sign }(x-y)R_\mu(x,y) \geq \frac{1}{C}\frac{1}{|x-y|}, \quad x/s < y < sx. \tag{3.20}$$

4. Boundedness properties of the Riesz transform

In this section we study L_p-boundedness properties for the Riesz transform R_μ. We will prove that R_μ is actually a Calderón-Zygmund singular integral operator. Then, L_p-boundedness properties can be deduced from the general theory. Our procedure is different that the ones employed in [4] and [13] to establish the L_p boundedness of Hankel conjugates.

First note that, according to (3.14), since $\mathcal{Q}_\mu$ is a bounded operator from $L_2(x^{2\mu+1}\,dx)$ into itself, the Riesz transform R_μ is bounded from L_2 into itself. With the next proposition, it is proved that the Riesz transform R_μ is a Calderón-Zygmund operator.

Proposition 4.1. *There exists $C > 0$ such that, for every $x, y \in (0,\infty)$, $x \neq y$,*

(i) $|\partial_x R_\mu(x,y)| + |\partial_y R_\mu(x,y)| \leq \dfrac{C}{|x-y|^2}$.

(ii) $|R_\mu(x,y)| \leq \dfrac{C}{|x-y|}$.

Proof. To prove (i) we write

$$R_\mu(x,y) = R_{\mu,1}(x,y) + R_{\mu,2}(x,y), \quad x,y \in (0,\infty), \quad x \neq y,$$

where $R_{\mu,1}$ and $R_{\mu,2}$ are defined in (3.2) and (3.3). We now analyze the derivatives of $R_{\mu,1}$ and $R_{\mu,2}$. Assume that $x, y \in (0,\infty)$, $x \neq y$. In this case, we can differenciate under the

integral sign and

$$\begin{aligned}\frac{\partial}{\partial x}R_{\mu,1}(x,y) &= \frac{2\mu+1}{\pi}\Bigg(-(xy)^{\mu+1/2}\int_0^{\pi/2}\frac{(\sin z)^{2\mu}}{(x^2+y^2-2xy\cos z)^{\mu+3/2}}\,dz\\ &- (\mu+\frac{3}{2})(xy)^{\mu+1/2}\int_0^{\pi/2}\frac{(\sin z)^{2\mu}(2x-2y\cos z)^2}{(x^2+y^2-2xy\cos z)^{\mu+5/2}}\,dz\\ &+ (\mu+\frac{1}{2})x^{\mu-1/2}y^{\mu+1/2}\int_0^{\pi/2}\frac{(\sin z)^{2\mu}(y\cos z-x)}{(x^2+y^2-2xy\cos z)^{\mu+3/2}}\,dz\Bigg)\\ &= I_1^1+I_1^2+I_1^3,\end{aligned}$$

Analogously,

$$\frac{\partial}{\partial x}R_{\mu,2}(x,y)=I_2^1+I_2^2+I_2^3,$$

where I_2^1, I_2^2 and I_2^3 are defined as I_1^1, I_1^2 and I_1^3 by changing the interval of integration by $(\pi/2,\pi)$. Applying the change of variables $z=\pi-x$ in I_2^1 and then that $1+\cos x\geq cx^2$, formula (3.7) gives that

$$\begin{aligned}|I_1^1|+|I_2^1| &\leq C(xy)^{\mu+\frac{1}{2}}\int_0^{\pi/2}\frac{z^{2\mu}\,dz}{(|x-y|^2+xyz^2)^{\mu+3/2}}\\ &\leq \frac{C}{|x-y|^2}\int_0^{\frac{\sqrt{xy}\pi}{2|x-y|}}\frac{u^{2\mu}}{(1+u^2)^{\mu+3/2}}\leq\frac{C}{|x-y|^2}.\end{aligned}$$

For the second term, one has

$$\begin{aligned}|I_1^2| &\leq C\Bigg((xy)^{\mu+\frac{1}{2}}\int_0^{\frac{\pi}{2}}\frac{(\sin z)^{2\mu}|x-y|^2}{(|x-y|^2+2xy(1-\cos z))^{\mu+5/2}}\,dz\\ &+ (xy)^{\mu+\frac{1}{2}}\int_0^{\frac{\pi}{2}}\frac{(\sin z)^{2\mu}(y(1-\cos z))^2}{(|x-y|^2+2xy(1-\cos z))^{\mu+5/2}}\,dz\Bigg)=I_1^{2,1}+I_1^{2,2}.\end{aligned}$$

By using again (3.7) we have

$$I_1^{2,1}\leq C(xy)^{\mu+\frac{1}{2}}\int_0^{\frac{\pi}{2}}\frac{z^{2\mu}|x-y|^2}{(|x-y|^2+xyz^2)^{\mu+5/2}}\,dz\leq\frac{C}{|x-y|^2}.$$

To study $I_1^{2,2}$ we consider two cases. Assume first that $y<2x$. Then (3.7) leads to

$$I_1^{2,2}\leq C(xy)^{\mu+\frac{1}{2}}\int_0^{\frac{\pi}{2}}\frac{z^{2\mu+2}y^2}{(|x-y|^2+2xy(1-\cos z))^{\mu+5/2}}\,dz\leq\frac{Cy}{x|x-y|^2}\leq\frac{C}{|x-y|^2}.$$

When $y>2x$, $|y-x|\sim y$ and

$$\begin{aligned}I_1^{2,2} &\leq C(xy)^{\mu+1/2}\int_0^{\frac{\pi}{2}}\frac{z^{2\mu}y^2}{(|x-y|^2+xyz^2)^{\mu+5/2}}\\ &\leq C(xy)^{\mu+1/2}\int_0^{\frac{\pi}{2}}\frac{z^{2\mu}}{(|x-y|^2+xyz^2)^{\mu+3/2}}\leq\frac{C}{|x-y|^2}.\end{aligned}$$

By the change of variable $z=\pi-x$ and similar arguments as the ones for I_1^2, we can see that $|I_2^2|\leq \frac{C}{|x-y|^2}$. We now study the third term I_1^3, also in the cases $y\leq 2x$ and $y>2x$. In the case $y\leq 2x$, since $|1-\cos z|\leq 1$, by (3.7), we have that

$$\begin{aligned}|I_1^3| &\leq Cx^{\mu-1/2}y^{\mu+1/2}\int_0^{\pi/2}\frac{(\sin z)^{2\mu}(|x-y|+y)}{(x^2+y^2-2xy\cos z)^{\mu+3/2}}\,dz\\ &\leq \frac{Cx^{\mu-1/2}y^{\mu+1/2}(|x-y|+y)}{|x-y|^2(xy)^{\mu+1/2}}\leq \frac{C}{|x-y|^2}.\end{aligned}$$

Again by the change of variable $z=\pi-x$ and similar arguments, one gets $|I_2^3|\leq \frac{C}{|x-y|^2}$ in the case $y\leq 2x$.

