LIMIT THEOREMS FOR STOCHASTIC RECURSIONS WITH MARKOV DEPENDENT COEFFICIENTS

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Abstract. We consider the stochastic recursion $X_n = A_n X_{n-1} + B_n$ for Markov dependent coefficients $(A_n, B_n) \in \mathbb{R}^+ \times \mathbb{R}$. We prove the central limit theorem, the local limit theorem and the renewal theorem for partial sums $S_n = X_1 + \cdots + X_n$.

1. Introduction

In this paper we consider the affine stochastic recursion on $\mathbb{R}$ defined by the action of the affine group on the real line:

$$\begin{aligned} X_0^x &= x, \\ X_n^x &= A_n X_{n-1}^x + B_n, \quad n \in \mathbb{Z}_+, \end{aligned} \tag{1.1}$$

where $\{(A_n, B_n)\}_{n\geq 1}$ is a stationary and ergodic sequence of random variables valued in $\mathbb{R}^+ \times \mathbb{R}$. It was proved by Brandt [2], that if $\mathbb{E}[\log A_1] < 0$ and $\mathbb{E}[\log |B_1|] < \infty$, then the recursion has a unique stationary measure ν. Moreover X_n^x converges in distribution to a random variable R with the law ν and the limit R does not depend on the starting point x. One can consider also the two-sided infinite affine recursion

$$X_n = A_n X_{n-1} + B_n, \quad n \in \mathbb{Z}, \tag{1.2}$$

where $\{(A_n, B_n)\}_{n\geq 1}$ is a stationary and ergodic sequence. Then under the assumptions stated above the extended recursion has a unique solution

$$X_n = \sum_{j=0}^{\infty} \left(\prod_{i=n+1-j}^{n} A_i \right) B_{n-j} \tag{1.3}$$

and all X_n's are distributed according to ν.

The recursion (1.1) has been studied for almost forty years mainly under the assumption of independence of the random coefficients (A_n, B_n). The most significant result is due to Kesten [13] (see also Goldie [6]) who proved that if $\mathbb{E}A^\alpha = 1$ for some $\alpha > 0$ (and of course a number of further assumptions are satisfied), the stationary measure ν is α-regularly varying. After this result an enormous number of further properties of the process $\{X_n\}$ has been proved, including limit theorems for partial sums $S_n = X_1 + \cdots + X_n$ (see [1, 4, 8, 18]).

During last several years substantial progress has been done in understanding the case of dependent coefficients (A_n, B_n). Here one should mention results of Collamore [5] and Roitershtein [17] who, under different sets of assumptions, independently proved the Kesten theorem for (A_n, B_n) depending on an underlying Markov chain. Collamore [5], indeed, considered more general recursions.

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The main purpose of the present paper is to prove limit theorems related to the partial sums S_n. We study here the case when the sequence $\{(A_n, B_n)\}_{n\in\mathbb{Z}}$ is a stationary sequence of random variables modulated by some Markov process. More precisely, let $(S, \mathcal{S})$ be a measurable space with countably generated σ-field $\mathcal{S}$ and $\{s_n\}_{n\in\mathbb{Z}}$ be a stationary Markov chain with transition kernel H and stationary measure π. For any measurable function f on S we denote by $\|f\|$ the essential supremum of f with respect to the measure π. We assume that there exists a kernel G on $S \times S \times \mathcal{B}$, where $\mathcal{B}$ is the Borel σ-field on $\mathbb{R}^+ \times \mathbb{R}$ such that the transition kernel $\widetilde{H}$ on $(S \times \mathbb{R}^+ \times \mathbb{R}, \mathcal{S} \times \mathcal{B})$ given by

$$\widetilde{H}((s, \zeta), U \times W) = \int_U H(s, dt)G(s, t, W)$$

defines a Markov modulated process (MMP) associated with $\{s_n\}$, i.e. a stationary Markov chain $\{(s_n, \zeta_n)\}_{n\in\mathbb{Z}}$ on a product space $S \times (\mathbb{R}^+ \times \mathbb{R})$, where $\zeta_n = (A_n, B_n)$ (compare [17]). Notice that (s_n, ζ_n) depends only on s_{n-1}. The stationary measure $\widetilde{\pi}$ of this Markov chain is given by

$$\widetilde{\pi}(U \times W) = \int_S \widetilde{H}((s, \xi), U \times W)\pi(ds) = \int_S \int_U H(s, dt)G(s, t, W)\pi(ds)$$

Given such a sequence we consider the real valued stationary process $\{X_n^x\}$ or $\{X_n\}$ defined by (1.1) or (1.2), respectively.

Our main results do not deeply depend on the structure of the underlying Markov chain s_n. The only property of the process we need is convergence of the powers of the Markov operator H to the stationary measure π, i.e. we assume that

$$\lim_{n\to\infty} H^n(f)(s) = \pi(f) \qquad \pi \text{ a.s.}, \tag{1.4}$$

for any bounded function f. This is satisfied for a large class of Markov chains, e.g. when s_n is an aperiodic Harris recurrent Markov chain (see [16], Corollary 6.7).

To avoid considerations of the degenerate case, when $X_n = x$ a.s for some $x \in \mathbb{R}$, we assume that

$$\mathbb{P}[A_0 x + B_0 = x] < 1 \qquad \text{for every } x \in \mathbb{R}. \tag{1.5}$$

Our main results are the following

Theorem 1.6 (Central Limit Theorem)**.** *Assume that* (1.4) *and* (1.5) *are satisfied and there exists* $\gamma > 2$ *such that*

$$\Lambda(\gamma) = \left\|\mathbb{E}_s[A_1^\gamma \dots A_n^\gamma]\right\| < 1 \quad \text{and } \left\|\mathbb{E}_s[B_1^\gamma]\right\| < \infty. \tag{1.7}$$

Then $\frac{1}{\sqrt{n}}(S_n - nm)$ *converges in distribution to the normal law* $N(0, \sigma^2)$*, for some* $\sigma^2 > 0$ *and* $m = \mathbb{E}X_0$*.*

Remark. The parameter σ^2 can be explicitly computed applying methods described in [4] (Lemma 6.2).

Theorem 1.8 (Local limit theorem)**.** *Under hypotheses of the Theorem above, for every compact set* $I \subset \mathbb{R}$ *with negligible boundary*

$$\lim_{n\to\infty} \sqrt{n}\mathbb{P}\big[S_n - nm \in I\big] = C_0 l(I),$$

where l *denotes the Lebesgue measure and* $C_0 = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\frac{t^2}{2}\sigma^2} dt$*.*

Theorem 1.9 (Renewal theorem)**.** *Let* $U(I) = \sum_{n\geq 1} \mathbb{E}\Big[\mathbf{1}_I(S_n)\Big]$*. Assume* $m > 0$*,* (1.4)*,* (1.5) *and* (1.7) *holds for some* $\gamma > 1$*, then we have*

$$\lim_{y\to\infty} U(I + y) = \frac{l(I)}{m},$$

for any compact set I with negligible boundary.

Proofs of our results stated above are based on the spectral method introduced in the fifties by Nagaev to study limit theorems for certain class of Markov chains and strongly developed during last years (see e.g. [8, 7, 9, 11]). We investigate spectral properties of the Markov operator P related to the Markov chain (X_n^x, s_n) and of P_t its perturbations by Fourier characters. On appropriately defined Banach spaces the operator P is quasi-compact and has a unique peripheral eigenvalue (i.e. the eigenvalue whose norm is equal to the spectral radius) equal to 1. It turns out that for small t the peripheral eigenvalue $k(t)$ of P_t is also unique and the corresponding eigenspace is one dimensional. Thus, the problem of description of the characteristic function $\mathbb{E}_s[e^{itS_n^x}] = P_t^n \mathbf{1}(x,s)$ which is asymptotically close to $k^n(t)$, can be reduced to problem of studying behavior of $k(t)$ close to 1 for small values of t.

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2. Fourier operators and their spectral properties

Given $x \in \mathbb{R}$ we consider on $\mathbb{R} \times S$ the Markov chain (X_n^x, s_n), where X_n^x is defined in (1.1) and $s_0 = s$ for some $s \in S$. We denote by P the corresponding Markov operator

$$Pf(x,s) = \mathbb{E}_s[f(X_1^x, s_1)]$$

and by $\widetilde{\nu}(f) = \mathbb{E}[f(X_0, s_0)]$ its stationary measure, for X_0 as in (1.3), where f is an arbitrary bounded measurable function on $\mathbb{R} \times S$.