Finally, suppose that $y>2x$. Note that in this case $|x-y|\sim y$. By using [11, (2.8)] we have

$$\begin{aligned}I^3 &= I_1^3+I_2^3=(\mu+\frac{1}{2})\frac{1}{x}R_\mu(x,y)\\ &= -(\mu+\frac{1}{2})\frac{1}{x}\left(\frac{2(\mu+1)}{\pi}x(xy)^{\mu+1/2}\int_0^{\pi}\frac{(\sin z)^{2\mu+2}}{(x^2+y^2-2xy\cos z)^{\mu+3/2}}\,dz+R_{\mu+1}(x,y)\right)\\ &= J^1+J^2.\end{aligned}$$

We write

$$J^1=-(\mu+\frac{1}{2})\frac{2(\mu+1)}{\pi}(xy)^{\mu+1/2}\left(\int_0^{\pi/2}+\int_{\pi/2}^{\pi}\right)\frac{(\sin z)^{2\mu+2}}{(x^2+y^2-2xy\cos z)^{\mu+3/2}}\,dz=J_1^1+J_2^1,$$

and

$$J^2=-(\mu+\frac{1}{2})\frac{1}{x}(R_{\mu+1,1}+R_{\mu+1,2})=J_1^2+J_2^2.$$

By using (3.7), as for I_1^1 and I_2^1, we conclude that

$$\begin{aligned}|J_1^1| &\leq C(xy)^{\mu+1/2}\int_0^{\pi/2}\frac{(\sin z)^{2\mu+2}}{(|x-y|^2+2xy(1-\cos z))^{\mu+3/2}}\,dz\leq\frac{C}{|x-y|^2},\\ |J_2^1| &\leq \frac{C}{|x-y|^2}.\end{aligned}$$

Also, as for I_1^2 and I_2^2, one has

$$\begin{aligned}|J_1^2| &\leq Cy(xy)^{\mu+1/2}\int_0^{\pi/2}\frac{(\sin z)^{2\mu+2}(|x-y|+y(1-\cos z))}{(|x-y|^2+2xy(1-\cos z))^{\mu+5/2}}\,dz\leq\frac{C}{|x-y|^2},\\ |J_2^2| &\leq \frac{C}{|x-y|^2}.\end{aligned}$$

Thus we conclude that

$$\left|\frac{\partial}{\partial x}R_\mu(x,y)\right|\leq\frac{C}{|x-y|^2}.$$

Next we estimate the derivative with respect to y of $R_\mu(x,y)$. Observe that in this case,

$$\begin{aligned}\frac{\partial}{\partial y}R_\mu(x,y) &= \frac{2\mu+1}{\pi}\Bigg(-(xy)^{\mu+1/2}\int_0^\pi \frac{(\sin z)^{2\mu}\cos z}{(|x-y|^2+2xy(1-\cos z))^{\mu+3/2}}\,dz\\ &- (\mu+\frac{3}{2})(xy)^{\mu+1/2}\int_0^\pi \frac{(\sin z)^{2\mu}(y\cos z-x)(2y-2x\cos z)}{(|x-y|^2+2xy(1-\cos z))^{\mu+5/2}}\,dz\\ &+ (\mu+\frac{1}{2})x^{\mu+1/2}y^{\mu-1/2}\int_0^\pi \frac{(\sin z)^{2\mu}(y\cos z-x)}{(|x-y|^2+2xy(1-\cos z))^{\mu+3/2}}\,dz\Bigg)\\ &= S^1+S^2+S^3.\end{aligned}$$

By proceeding as for I_1^1 and I_2^1, we get $|S^1|\le \frac{C}{|x-y|^2}$. In order to estimate S^2 we note that

$$(x-y\cos z)(y-x\cos z) = -|x-y|^2+y(y-x)(1-\cos z))+yx(1-\cos z)^2+x(x-y)(1-\cos z),$$

and we write $S^2=\sum_{j=1}^4 S^{2,j}$, where $S^{2,j}$, $j=1,2,3,4$, are defined in the obvious way.

$S^{2,1}$ can be studied like $I_1^{2,1}$ and $I_2^{2,1}$. To analyze $S^{2,2}$ we consider two cases. If $y>2x$, then $y\sim|x-y|$ and we proceed as for $S^{2,1}$. Otherwise, we argue as for the former estimate for J_1^2. $S^{2,3}$ coincides with $S^{2,2}$ when x and y are interchanged. To estimate $S^{2,4}$, we consider again two regions: $y>2x$ and $y<2x$, and follow the same reasoning as above. Thus we conclude that $|S^2|\le C|x-y|^{-2}$.

Finally, observe that the estimate for S^3 can be obtained by proceeding as in the study of I^3, just by interchanging the role of x and y. With this, we obtain that

$$\left|\frac{\partial}{\partial y}R_\mu(x,y)\right|\le\frac{C}{|x-y|^2}.$$

Let us now prove (ii). We write

$$\begin{aligned}|R_\mu(x,y)| &\le C(xy)^{\mu+1/2}\Bigg(\int_0^\pi \frac{(\sin z)^{2\mu}|x-y|}{(|x-y|^2+2xy(1-\cos z))^{\mu+3/2}}\,dz\\ &+ \int_0^\pi \frac{(\sin z)^{2\mu}y(1-\cos z)}{(|x-y|^2+2xy(1-\cos z))^{\mu+3/2}}\,dz\Bigg)=\Sigma^1+\Sigma^2\end{aligned}$$

Note that $\Sigma^1\le C|x-y|(I_1^1+I_2^1)$, thus by the proof of (i) $\Sigma^1\le C|x-y|^{-1}$. On the other hand, if $y>2x$, then $y\sim|y-x|$ and $\Sigma^2\le C\Sigma^1\le C|x-y|^{-1}$. In the case $y\le 2x$, then $y\le C\sqrt{xy}$ and we consider

$$\Sigma_1^2=(xy)^{\mu+1/2}\int_0^{\pi/2}\frac{(\sin z)^{2\mu}y(1-\cos z)}{(|x-y|^2+2xy(1-\cos z))^{\mu+3/2}}\,dz$$

and $\Sigma_2^2=\Sigma^2-\Sigma_1^2$. By using (3.7) we get

$$\Sigma_1^2\le C(xy)^{\mu+1}\int_0^{\pi/2}\frac{z^{2\mu+1}}{(|x-y|^2+xyz^2)^{\mu+3/2}}\,dz\le\frac{C}{|x-y|}.$$

Finally, since $1-\cos z\ge 1$ for $z\in(\pi/2,\pi)$, one easily gets $\Sigma_2^2\le C|x-y|^{-1}$. Thus we conclude that (ii) holds. ∎

Now, the general theory of Calderón-Zygmund singular integrals allows us to obtain L_p-boundedness properties for the Riesz transform R_μ.

Theorem 4.2. *The Riesz transform R_μ can be uniquely extended to $L_p(w)$ as a bounded operator from $L_p(w)$ into itself, for every $1 < p < \infty$ and $w \in A_p(0,\infty)$. Also, if $w \in A_1(0,\infty)$ then, R_μ uniquely extends to $L_1(w)$ as a bounded operator from $L_1(w)$ into $L_{1,\infty}(w)$.*

Define the maximal operator associated to R_μ by

$$R_{\mu,*}(f) = \sup_{\varepsilon>0} |R_{\mu,\varepsilon}(f)|,$$

where, for every $\varepsilon > 0$,

$$R_{\mu,\varepsilon}(f)(x) = \int_{|x-y|>\varepsilon} R_\mu(x,y) f(y)\, dy.$$

The next theorem follows from Theorem 4.2, according to [7, Corollary 7.13].

Theorem 4.3. *For every $1 < p < \infty$ and $w \in A_p(0,\infty)$, $R_{\mu,*}$ is a bounded operator from $L_p(w)$ into itself. Moreover, if $w \in A_1(0,\infty)$, $R_{\mu,*}$ is a bounded operator from $L_1(w)$ into $L_{1,\infty}(w)$.*

Since C_c^∞ is a dense subset of $L_p(w)$, $1 \le p < \infty$ and $w \in A_p(0,\infty)$, Theorems 3.3 and 4.3 imply the following result.