On functions on $\mathbb{R} \times S$ we introduce the seminorm

$$[f]_{\varepsilon,\lambda} = \left\| \sup_{x \neq y} \frac{|f(x,s) - f(y,s)|}{|x-y|^{\varepsilon}(1+|x|)^{\lambda}(1+|y|)^{\lambda}} \right\|$$

and the two norms

$$|f|_{\theta} = \left\| \sup_{x} \frac{|f(x,s)|}{(1+|x|)^{\theta}} \right\|$$

$$\|f\|_{\theta,\varepsilon,\lambda} = |f|_{\theta} + [f]_{\varepsilon,\lambda}.$$

Given γ, without any lose of generality, we may assume that the parameters $\varepsilon, \lambda, \gamma$ satisfy

$$\varepsilon < 1, \lambda + 3\varepsilon < \theta < 2\lambda < \gamma - 3\varepsilon, 1 + \varepsilon < \gamma. \tag{2.1}$$

Moreover we assume that all the assumptions of the limits theorems are satisfied, and from now we will refer to them without any further saying.

The norms define the Banach spaces

$$\mathbb{C}_{\theta} = \{f \colon |f|_{\theta} < \infty\},$$

$$\mathbb{B}_{\theta,\varepsilon,\lambda} = \{f \colon \|f\|_{\theta,\varepsilon,\lambda} < \infty\}.$$

On those Banach spaces we consider a family of Fourier operators related to the Markov kernel P. For $t \in \mathbb{R}$ we define

$$P_t f(x,s) = \mathbb{E}_s\left[e^{itX_1^x} f(X_1^x, s_1)\right].$$

We notice that by the Markov property

$$(P_t^n f)(x,s) = \mathbb{E}_s\left[e^{itS_n^x} f\left(X_n^x, s_n\right)\right], \tag{2.2}$$

where $S_n^x = \sum_{k=1}^n X_k^x$. In particular if $\mathbf{1}$ is the function on $\mathbb{R} \times S$ identically equal to 1, then

$$(P_t^n \mathbf{1})(x,s) = \mathbb{E}_s\left[e^{itS_n^x}\right]$$

i.e. $P_t^n\mathbf{1}$ is just the characteristic function of S_n^x, that explains the role of the operator P_t in studying limit theorems related to S_n^x.

Notice that by the Jensen inequality, (1.7) implies $\mathbb{E}[\log A_0] < 0$. Therefore the sequence $\{(A_n, B_n)\}$ satisfies the assumptions of Brandt's theorem [2]. Hence (1.2) has a unique solution given by (1.3).

Lemma 2.3. *For every $\theta < \gamma$*

$$\Big\| \sup_{n\in\mathbb{N}} \mathbb{E}_s\Big[\big|X_n^0\big|^\theta\Big]\Big\| < C(\theta) < \infty.$$

Proof. We consider two cases. If $\theta > 1$, then we use the Minkowski inequality. By the Markov property and (1.7) we have

$$\begin{aligned}
\Big\|\Big(\mathbb{E}_s\Big[\big|X_n^0\big|^\theta\Big]\Big)^{\frac{1}{\theta}}\Big\| &\le \Big\|\sum_{j=1}^n \Big(\mathbb{E}_s\Big[|B_j|^\theta \prod_{i=j+1}^n A_i^\theta\Big]\Big)^{\frac{1}{\theta}}\Big\| \\
&\le \sum_{j=1}^n \Big\|\mathbb{E}_s\Big[|B_j|^\theta \mathbb{E}_s\Big[\mathbb{E}_{s'}\Big[\prod_{i=1}^{n-i} A_i^\theta\Big]\Big| s_j = s'\Big]\Big]\Big\| \\
&\le C_\theta \sum_{j=1}^\infty \rho_\theta^{\frac{j}{\theta}} \le C(\theta).
\end{aligned}$$

For $\theta \le 1$ we can repeat the calculations above, but we use the inequality $(a+b)^\theta \le a^\theta + b^\theta$ valid for any $a, b > 0$, instead of the Minkowski inequality. □

Lemma 2.4. *For every $t \in \mathbb{R}$*

$$|P_t^n f|_\theta \le C_1 |f|_\theta.$$

Moreover there exists $\rho < 1$ such that

$$[P_t^n f]_{\varepsilon,\lambda} \le C_2 \rho^n [f]_{\varepsilon,\lambda} + C_3 |t|^\varepsilon |f|_\theta.$$

Finally for every η satisfying $\lambda + 2\varepsilon \le \eta \le \theta$, $\delta < \varepsilon$ and $s, t \in \mathbb{R}$

$$|(P_s - P_t)f|_\eta \le C_4 |s-t|^\delta \|f\|_{\theta,\varepsilon,\lambda}.$$

We omit the proof since the arguments are exactly as in the i.i.d. case. See e.g. [4] for more details. One has just to use (1.7) and Lemma 2.3.

Lemma 2.5. *The unique eigenvalue of modulus 1 of P acting on $\mathcal{B}_{\theta,\varepsilon,\lambda}$ is 1 and the corresponding eigenspace is $\mathbb{C}\mathbf{1}$*

Proof. Assume that $Pf = zf$ for some nonzero $f \in B_{\theta,\varepsilon,\lambda}$ and $z \in \mathbb{C}$, $|z| = 1$. Then for π almost every s and every x by Lemma 2.4 we have

$$\big|f(x,s) - f(0,s)\big| = \big|P^n f(x,s) - P^n f(0,s)\big| \le C\rho^n [f]_{\varepsilon,\lambda} \cdot |x|^\varepsilon (1+|x|)^\lambda.$$

Since $\rho < 1$, the value above tends to 0 as n tends to ∞. Therefore $f(0,s) = f(x,s)$ for s, π a.s. and the function f depends only on the second coordinate. The operator P acting on functions defined on S coincides with the transition probability H. Therefore, by (1.4), we have:

$$z^n f(s) = P^n f(s) = H^n f(s) \to \pi(f).$$

So, z must be equal 1 and then

$$f(s) = H^n f(s) \to \pi(f),$$

hence $f(s) = \pi(f)$, π a.s. □

In view of the last lemma we may use the Ionescu-Tulcea Marinescu theorem [12] (see also [9]) for the operator P. It says that the operator P on $\mathcal{B}_{\theta,\varepsilon,\lambda}$ is quasi-compact, i.e. in our case the Banach space $\mathcal{B}_{\theta,\varepsilon,\lambda}$ can be decomposed into a sum of two closed P invariant subspaces: $\mathcal{B}_{\theta,\varepsilon,\lambda} = \{\mathbb{C}\mathbf{1}\} \oplus \mathcal{H}$, where $\mathcal{H} = \{f \in B_{\theta,\varepsilon,\lambda} : \widetilde{\nu}(f) = 0\}$, and moreover $r(P_{|\mathcal{H}}) < 1$. For our purpose we need a uniform control of spectrums $\sigma(P_t)$ of P_t for small t and this is provided by the Keller, Liverani theorem [14].

Proposition 2.6. *There exist $t_0 > 0$, $\delta > 0$, $\rho < 1 - \delta$ such that for every $|t| < t_0$:*

- *The spectrum of P_t acting on $\mathcal{B}_{\theta,\varepsilon,\lambda}$ is contained in $D = \{z\colon |z| \le \rho\} \cup \{z\colon |z-1| < \delta\}$.*
- *The set $\sigma(P_t) \cap \{z\colon |z-1| < \delta\}$ consists of exactly one eigenvalue $k(t)$, the corresponding eigenspace is one dimensional and moreover $\lim_{t\to 0} k(t) = 1$.*
- *If π_t is the projection of P_t onto the eigenspace mentioned above, then there exists an operator Q_t with the spectral radius at most ρ, $\pi_t Q_t = Q_t \pi_t = 0$ and for every n*

$$P_t^n f = k(t)^n \pi_t(f) + Q_t^n(f), \quad f \in \mathcal{B}_{\theta,\varepsilon,\lambda}.$$

For any z belonging to the complement of D:

$$\|(z - P_t)^{-1} f\|_{\theta,\varepsilon,\lambda} \le C \|f\|_{\theta,\varepsilon,\lambda}$$

for some constant C independent on t.