Corollary 4.4. *For every $1 \le p < \infty$ and $w \in A_p(0,\infty)$,*

$$R_\mu(f)(x) = \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} R_\mu(x,y) f(x) dx, \quad \text{for a.e. } x \in (0,\infty).$$

5. Oscillation and variation of the Riesz transforms R_μ.

Suppose that $(X, \mathcal{F}, \mu)$ denotes a σ-finite measure space and that $\mathcal{T} = \{T_r\}_{r>0}$ is a family of bounded operators form $L_p(X, \mathcal{F}, \mu)$ into itself for some $1 < p < \infty$. Assume also that the limit $Tf = \lim_{r\to 0} T_r f$ exists, in some sense, for each $f \in L_p(X, \mathcal{F}, \mu)$. Then a natural question appears: to measure the speed of convergence of $T_r f$ to Tf. Inspired by the classical method of considering square functions to measure the speed of convergence, two kinds of operators have been introduced: the oscillation and the variation operators ([5], [6] and [10]). Let $\{t_i\}_i$ be a fixed decreasing sequence converging to zero. We define the oscillation operator as follows

$$\mathcal{O}(\mathcal{T}f)(x) = \left(\sum_{i=1}^{\infty} \sup_{t_{i+1}\le \varepsilon_{i+1} < \varepsilon_i \le t_i} |T_{\varepsilon_{i+1}} f(x) - T_{\varepsilon_i} f(x)|^2 \right)^{1/2}.$$

It is not hard to see that $\mathcal{O}(\mathcal{T}f)$ is equivalent to $\mathcal{O}'(\mathcal{T}f)$, being

$$\mathcal{O}'(\mathcal{T}f)(x) = \left(\sum_{i=1}^{\infty} \sup_{t_{i+1}\le \delta_i \le t_i} |T_{t_{i+1}} f(x) - T_{\delta_i} f(x)|^2 \right)^{1/2}.$$

For every $\rho > 2$ the variation operator $\mathcal{V}_\rho$ is defined by

$$\mathcal{V}_\rho(\mathcal{T}f)(x) = \sup_{\varepsilon} \left(\sum_{i=1}^{\infty} |T_{\varepsilon_{i+1}} f(x) - T_{\varepsilon_i} f(x)|^\rho \right)^{1/\rho},$$

where the supremum is taken over all decreasing sequences of real numbers $\{\varepsilon_i\}$ converging to zero. The oscillation and variation operators can be understood as norms of linear operators acting on suitable Banach spaces. Since our results only refer to operators acting on functions defined on $(0,\infty)$, in the sequel we will restrict ourselves to this situation. We denote by $(X, \|.\|_X)$ the Banach spaces that consists of all those functions h defined on $(0,\infty)\times\mathbf{N}$ such that

$$\|h\|_X = \left(\sum_{i=1}^{\infty}(\sup_{s\in(0,\infty)}|h(s,i)|)^2\right)^{1/2} < \infty.$$

Let $\mathcal{T} = \{T_t\}_{t>0}$ be a family of operators defined on L_p for some $1<p<\infty$. Fix $\{t_i\}_i$ a sequence of real numbers that is decreasing and converging to zero and define, for every $i\in\mathbf{N}$, the interval $I_i = (t_{i+1}, t_i]$. Then the operator $U(\mathcal{T})$ is defined as follows

$$f \longrightarrow U(\mathcal{T})(f) = \{(T_{t_{i+1}}f - T_s f)\chi_{I_i}\}_{i\in\mathbf{N}}.$$

Thus, it is clear that $\mathcal{O}'(\mathcal{T}f)(x) = \|U(\mathcal{T})(f)(x)\|_X = \|\{(T_{t_{i+1}}f(x) - T_s f(x))\chi_{I_i}\}_{i\in\mathbf{N}}\|_X$. To prove that the oscillation operator $\mathcal{O}$ of $\mathcal{T}$ is bounded from L_p into itself, it suffices to see that the operator $U(\mathcal{T})$ is bounded from L_p into $L_p(X)$ (being $L_p(X)$ the usual Bochner-Lebesgue space of functions with values in X).

For the ρ-variation, let

$$\Theta = \{\varepsilon = \{\varepsilon_i\}_{i\in\mathbf{N}} : \varepsilon_i\in\mathbf{R},\ i\in\mathbf{N},\ \ \varepsilon_i\downarrow 0\}.$$

Consider, for every $1\le\rho<\infty$ the Banach space $(X_\rho, \|.\|_\rho)$, whose elements are functions h defined in $\mathbf{N}\times\Theta$ such that

$$\|h\|_\rho = \sup_{\varepsilon\in\Theta}\left(\sum_{i=0}^{\infty}|h(i,\varepsilon)|^\rho\right)^{1/\rho} < \infty.$$

Now we define the operator $\mathcal{V}(\mathcal{T})(f) = \{T_{\varepsilon_{i+1}}f - T_{\varepsilon_i}f\}_{\varepsilon\in\Theta}$. This operator is defined for functions in L_p and takes values in X_ρ. Thus, the variation operator $\mathcal{V}_\rho$ appears as $\mathcal{V}_\rho(Tf) = \|\mathcal{V}(\mathcal{T})f\|_{X_\rho}$.

According to Corollary 4.4, for every $f\in L_p$, $1\le p<\infty$, $R_\mu(f)(x) = \lim_{\varepsilon\to 0} R_{\mu,\varepsilon}(f)(x)$, for almost every $x\in(0,\infty)$. We now analyze the speed of convergence in the above limit by studying the boundedness of the oscillation and variation operators associated with the family $R_\mu = \{R_{\mu,\varepsilon}\}_{\varepsilon>0}$. The corresponding analysis for the Hilbert transform was made in [6]. More recently, Gillespie and Torrea [10, Theorem 1.5] have obtained a weighted version of the previous result in [6] by proving that the oscillation and variation operators associated with the Hilbert transform are bounded from $L_p(v)$ into itself for every $1<p<\infty$ and $v\in A_p$.

Theorem 5.1. *The operators $\mathcal{O}(R_\mu)$ and $\mathcal{V}_\rho(R_\mu)$ are bounded from $L_p(v)$ into itself, provided that $1<p<\infty$ and $v\in A_p(0,\infty)$. Moreover, the operators $\mathcal{O}(R_\mu)$ and $\mathcal{V}_\rho(R_\mu)$ are bounded from L_1 into $L_{1,\infty}$.*

Proof. Assume that $1 < p < \infty$ and $v \in A_p(0,\infty)$. Let $\{t_i\}_{i\in\mathbf{N}}$ be a decreasing sequence in $(0,\infty)$ such that $t_i \downarrow 0$. According to (3.15), (3.16) and (3.17) we can write

$$\begin{aligned}
|R_{\mu,t_{i+1}}f(x) - R_{\mu,s}f(x)| &= \left|\int_{\{t_{i+1}<|x-y|<s\}} R_\mu(x,y)f(y)\,dy\right| \\
&\leq C\Bigg(x^{-\mu-3/2}\int_0^{x/2}\chi_{\{t_{i+1}<|x-y|<s\}}(y)y^{\mu+1/2}|f(y)|\,dy \\
&+ x^{\mu+3/2}\int_{2x}^{\infty}\chi_{\{t_{i+1}<|x-y|<s\}}(y)y^{-\mu-5/2}|f(y)|\,dy \\
&+ \left|\int_{x/2}^{2x}\frac{1}{x-y}\chi_{\{t_{i+1}<|x-y|<s\}}(y)f(y)\,dy\right| \\
&+ \int_{x/2}^{2x}\frac{1}{y}\left(1+\log_+\frac{\sqrt{xy}}{|x-y|}\right)\chi_{\{t_{i+1}<|x-y|<s\}}(y)|f(y)|\,dy\Bigg).
\end{aligned}$$