The identity embedding of $\mathcal{B}_{\theta,\varepsilon,\lambda}$ into $\mathcal{B}_{\theta,\varepsilon,\lambda+\varepsilon}$ is continuous and the decomposition $P_t = k(t)\pi_t + Q_t$ coincides on both spaces and

$$\|(\pi_t - \pi_0) f\|_{\theta,\varepsilon,\lambda+\varepsilon} \le C|t|^{\varepsilon} \|f\|_{\theta,\varepsilon,\lambda}.$$

Finally

$$|k(t) - 1| \le C|t|^{\varepsilon}.$$

We omit the proof of the Proposition, see [8] and [4] for details.

Lemma 2.7. *For every $t \neq 0$, the spectral radius of P_t is strictly smaller than 1.*

Proof. Assume that there exists a nonzero function f belonging to $\mathcal{B}_{\theta,\varepsilon,\lambda}$ and z of modulus 1 such that

$$P_t f = z f.$$

The function f is bounded, since by Lemma 2.3

$$|f(x,s)| = |P_t f(x,s)| \le P(|f|)(x,s) \le |f|_\theta \mathbb{E}_s\big[(1 + |X_1^x|^\theta)\big] < \infty.$$

Therefore for every n

$$\widetilde{\nu}\big(P^n|f| - |f|\big) = 0.$$

However, the integrated function is positive:

$$|f(x,s)| = |z^n f(x,s)| = |P_t^n f(x,s)| \le P^n(|f|)(x,s).$$

Therefore for every n

$$|f(x,s)| = P^n(|f|)(x,s) \qquad \widetilde{\nu} \text{ a.s.}$$

In view of Lemma 2.5, $|f|$ must be constant.

Next the convexity argument implies that for every n, s π a.s. and every x

$$z^n f(x,s) = e^{itS_n^x} f(X_n^x, s_n) \qquad \mathbb{P} \text{ a.s.}$$

Hence

$$\frac{f(x,s)}{f(y,s)} e^{it(y-x)\sum_{j=0}^n A_1 \dots A_j} = \frac{f(X_n^x, s_n)}{f(X_n^y, s_n)} \qquad \mathbb{P} \text{ a.s..}$$

Notice that the left hand-side has a limit $\mathbb{P}$ a.s. as n tends to infinity. Since

$$\lim_{n\to\infty} \mathbb{E}\left|\frac{f(X_n^x, s_n)}{f(X_n^y, s_n)} - 1\right| = 0$$

(this can be proved exactly as in [4], Lemma 3.14), the limit of the right hand side is 1, $\mathbb{P}$ a.s. Therefore

$$\frac{f(x,s)}{f(y,s)} = e^{it(x-y)\sum_{j=0}^{\infty} A_1 \dots A_j}.$$

Since the left hand side is nonrandom and the right one is random and nonconstant, t must be equal 0. Thus, $zf = Pf$ and in view of Lemma 2.5 z must be 1, and f must be constant. □

Now our aim is to prove the following proposition, that will be crucial in proving limit theorems.

Proposition 2.8. *If $\gamma > 1$, then there exists $\delta > 0$ such that*

$$k(t) = 1 + itm + o(t^{1+\delta})$$

for $m = \int_{\mathbb{R}} x\nu(dx) = \mathbb{E}X_0$ and $\varepsilon < \delta < \gamma - 1$.

If $\gamma > 2$, then

$$k(t) = 1 + itm - t^2\sigma^2/2 + o(t^2)$$

for some $\sigma^2 > 0$.

Proof. First applying rather standard arguments (see [9, 10]) we will prove the second part of the proposition.

Assume $\gamma > 2$. Let us fix two triples $(\theta, \varepsilon, \lambda)$ and $(\theta', \varepsilon, \lambda')$ both satisfying (2.1) and additionally such that $\theta > \theta' + 2$ and $\lambda > \lambda' + 2$. For $k = 1, 2$ let us define

$$L_{k,t} f(x,s) = \mathbb{E}_s\big[(iX_1^x)^k e^{itX_1^x} f(X_1^x, s_1)\big],$$

then $L_{k,t}$ are bounded operators from $\mathcal{B}_{\theta',\varepsilon,\lambda'}$ to $\mathcal{B}_{\theta,\varepsilon,\lambda}$ and (compare [10] Proposition 6.3)

$$\lim_{h\to 0} \frac{1}{|h|^n} \left\| P_{t+h} - P_t - \sum_{k=1}^{n} \frac{h^k}{k!} L_{k,t} \right\|_{\mathcal{B}_{\theta',\varepsilon,\lambda'}, \mathcal{B}_{\theta,\varepsilon,\lambda}} = 0. \tag{2.9}$$