Then

$$\mathcal{O}'(R_\mu f)(x) \leq C\sum_{j=1}^{4}\|\{I_j(s,i)(x)\}_i\|_X,$$

where

$$\begin{aligned}
I_1(s,i)(x) &= \frac{1}{x}\int_0^x \chi_{\{t_{i+1}<|x-y|<s\}}(y)|f(y)|\,dy \\
I_2(s,i)(x) &= \int_x^\infty \frac{1}{y}\chi_{\{t_{i+1}<|x-y|<s\}}(y)|f(y)|\,dy \\
I_3(s,i)(x) &= \int_0^\infty \frac{1}{x-y}\chi_{\{t_{i+1}<|x-y|<s\}}(y)f(y)\,dy \\
I_4(s,i)(x) &= \int_{x/2}^{2x}\frac{1}{y}\left(1+\log_+\frac{\sqrt{xy}}{|x-y|}\right)\chi_{\{t_{i+1}<|x-y|<s\}}(y)|f(y)|\,dy.
\end{aligned}$$

Assume that $f \in L_{1,loc}([0,\infty))$. By Minkowski's inequality, we obtain that

$$\|\{I_1(s,i)(x)\}_i\|_X \leq \frac{1}{x}\int_0^x\sum_{i=1}^{\infty}\sup_{t_{i+1}<s\leq t_i}\chi_{\{t_{i+1}<|x-y|<s\}}(y)|f(y)|\,dy \leq \frac{1}{x}\int_0^x|f(y)|\,dy.$$

In a similar way we obtain that

$$\|\{I_2(s,i)(x)\}_i\|_X \leq \int_x^\infty \frac{1}{y}|f(y)|\,dy,$$

and

$$\|\{I_4(s,i)(x)\}_i\|_X \leq \int_{x/2}^{2x}\frac{1}{y}\left(1+\log_+\frac{\sqrt{xy}}{|x-y|}\right)|f(y)|\,dy.$$

As it is known the Hardy operators T_j, $j=1,2$, defined by

$$T_1(f)(x) = \frac{1}{x}\int_0^x f(y)dy \quad \text{and} \quad T_2(f)(x) = \int_x^\infty \frac{1}{y}f(y)\,dy,$$

are bounded from $L_p(v)$ into itself. Also, by proceeding as in [2, p. 18-19] we can see that the operator

$$T_4(f)(x) = \int_{x/2}^{2x} \frac{1}{y}\left(1 + \log_+ \frac{\sqrt{xy}}{|x-y|}\right)|f(y)|\,dy$$

is bounded from $L_p(v)$ into itself. Therefore,

$$\|\|\{I_j(s,i)(x)\}_i\|_X\|_{L_p(v)} \leq C\|f\|_{L_p(v)}, \quad \text{for } j = 1,2,4 \text{ and } f \in L_p(v).$$

Finally, by using [10, Theorem 1.5], we get

$$\|\|\{I_3(s,i)(x)\}_i\|_X\|_{L_p(v)} \leq C\|f\|_{L_p(v)}, \quad f \in L_p(v).$$

Thus we conclude that the oscillation operator $\mathcal{O}(R_\mu)$ is bounded from $L_p(v)$ into itself.

In a similar way, by using again [10, Theorem 1.5], it can be proved that the variation operators $\mathcal{V}_\rho(R_\mu)$, $\rho > 2$, are bounded from $L_p(v)$ into itself.

On the other hand, since the operators T_j, $j = 1,2,4$, are bounded from L_1 into $L_{1,\infty}$, [6, Theorems 1.1 and 1.2] we get that the oscillation operator $\mathcal{O}(R_\mu)$ and the variation operators $\mathcal{V}_\rho(R_\mu)$, $\rho > 2$, are of weak type $(1,1)$. ∎

6. Weighted inequalities.

In this section we study the conditions for a pair of positive measurable functions (u,v) such that the following inequality holds

$$\int_0^\infty |R_\mu(f)(x)|^p u(x)dx \leq C\int_0^\infty |f(x)|^p v(x)dx, \quad f \in L^p(v), \tag{6.1}$$

where $1 < p < \infty$.

According to Theorem 4.2, (6.1) holds whenever $u = v \in A_p(0,\infty)$. Moreover, by using [2, Theorem 1] we can obtain a complete characterization of the measurable functions u for which (6.1) is satisfied when $u = v$.

In the general case of different weights we consider the following weak problem: find conditions for a weight v (respectively u) such that (6.1) holds true for some weight u (respectively v). Note that, since R_μ is a Calderón-Zygmund singular integral operator (Proposition 4.1), from [9, Theorem 6.4] we deduce that if $1 < p < \infty$ and

$$\int_0^\infty v(x)^{1-p'}(1+x)^{-p'}dx < \infty \quad \left(\text{respectively } \int_0^\infty u(x)(1+x)^{-p}dx < \infty\right),$$

then there exists a positive measurable function u (respectively v) for which (6.1) holds. Next we characterize the positive measurable functions v (respectively u) satisfying (6.1) for some positive measurable function u (respectively v).

We will need the following result established in [8].

Theorem 6.1. *Let $(Y, d\nu)$ be a measure space, F, G be Banach spaces, and $\{A_k\}_{k\in\mathbf{Z}}$ be a sequence of pairwise disjoint measurable sets of Y such that $Y = \cup_{k\in\mathbf{Z}} A_k$. Consider $0 < s < p < \infty$ and T a sublinear operator which satisfies the following vector valued inequality*

$$\left\|\left(\sum_j \|Tf_j\|_G^p\right)^{1/p}\right\|_{L^s(A_k,d\nu)} \leq C_k\left(\sum_j \|f_j\|_F^p\right)^{1/p}, \quad k \in \mathbf{Z},$$

where, for every $k \in \mathbf{Z}$, C_k only depends of F, G, p and s. Then there exist a positive measurable function u on Y such that

$$\left(\int_Y \|Tf(x)\|_G^p u(x)d\nu(x)\right)^{1/p} \leq C\|f\|_F,$$

where C only depends of F, G, p and s.

■

Let us introduce, for every $1 < p < \infty$, the set of functions D_p as follows:

$$D_p = \Bigg\{ w \geq 0 : \text{ measurable, such that for any } a > 0$$

$$\int_0^a x^{(\mu+1/2)p/(p-1)} w(x)^{-1/(p-1)} dx < \infty \text{ and } \int_a^\infty x^{-(\mu+5/2)p/(p-1)} w(x)^{-1/(p-1)} dx < \infty \Bigg\}.$$

Theorem 6.2. *Let $1 < p < \infty$ and let v be a measurable function such that $0 < v < \infty$. The following assertions are equivalent:*

(i) v is in D_p.

(ii) For every $f \in L_p(v)$, there exists $\lim_{\varepsilon\to 0} R_{\mu,\varepsilon}(f)(x)$, for almost all $x \in (0,\infty)$.

(iii) For every $f \in L_p(v)$, $R_{\mu,*}(f)(x) = \sup_{\varepsilon>0} |R_{\mu,\varepsilon}(f)(x)| < \infty$, for almost all $x \in (0,\infty)$.

(iv) There exist a positive measurable function u and $C > 0$ such that, for every $f \in L_p(v)$,

$$\int_0^\infty |R_{\mu,*}(f)(x)|^p u(x)dx \leq C \int_0^\infty |f(x)|^p v(x)dx.$$

(v) There exist a positive measurable function u and $C > 0$ such that, for every $f \in L_p(v)$,

$$\int_0^\infty |R_\mu(f)(x)|^p u(x)dx \leq C \int_0^\infty |f(x)|^p v(x)dx.$$

Proof. $(ii) \Rightarrow (iii)$ and $(iv) \Rightarrow (v)$ are clear.

$(i) \Rightarrow (iv)$. Assume that $v \in D_p$ and take $0 < s < 1$. We define $I_0 = (\frac{1}{2}, 2)$, and, for every $k \in \mathbf{N}$, $I_k = (2^{-k-1}, 2^{-k}] \cup [2^k, 2^{k+1})$ and $B_k = (2^{-k-2}, 2^{k+2})$. Let $k \in \mathbf{N}$ and f be a measurable function on $(0,\infty)$. We write $f = f^1 + f^2 + f^3$, where $f^1 = f\chi_{B_k}$, $f^2 = f\chi_{(0,2^{-k-2}]}$ and $f^3 = f\chi_{[2^{k+2},\infty)}$. Observe that for $x \in I_k$ and $y \in (0, 2^{-k-2})$ then $\frac{x}{y} > 2$, and, according to (3.15), $|R_\mu(x,y)| \leq Cx^{-\mu-3/2}y^{\mu+1/2}$. Hence, for every $\varepsilon > 0$ and $x \in I_k$,

$$\begin{aligned} |R_{\mu,\varepsilon}(f^2)(x)| &\leq C2^{(k+1)(\mu+3/2)} \int_0^{2^{-k-2}} y^{\mu+1/2} v(y)^{1/p} v(y)^{-1/p} |f(y)|\, dy \\ &\leq C2^{(k+1)(\mu+3/2)} \left(\int_0^\infty v(y)|f(y)|^p\, dy\right)^{1/p} \left(\int_0^{2^{-k-2}} y^{(\mu+1/2)p'} v(y)^{-p'/p}\, dy\right)^{1/p'}. \end{aligned}$$

Thus we conclude that

$$\left\| \left(\sum_j |R_{\mu,*}(f_j^2)|^p \right)^{1/p} \right\|_{L^s(I_k,dx)} \leq C_k \left(\sum_j \|f_j\|_{L^p(v)}^p \right)^{1/p}. \tag{6.2}$$

Also if $x \in I_k$ and $y > 2^{k+2}$ then $\frac{x}{y} < \frac{1}{2}$, and, according to (3.16), $|R_\mu(x,y)| \leq Cx^{\mu+3/2}y^{-\mu-5/2}$. Hence, for every $\varepsilon > 0$, one can get in a similar way that

$$\left\| \left(\sum_j |R_{\mu,*}(f_j^3)|^p \right)^{1/p} \right\|_{L^s(I_k,dx)} \leq C_k \left(\sum_j \|f_j\|^p_{L^p(v)} \right)^{1/p}. \tag{6.3}$$

By using standard techniques, one can easily obtain a ℓ^p-valued (self)-improvement of Theorem 4.3 **poner referencia** that, together with Kolmogorov's inequality ([9, p. 485]), give us

$$\begin{aligned} \left\| \left(\sum_j |R_{\mu,*}(f_j^1)|^p \right)^{1/p} \right\|_{L^s(I_k,dx)} &\leq C_k \left\| \left(\sum_j |R_{\mu,*}(f_j^1)|^p \right)^{1/p} \right\|_{L_{1,\infty}(I_k,dx)} \\ &\leq C_k \int_{I_k} \left(\sum_j |f_j(x)|^p \right)^{1/p} v(x)^{1/p} v(x)^{-1/p} dx \leq C_k \left(\sum_j \|f\|^p_{L^p(v)} \right)^{1/p}. \end{aligned}$$

Then

$$\left\| \left(\sum_j |R_{\mu,*}(f_j^1)|^p \right)^{1/p} \right\|_{L^s(I_k,dx)} \leq C_k \left(\sum_j \|f_j\|^p_{L^p(v)} \right)^{1/p}. \tag{6.4}$$

By combining now (6.2), (6.3) and (6.4) we conclude that, for every $k \in \mathbf{N}$,

$$\left\| \left(\sum_j |R_{\mu,*}(f_j)|^p \right)^{1/p} \right\|_{L^s(I_k,dx)} \leq C_k \left(\sum_j \|f_j\|^p_{L^p(v)} \right)^{1/p}.$$

Hence, from Theorem 6.1, we deduce that there exists a weight u such that (6.1) holds true.

$(v) \Rightarrow (i)$. Duality arguments allow us to conclude that if for a pair of weights (u,v) (6.1) holds then,

$$\int_0^\infty |R_\mu^*(f)(x)|^{p'} v(x)^{1-p'} dx \leq C \int_0^\infty |f(x)|^{p'} u(x)^{1-p'} dx, \quad f \in L^{p'}(v^{1-p'}), \tag{6.5}$$

where R_μ^* denotes the adjoint operator of R_μ and it is given by

$$R_\mu^*(f)(x) = \int_0^\infty R_\mu(y,x) f(y)\, dy.$$

Assume that (6.5) holds for some weight u. Let $a > 0$ and choose a bounded set $J \subset (ak, \infty)$, where k is a suitable positive constant given in (3.18) and (3.19), having positive Lebesgue measure and such that $\int_J u(x)^{1-p'} dx < \infty$. According to (3.18), since $\frac{x}{y} < \frac{1}{k}$, provided that $x \in (0,a)$ and $y \in J$,

$$|R_\mu^*(\chi_J)(x)| = \left| \int_J R_\mu(y,x)\, dy \right| \geq Cx^{\mu+1/2} \int_J y^{-\mu-3/2}\, dy, \quad x \in (0,a).$$

Then

$$\begin{aligned}\int_0^a x^{(\mu+1/2)p'} v(x)^{1-p'}\,dx &\leq C\int_0^a |R_\mu^*(\chi_J)(x)|^{p'} v(x)^{1-p'}\,dx\\ &\leq C\int_0^\infty \chi_J(x)u(x)^{1-p'}\,dx \leq C\int_J u(x)^{1-p'}\,dx.\end{aligned}$$

Now we take a bounded set $J\subset(0,a/k)$ having positive Lebesgue measure and such that $\int_J u(x)^{1-p'}dx<\infty$. According to (3.19), since $\frac{x}{y}>k$, provided that $x\in(a,\infty)$ and $y\in J$,

$$|R_\mu^*(\chi_J)(x)| = \left|\int_J R_\mu(y,x)\,dy\right| \geq Cx^{-\mu-5/2}\int_J y^{\mu+3/2}\,dy,\ \ x\in(a,\infty).$$

Then

$$\begin{aligned}\int_a^\infty x^{-(\mu+5/2)p'} v(x)^{1-p'}\,dx &\leq C\int_a^\infty |R_\mu^*(\chi_J)(x)|^{p'} v(x)^{1-p'}\,dx\\ &\leq C\int_0^\infty \chi_J(x)u(x)^{1-p'}\,dx \leq C\int_J u(x)^{1-p'}\,dx.\end{aligned}$$

And with this we prove that $v\in D_p$.

$(iv)\Rightarrow(ii)$. Since the measure $v\,dx$ is σ-finite on $(0,\infty)$, $L_p(v)\cap L_2(dx)$ is a dense subset of $L_p(v)$. Moreover, according to Corollary 4.4, for every $f\in L_2(dx)$ there exists $\lim_{\varepsilon\to0}R_{\mu,\varepsilon}f$ almost everywhere. Then, for every $f\in L_p(v)$, there exists $\lim_{\varepsilon\to0}R_{\mu,\varepsilon}f$ almost everywhere, provided that (iv) holds. To see this it is sufficient to argue as in [7, p. 27 and 28].