Let $h_t \in B_{\theta',\varepsilon,\lambda'} \subset \mathcal{B}_{\theta,\varepsilon,\lambda}$ be the eigenfunction of P_t:

$$P_t(h_t) = k(t) h_t$$

such that $\widetilde{\nu}(h_t) = 1$. Then $h_t = \frac{\pi_t(1)}{\widetilde{\nu}(\pi_t(1))}$. Notice that

$$\widetilde{\nu}(\chi_t h_t) = k(t), \tag{2.10}$$

where $\chi_t(x,s) = e^{itx}$. We will prove that $t \to h_t$ is twice differentiable. Indeed it is enough to prove that π_t has second derivative.

To compute the first derivative of π_t we will use the following formula (see [14])

$$\pi_t = \frac{1}{2\pi i} \int_{|z-1|=\delta} (z - P_t)^{-1} dz.$$

Then for $f \in \mathcal{B}_{\theta',\varepsilon,\lambda'}$ we write

$$\begin{aligned}
\frac{1}{h}(\pi_{t+h}-\pi_t)f &= \frac{1}{h}\frac{1}{2\pi i}\int_{|z-1|=\delta}\Big((z-P_{t+h})^{-1}-(z-P_t)^{-1}\Big)fdz \\
&= \frac{1}{h}\frac{1}{2\pi i}\int_{|z-1|=\delta}(z-P_{t+h})^{-1}(P_{t+h}-P_t)(z-P_t)^{-1}fdz \\
&= \frac{1}{2\pi i}\int_{|z-1|=\delta}(z-P_t)^{-1}L_{1,t}(z-P_t)^{-1}fdz \\
&\quad +\frac{1}{2\pi i}\int_{|z-1|=\delta}\Big((z-P_{t+h})^{-1}-(z-P_t)^{-1}\Big)L_{1,t}(z-P_t)^{-1}fdz \\
&\quad +\frac{1}{2\pi i}\int_{|z-1|=\delta}(z-P_{t+h})^{-1}\Big(\frac{P_{t+h}-P_t-hL_{1,t}}{h}\Big)(z-P_t)^{-1}fdz
\end{aligned}$$

By Proposition 2.6 and (2.9) the second integral and the third one go to 0. So, the derivative $\pi_t^{(1)}$ of π_t is a bounded operator from $\mathcal{B}_{\theta',\varepsilon,\lambda'}$ to $\mathcal{B}_{\theta'+1,\varepsilon,\lambda'+1}$ and

$$\pi_t^{(1)}f = \frac{1}{2\pi i}\int_{|z-1|=\delta}(z-P_t)^{-1}L_{1,t}(z-P_t)^{-1}fdz$$

In the same we may compute the second derivative of π_t

$$\pi_t^{(2)}f = \frac{1}{2\pi i}\int_{|z-1|=\delta}(z-P_t)^{-1}L_{2,t}(z-P_t)^{-1}fdz$$

Therefore h_t and hence $k(t)$ are twice differentiable at 0. In particular it can be expanded in a Taylor series

$$k(t) = 1 + k'(0)t + \frac{k''(0)}{2}t^2 + o(t^2).$$

To compute the derivative of k denote by ζ the derivative of $t \to h_t$ at 0. Then differentiating (2.10) we obtain

$$k'(0) = im + \widetilde{\nu}(\zeta).$$

However differentiating the equation $\widetilde{\nu}(h_t) = 1$ at zero we obtain $\widetilde{\nu}(\zeta) = 0$. Hence

$$k'(0) = im.$$

Proceeding as in [4] (Lemmas 6.2 and 6.7) one can explicitly compute the value of $k''(0)$ i.e. of $\sigma^2 = -k''(0)$ in terms of the function ζ, and then to prove that Lemma 2.7 implies $\sigma^2 > 0$.

Assume now $\gamma > 1$. In order to prove the proposition we proceed as in [15]. Although this paper was written under much stronger assumptions, including heavy tail of ν, the method can be adapted to our situation.