$(iii)\Rightarrow(i)$. Assume that (iii) holds. First, we see that the sublinear operator $R_{\mu,*}$, defined on $L_p(v)$, is continuous in measure. Since $R_{\mu,*}f(x)<\infty$ a.e., for every $f\in L_p(v)$, according to [9, Proposition VI.1.4], it is sufficient to show that, for every $\varepsilon>0$, $R_{\mu,\varepsilon}$ is continuous in measure. In fact, we shall see that the sublinear operators

$$S_{\mu,\varepsilon}f(x) = \int_{|x-y|>\varepsilon} |R_\mu(x,y)||f(x)|dx$$

are continuous in measure. We now define, for each $m\in\mathbf{N}$ and $l\in\mathbf{Z}$, the set $A_{m,l}=\{x\in(m,m+1): 2^l<v(x)<2^{l+1}\}$. Thus, we can write, for every $m,n\in\mathbf{N}$ and $l,k\in\mathbf{Z}$,

$$S_{\mu,\varepsilon}f(x) = \sum_{m,l}\sum_{k,n}\chi_{A_{m,l}}(x)\int_{|x-y|>\varepsilon}\chi_{A_{n,k}}(y)|R_\mu(x,y)||f(y)|\,dy = \sum_{m,l}\sum_{k,n} S_{\mu,\varepsilon}^{(m,l),(n,k)}(f)(x).$$

Fix $m,n\in\mathbf{N}$ and $l,k\in\mathbf{Z}$. By Proposition 4.1, (ii) and Holder's inequality, we have

$$\begin{aligned}S_{\mu,\varepsilon}^{(m,l),(n,k)}(f)(x) &\leq \frac{C}{\varepsilon}\int_{A_{n,k}}|f(y)|v^{1/p}v^{-1/p}\,dy\\ &\leq \frac{C}{\varepsilon}\left(\int_{A_{n,k}} v^{-1/(p-1)}\,dy\right)^{1/p'}\|f\|_{L_p(v)},\ \ x\in A_{m,l}\end{aligned}$$

Hence, $S_{\mu,\varepsilon}^{(m,l),(n,k)}$ is bounded from $L_p(v)$ into $L_\infty(dx)$ and then, $S_{\mu,\varepsilon}^{(m,l),(n,k)}$ is continuous in measure. According to our hypothesis, if $f\in L_p(v)$ then $S_{\mu,\varepsilon}f(x)<\infty$ for almost every $x\in(0,\infty)$. From Proposition VI.1.4 in [9] it follows that $S_{\mu,\varepsilon}$ is continuous in measure ($S_{\mu,\varepsilon}f(x)$ is the supremum of finite sums of continuous-in-measure operators).

Since $|R_{\mu,\varepsilon}f| \leq S_{\mu,\varepsilon}f$, $R_{\mu,\varepsilon}$, and by the same reasoning $R_{\mu,*}$, are also continuous-in-measure operators. Nikishin's theorem ([9, Corollary VI.2.7]) implies that there exists a measurable function $u > 0$, a.e., such that

$$\int_{\{x\in(0,\infty):R_{\mu,*}f(x)>\lambda\}} u(x)dx \leq \left(\|f\|_{L_p(v)}/\lambda\right)^q, \quad f \in L_p(v) \quad \text{and} \quad \lambda > 0, \tag{6.6}$$

where $q = \inf(p, 2)$. Assume first that $1 < p \leq 2$. Then we have that

$$\int_{\{x\in(0,\infty):R_{\mu,*}f(x)>\lambda\}} u(x)dx \leq \left(\|f\|_{L_p(v)}/\lambda\right)^p, \quad f \in L_p(v) \quad \text{and} \quad \lambda > 0, \tag{6.7}$$

u being a positive measurable function . By Corollary 4.4, $R_\mu(f)(x) = \lim_{\varepsilon\to 0} R_{\mu,\varepsilon}(f)(x)$, a.e., for every $f \in L_2$. Then, since $L_2 \cap L_p(v)$ is a dense subset in $L_p(v)$, (6.7) allows us to extend R_μ to $L_p(v)$ and the extended operator satisfies

$$\int_{\{x\in(0,\infty):R_{\mu}f(x)>\lambda\}} u(x)dx \leq \left(\|f\|_{L_p(v)}/\lambda\right)^p, \quad f \in L_p(v) \ \textit{and} \ \lambda > 0. \tag{6.8}$$

From (6.8) we deduce that $u \in L_{1,loc}$. Indeed, let $x_0 > 0$. We define, for every $k \in \mathbf{N}$, the set $C_k = B(x_0, 2^{-k}x_0)\backslash B(x_0, 2^{-k-1}x_0)$, where as usual $B(x, r) = \{y \in (0, \infty) : |y-x| < r\}$, $r, x > 0$. We choose $k_0, l_0 \in \mathbf{N}$ such that $\frac{1+2^{-k_0-l_0}}{1-2^{-k_0}} < \frac{1}{l}$, l being a positive constant for which

$$R_\mu(x, y) \geq C\frac{\operatorname{sign}(x-y)}{x-y}, \quad \frac{1}{l}x < y < lx$$

(see (3.20)), and we take a measurable set $A \subset C_{k_0} \cap (0, x_0)$ such that $|A| > 0$ and $\int_A v\,dy < \infty$. Note that this set A exists because $0 < v < \infty$ a.e.. Assume now that $x \in B(x_0, 2^{-k_0-l_0}x_0)$ and $y \in A$. Then $y > x/l$ and

$$R_\mu(x, y) \geq \frac{C}{x-y} \geq \frac{C}{x_0(2^{-k_0-1} - 2^{-k_0-l_0})} = C_0.$$

Hence

$$R_\mu(\chi_A)(x) \geq \int_A C_0\,dx = C_0|A| > 0, \quad x \in B(x_0, 2^{-k_0-l_0}x_0).$$

By taking $0 < \lambda_0 < C_0|A|$, from (6.8) we deduces that

$$\begin{aligned}\int_{B(x_0,2^{-k_0-l_0}x_0)} u(z)\,dz &\leq \int_{\{z\in(0,\infty):|R_\mu(\chi_A)(z)|>\lambda_0\}} u(z)\,dz \\ &\leq C\left(\frac{\|\chi_A\|_{L_p(v)}}{\lambda_0}\right)^p \leq \frac{C}{\lambda_0^p}\int_A v\,dy < \infty.\end{aligned}$$

Thus, $u \in L_{1,loc}$. Since $u \in L_{1,loc}$ the measure $u\,dx$ is σ-finite and from Kolmogorov's inequality ([9, Lemma V.2.8]), if $1 < r < p$,

$$\sup_{E:0<\int_E v<\infty} \left(\int_E u\,dy\right)^{1/p-1/r} \left(\int_E |R_\mu(f)(x)|^r u(x)dx\right)^{1/r} \leq C\|R_\mu f\|_{L_{p,\infty}(u)} \leq C\|f\|_{L_p(v)}.$$

Choose a measurable set A such that $|A|<\infty$, $0<\int_A u\,dy<\infty$ and $\int_A u^{1-r'}(x)dx<\infty$. It is not hard to see that this set A can be found. Since the set $M_p(v)=L_p\cap L_p(v^{-1})\cap L_\infty$ is dense in L_p, we can write

$$\|R_\mu^*(\chi_A)\|_{L_p'(v^{1-p'})}=\|R_\mu^*(\chi_A)v^{-1/p}\|_{L_{p'}}=\sup_{g\in M_p(v),\|g\|_p=1}\left|\int_0^\infty R_\mu^*(\chi_A)(x)v(x)^{-1/p}g(x)dx\right|.$$

Take $g\in M_p(v)$. It is clear that $\chi_A\in L_{p'}$ and then $R_\mu^*(\chi_A)\in L_{p'}$ (Theorem 4.2). Also $gv^{-1/p}\in L_p$. Therefore

$$\begin{aligned}\left|\int_0^\infty R_\mu^*(\chi_A)(x)v(x)^{-1/p}g(x)dx\right| &= \left|\int_0^\infty \chi_A(x)R_\mu(v^{-1/p}g)(x)dx\right|\\ &\le \left(\int_A |R_\mu(v^{-1/p}g)(x)|^r u(x)dx\right)^{1/r}\left(\int_A u(x)^{-r'/r}dx\right)^{1/r'}\\ &\le C\left(\int_A u(z)\,dz\right)^{1/r-1/p}\|v^{-1/p}g\|_{L_p(v)}\left(\int_A u(x)^{-r'/r}dx\right)^{1/r'}\\ &\le C\left(\int_A u(z)\,dz\right)^{1/r-1/p}\|g\|_{L_p}\left(\int_A u(x)^{-r'/r}dx\right)^{1/r'}.\end{aligned}$$