Let $\delta_t f(x,s) = f(tx,s)$. We define the second family of Fourier operators on $\mathcal{B}_{\theta,\varepsilon,\lambda}$:

$$\begin{aligned}
T_t f(x,s) &= \delta_t^{-1}P_t\delta_t f(x,s) = \mathbb{E}_s\Big[e^{i(A_1x+tB_1)}f(A_1x+tB_1,s_1)\Big], \qquad t \neq 0, \\
Tf(x,s) &= T_0 f(x,s) = \mathbb{E}_s\Big[e^{i(A_1x)}f(A_1x,s_1)\Big].
\end{aligned}$$

Notice that for $t \neq 0$, a function f is an eigenvalue of P_t if and only if $\delta_t^{-1}f$ is an eigenvalue of T_t. Moreover the function

$$h(x,s) = \mathbb{E}_s\Big[e^{ix\sum_{j=1}^{\infty}A_1\ldots A_j}\Big]$$

is the unique eigenfunction of T: $Th = h$ corresponding to eigenvalue 1 and there are no other eigenvalues of modulus 1. Indeed, by the Markov property

$$\begin{aligned} Th(x,s) &= \mathbb{E}_s\Big[e^{iA_1x}h(A_1x,s_1)\Big] \\ &= \mathbb{E}_s\Big[e^{iA_1x}\mathbb{E}_{s_1}\Big[e^{iA_1x\sum_{j=2}^{\infty}A_2\dots A_j}\Big]\Big] \\ &= \mathbb{E}_s\Big[e^{ix\sum_{j=1}^{\infty}A_1\dots A_j}\Big] = h(x,s). \end{aligned}$$

To prove uniqueness one has to argue as in proof of Lemma 2.5.

One can prove that for $\theta, \varepsilon, \lambda$ satisfying (2.1) the family T_t satisfies Proposition 2.6 and in particular for small values of t, $T_tf = k(t)\widetilde{\pi}(t)f + \widetilde{Q}(t)f$ with the same eigenvalue $k(t)$ as for P_t.

Notice that $h_t = \delta_t\widetilde{\pi}_t h$ is an eigenfunction of P_t and since $\widetilde{\nu}P = \widetilde{\nu}$ we have

$$(k(t)-1)\widetilde{\nu}(h_t) = \widetilde{\nu}P_t(h_t) - \widetilde{\nu}(h_t) = \widetilde{\nu}((\chi_t - 1)h_t)$$

Therefore, denoting by g the function $g(x,s) = x$, we obtain

$$\begin{aligned} k(t) - 1 - itm &= \frac{1}{\widetilde{\nu}(h_t)}\Big[\widetilde{\nu}((\chi_t-1)h_t) - itm + itm(1-\widetilde{\nu}(h_t))\Big] \\ &= \frac{1}{\widetilde{\nu}(h_t)}\Big[\widetilde{\nu}(\chi_t - 1 - itx) + \widetilde{\nu}\big((\chi_t-1)(h_t-1)\big) + itm\widetilde{\nu}(1-h_t)\Big] \end{aligned}$$

Now, reasoning exactly in the same way as in [15] (see the proof of Theorem 6.3 and the last inequality in it for the conclusion) we obtain that for any ρ, δ such that $0 < \rho < \gamma - 1$ and $\varepsilon < \delta < \gamma - 1$ there exists C such that

$$|h_t(x,s) - 1| \le Ct^{\delta}(1+|x|)^{\rho}.$$

Then, since $\widetilde{\nu}(h_t)$ converges to 1 as t tends to 0 and $\widetilde{\nu}(x^{1+\rho}) < \infty$ we obtain

$$\begin{aligned} |k(t) - 1 - itm| &\le C\Big[\widetilde{\nu}((tx)^{1+\delta}) + \widetilde{\nu}(tx\cdot t^{\delta}(1+|x|)^{\rho}) + tm\widetilde{\nu}(t^{\delta}(1+|x|)^{\rho})\Big] \\ &\le Ct^{1+\delta}. \end{aligned}$$

□

3. Limit theorems

Now we are going to prove limit theorems related to partial sums $S_n = X_1 + \dots + X_n$. In fact the main work has been done in the previous section and in view of Proposition 2.8 the proof is rather classical (see e.g. [9]). However for reader's convenience we present here some details.