Hence, we obtain

$$\|R_\mu^*(\chi_A)\|_{L_{p'}(v^{1-p'})}\le\left(\int_A u(z)\,dz\right)^{1/r-1/p}\left(\int_A u(x)^{-r'/r}dx\right)^{1/r'}. \tag{6.9}$$

Next step is proving that $v\in D_p$. Let $a>0$ and take measurable set $A\subset(k,\infty)$, where k is a suitable positive constant given in (3.18) and (3.19), such that, as above, $|A|<\infty$, $0<\int_A u\,dy<\infty$ and $\int_A u^{1-r'}(x)dx<\infty$. By proceeding as in the proof of $(v)\Rightarrow(i)$ we get

$$|R_\mu^*(\chi_A)(x)|\ge Cx^{\mu+1/2}\int_A y^{-\mu-3/2}\,dy,\quad x\in(0,a).$$

Then (6.9) leads to

$$\int_0^a x^{(\mu+1/2)p'}v(x)^{1-p'}dx\le C\int_0^a|R_\mu^*(\chi_A)(x)|^{p'}v(x)^{1-p'}dx\le C\|R_\mu^*(\chi_A)\|^{p'}_{L_{p'}(v^{1-p'})}<\infty.$$

Also by using (6.9) and by proceeding as in the proof of $(v)\Rightarrow(i)$ we obtain that

$$\int_a^\infty x^{-(\mu+5/2)p'}v(x)^{1-p'}dx<\infty.$$

Thus we have proved that $v\in D_p$ when $1<p\le 2$.

Assume now that $p>2$. In this case, from (6.6) we infer that

$$\int_{\{x\in(0,\infty):R_\mu f(x)>\lambda\}}u(x)dx\le\left(\|f\|_{L_p(v)}/\lambda\right)^2,\quad f\in L_p(v)\quad\text{and}\quad\lambda>0. \tag{6.10}$$

Define the functions

$$w(x)=\begin{cases} v^{1/(p-1)}x^{(\mu+1/2)(2-p')}, & 0<x<1\\ v^{1/(p-1)}x^{-(\mu+5/2)(2-p')}, & x>1,\end{cases}$$

and

$$\alpha(x) = \begin{cases} x^{\mu+1/2}, & 0 < x < 1 \\ x^{-(\mu+5/2)}, & x > 1. \end{cases}$$

Then $L_2(w) \subset L_p(v) + L_1(\alpha)$. Indeed, let $f \in L_2(w)$. We write $f = g + h$ where

$$g(x) = \begin{cases} f(x), & |f(x)| \leq (w(x)/v(x))^{1/(p-2)} \\ 0, & \text{otherwise}, \end{cases}$$

and $h = f - g$. Since $p > 2$ we can write

$$\int_0^\infty |g(x)|^p v(x)dx \leq \int_{\{x:|f(x)|\leq (w(x)/v(x))^{1/(p-2)}\}} |f(x)|^2 |f(x)|^{p-2} v(x)dx \leq \int_0^\infty |f(x)|^2 w(x)dx.$$

Hence $g \in L_p(v)$.

On the other hand, we get

$$\begin{aligned} \int_0^\infty |h(x)|\alpha(x)dx &= \int_{\{x:|f(x)|>(w(x)/v(x))^{1/(p-2)}\}} |f(x)|\alpha(x)dx \\ &\leq \int_{\{x:|f(x)|>(w(x)/v(x))^{1/(p-2)}\}} |f(x)|^2 w(x)^{-1/(p-2)} v(x)^{1/(p-2)} \alpha(x)dx \\ &\leq \int_0^\infty |f(x)|^2 w(x)dx, \end{aligned}$$

that proves that $h \in L_1(\alpha)$.

Our next aim is to show that, for every $f \in L_2(w)$, $R_{\mu,*}(f)(x) < \infty$, for almost every $x \in (0, \infty)$. According to our hypothesis, it is sufficient to see that $R_{\mu,*}(h)(x) < \infty$, a.e., for each $h \in L_1(\alpha)$. Let $h \in L_1(\alpha)$. By (3.15), (3.16) and (3.17) we can write, for every $\varepsilon > 0$,

$$\begin{aligned} |R_{\mu,\varepsilon}h(x)| &\leq C\Bigg(x^{-\mu-3/2} \int_0^x y^{\mu+1/2}|h(y)|\,dy + \left| \int_{(x/2,2x)\cap\{|x-y|>\varepsilon\}} \frac{h(y)}{x-y}\,dy \right| \\ &+ \int_{x/2}^{2x} \frac{1}{y}\left(1 + \log_+ \frac{\sqrt{xy}}{|x-y|}\right)|h(y)|\,dy + x^{\mu+3/2} \int_x^\infty y^{-\mu-5/2}|h(y)|\,dy \Bigg) \\ &\leq C\Bigg(\sum_{j=1}^3 T_j(h)(x) + |H_{0,\varepsilon}(h)(x)| \Bigg), \end{aligned}$$

where

$$\begin{aligned} T_1(h)(x) &= x^{-\mu-3/2} \int_0^x y^{\mu+1/2}|h(y)|\,dy \\ T_2(h)(x) &= x^{\mu+3/2} \int_x^\infty y^{-\mu-5/2}|h(y)|\,dy \\ T_3(h)(x) &= \int_{x/2}^{2x} \frac{1}{y}\left(1 + \log_+ \frac{\sqrt{xy}}{|x-y|}\right)|h(y)|\,dy \\ H_{0,\varepsilon}(f)(x) &= \int_{(x/2,2x)\cap\{|x-y|>\varepsilon\}} \frac{h(y)}{x-y}\,dy. \end{aligned}$$

It is not hard to see that $T_j(h)(x) < \infty$, $j = 1, 2$, for every $x \in (0, \infty)$. On the other hand, if $x \in (0, \infty)$ there exists $C(x) > 0$ such that

$$T_3(h)(x) \leq C(x) \int_{1/2}^{2} |h_1(ux)| \left(1 + \log_+ \frac{\sqrt{u}}{|1-u|}\right) \frac{du}{u},$$

where $h_1 = h\alpha$. Moreover,

$$\begin{aligned} \int_0^\infty \int_{1/2}^{2} |h_1(ux)| \left(1 + \log_+ \frac{\sqrt{u}}{|1-u|}\right) \frac{du}{u} dx &= \int_{1/2}^{2} \int_0^\infty |h_1(ux)| dx \left(1 + \log_+ \frac{\sqrt{u}}{|1-u|}\right) \frac{du}{u} \\ &= \int_{1/2}^{2} \left(1 + \log_+ \frac{\sqrt{u}}{|1-u|}\right) \frac{du}{u^2} \|h_1\|_{L_1} < \infty. \end{aligned}$$

Hence $T_3(h)(x) < \infty$, for almost all $x \in (0, \infty)$.

To prove that $\sup_{\varepsilon>0} |H_{0,\varepsilon}(h)(x)| < \infty$, for almost every $x \in (0, \infty)$, we will show that $\lim_{\varepsilon\to 0} H_{0,\varepsilon}h(x)$ exists for almost all $x \in (0, \infty)$. In order to establish the last assertion it is sufficient to see that the operator L defined by

$$h \longrightarrow L(h) = \lim_{\varepsilon\to 0} H_{0,\varepsilon}h(x)$$

is bounded from $L_1(\alpha)$ into $L_{1,\infty}(\alpha)$. According to [3, Lemma 1] we have the boundedness of the operator L provided that α is in $A_{1,loc}$, that is, if there exists a constant $C > 0$ for which

$$\int_a^b \alpha(x) dx \sup_{x\in(a,b)} \frac{1}{\alpha(x)} \leq C(b - a),$$

for every $0 < a < b < 2a$. This property for α can be established by using the mean value theorem in a suitable way. Hence we conclude that $\sup_{\varepsilon>0} |H_{0,\varepsilon}(h)(x)| < \infty$, for almost every $x \in (0, \infty)$.

Thus it is proved that, for every $f \in L_2(w)$, $R_{\mu,*}(f)(x) < \infty$, for almost every $x \in (0, \infty)$. Then, since we already have shown that $(iii) \Rightarrow (i)$ when $p = 2$, $w \in D_2$, that is, $v \in D_p$ and the proof of this theorem is now completed. ∎

Duality arguments would allow us to deduce from Theorem 6.2 necessary and sufficient conditions on a measurable function u in order to guarantee that there exists a positive measurable function v such that (6.1) holds true.

7. Appendix

It is clear that $\Delta_\mu = \Delta_{-\mu}$. Then, we also have for Δ_μ the decomposition $\Delta_\mu = B_\mu^* B_\mu$, where $B_\mu = x^{-\mu+1/2} D x^{\mu-1/2}$. According to the general idea described in the introduction, we can define other Riesz transform, that will be denoted by T_μ, associated with Δ_μ as

$$T_\mu = B_\mu \Delta_\mu^{-1/2}.$$

In this case, by proceeding as for R_μ, we get that for every $f \in C_c^\infty$

$$T_\mu(f)(x) = \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} T_\mu(x, y) f(y)\, dy, \quad x \in (0, \infty),$$

being, for every $x, y \in (0,\infty)$, $x \neq y$,

$$\begin{aligned} T_\mu(x,y) &= x^{-\mu+1/2}\frac{d}{dx}\left(x^{\mu-1/2}\int_0^\infty P_\mu(t,x,y)\,dt\right) \\ &= x^{-\mu+1/2}\frac{d}{dx}\left(x^{\mu-1/2}\int_0^\infty\int_0^\infty e^{-zt}(xz)^{1/2}J_\mu(xz)(yz)^{1/2}J_\mu(yz)\,dz\,dt\right) \\ &= \int_0^\infty\int_0^\infty e^{-zt}(xz)^{1/2}J_{\mu-1}(xz)(yz)^{1/2}J_\mu(yz)z\,dz\,dt \\ &= -R_{\mu-1}(y,x). \end{aligned}$$

Hence, $T_\mu = -R_{\mu-1}$. Then, the results obtained in this paper can be transferred to the Riesz transform T_μ, provided that $\mu > 1/2$.

On the other hand, the Riesz transform defined in [4] can be written as

$$\mathcal{R}_\mu = \frac{d}{dx}\Delta_\mu^{-1/2},$$

and a straightforward manipulation allows us to get

$$\mathcal{R}_\mu = \frac{1}{2\mu}\left((\mu-1/2)R_\mu + (\mu+1/2)T_\mu\right).$$

By using this relation and our results in Sections 3-6, we can obtain new results for the Riesz transforms $\mathcal{R}_\mu$. In particular, from Theorem 4.2, it follows that, when $\mu > 1/2$, $\mathcal{R}_\mu$ is a Calderón-Zygmund singular integral operator. Then, for every $w \in A_p(0,\infty)$, $\mathcal{R}_\mu$ defines a bounded operator from $L_p(w)$ into itself, $1 < p < \infty$, and from $L_1(w)$ into $L_{1,\infty}(w)$. Thus we improve [4, Theorem 3.4].

References

[1] **K.F. Andersen**, *Norm inequalities for ultraspherical and Hankel conjugate functions*, Can. J. Math., 27 (1975), 162-171.

[2] **K.F. Andersen and R.A. Kerman**, *Weighted norm inequalities for generalized Hankel conjugate transformations*, Studia Math., 71 (1981), 15- 26.

[3] **K.F. Andersen and R.A. Muckenhoupt**, *Weighted weak type Hardy inequalities with applications to Hilbert transforms and maximal functions*, Studia Math., 72 (1982), 9-26.

[4] **J.J. Betancor and K. Stempak**, *On Hankel conjugate functions*, Stud. Sci. Math. Hungarica, 41 (2004), 59-91.

[5] **D. Buraczewski, M. Martínez, J.L. Torrea and R. Urban**, *On the Riesz transform for ultraspherical expansions*, preprint, 2003.

[6] **J.T. Campbell, R.L. Jones, K. Reinhold and M. Wierdl**, *Oscillation and variation for the Hilbert transform*, Duke Math. J., 105 (2000), 59-83.

[7] **J. Duoandikoetxea**, *Fourier Analysis*, Graduate Studies in Math., 29, Amer. Math. Soc., Providence, Rhode Islands, 2001.

[8] **L.M. Fernández-Cabrera and J.L. Torrea**, *Vector valued inequalities with weights*, Publ. Mat., 37 (1) (1993), 177-208.

[9] **J. García Cuerva and J.L. Rubio de Francia**, *Weighted norm inequalities and related topics*, North-Holland Mathematics Studies, 116, North-Holland Publishing Co., Amsterdam, 1985.

[10] **A.T. Gillespie and J.L. Torrea**, *Dimension free estimates for the oscillation of Riesz transforms*, to appear in *Israel J. Math.*.

[11] **R.A. Kerman**, *Generalized Hankel conjugate transformations on rearrangement invariant spaces*, Trans. Amer. Math. Soc., 232 (1977), 111-130.

[12] **R.A. Kerman**, *Boundedness criteria for generalized Hankel conjugate transformations*, Canad. J. Math., 30 (1978), 147-153.

[13] **B. Muckenhoupt and E. Stein**, *Classical expansions and their relations to conjugate harmonic functions*, Trans. Amer. Math. Soc., 118 (1965), 17-92.

[14] **S. Philipp**, *Hankel transform and GASP*, Trans. Amer. Math. Soc., 176 (1973), 59-72.

[15] **E.M. Stein**, *Topics in harmonic analysis related to the Littlewood Paley theory*, Princeton Univ. Press, Princeton, N.J., 1970.

[16] **K. Stempak**, *Conjugate expansions for ultraspherical functions*, Tohoku Math. J., II. Ser. 45 (4) (1993), 461-469.

[17] **A.H. Zemanian**, *A distributional Hankel transformation*, J. SIAM Appl. Math., 14 (1966), 561-576.

[18] **A.H. Zemanian**, *Generalized integral transformations*, Interscience Publishers, New York, 1968.

Departamento de Análisis Matemático, Universidad de La Laguna, 38271 La Laguna, Tenerife, Spain
E-mail address: jbetanco@ull.es

Institute of Mathematics, University of Wroclaw, pl. Grunwaldzki 2/4, 50-384 Wroclaw, Poland
E-mail address: dbura@math.uni.wroc.pl

Departamento de Análisis Matemático, Universidad de La Laguna, 38271 La Laguna, Tenerife, Spain
E-mail address: jcfarina@ull.es

Departamento de Matemáticas, Universidad Autónoma de Madrid, Ciudad Universitaria de Canto Blanco, 28049 Madrid, Spain
E-mail address: teresa.martinez@uam.es

Departamento de Matemáticas, Universidad Autónoma de Madrid, Ciudad Universitaria de Canto Blanco, 28049 Madrid, Spain
E-mail address: joseluis.torrea@uam.es