Proof of Theorem 1.6. By Proposition 2.6, for fixed t we have

$$\begin{aligned} \lim_{n\to\infty}\mathbb{E}\Big[e^{it\frac{S_n - mn}{\sqrt{n}}}\Big] &= \lim_{n\to\infty}\int \mathbb{E}_s\Big[e^{it\frac{S_n^x - mn}{\sqrt{n}}}\Big]\widetilde{\nu}(dx\,ds) = \lim_{n\to\infty}\int e^{-itm\sqrt{n}}P^n_{\frac{t}{\sqrt{n}}}(1)(x,s)\widetilde{\nu}(dx\,ds) \\ &= \lim_{n\to\infty}\Big(e^{-i\frac{tm}{\sqrt{n}}}k\big(t/\sqrt{n}\big)\Big)^n \cdot \lim_{n\to\infty}\int \pi_{\frac{t}{\sqrt{n}}}(1)(x,s)\widetilde{\nu}(dx\,ds) + \lim_{n\to\infty}\int Q^n_{\frac{t}{\sqrt{n}}}(1)(x,s)\widetilde{\nu}(dx,ds). \end{aligned}$$

and

$$\begin{aligned}
\lim_{n\to\infty}\left|\int \pi_{\frac{t}{\sqrt{n}}}(1)(x,s)\widetilde{\nu}(ds\,dx)-1\right| &= \lim_{n\to\infty}\left|\int\Big(\pi_{\frac{t}{\sqrt{n}}}(1)(x,s)-\pi(1)(x,s)\Big)\widetilde{\nu}(ds\,dx)\right| \\
&\le C\lim_{n\to\infty}\left|\int |t/\sqrt{n}|^{\varepsilon}\big(1+|x|^{\theta}\big)\widetilde{\nu}(dx\,ds)\right| \\
&= 0, \\
\lim_{n\to\infty}\left|\int Q^n_{\frac{t}{\sqrt{n}}}(1)(x,s)\widetilde{\nu}(ds\,dx)\right| &\le C\lim_{n\to\infty}\rho^n\int\big(1+|x|^{\theta}\big)\nu(dx)=0
\end{aligned}$$

Therefore it is sufficient to prove

$$\lim_{n\to\infty}\Big(e^{-i\frac{tm}{\sqrt{n}}}k\big(t/\sqrt{n}\big)\Big)^n = e^{\lim_{n\to\infty}\left(n(e^{-i\frac{tm}{\sqrt{n}}}k(t/\sqrt{n})-1)\right)} = e^{-\frac{t^2}{2}\sigma^2}. \tag{3.1}$$

For this purpose applying Proposition 2.8 we write

$$\begin{aligned}
&\lim_{n\to\infty}\left(n\Big(e^{-i\frac{tm}{\sqrt{n}}}k(t/\sqrt{n})-1\Big)\right) \\
&\quad= \lim_{n\to\infty}\left[n\Big(1-\frac{itm}{\sqrt{n}}+\frac{t^2m^2}{n}+O(n^{-\frac{3}{2}})\Big)\Big(1+i\frac{tm}{\sqrt{n}}-\frac{t^2\sigma^2}{2n}+o(1/n)\Big)-1\right]=-\frac{t^2}{2}\sigma^2.
\end{aligned}$$

□

Proof of Theorem 1.8. In view of Theorem 10.7 [3] it is sufficient to prove

$$\lim_{n\to\infty} n^{\frac{1}{2}}\mathbb{E}\Big[h(S_n-nm)\Big]=C_0I(h)$$

for any $h\in L^1$ such that the Fourier transform of h is compactly supported. Take M such that $\mathrm{supp}\widehat{h}\subset[-M,M]$ and fix $0<\delta<t_0$. By the Fourier inversion formula and Proposition 2.6 we have

$$\begin{aligned}
\sqrt{n}\mathbb{E}\big[h(S_n-nm)\big] &= \frac{\sqrt{n}}{2\pi}\int_{\mathbb{R}}\mathbb{E}\Big[e^{it(S_n-nm)}\widehat{h}(t)\Big]dt \\
&= \frac{\sqrt{n}}{2\pi}\int\int_{\mathbb{R}} e^{-itnm}P^n_t(1)(x,s)\widehat{h}(t)dt\widetilde{\nu}(dx\,ds) \\
&= \frac{\sqrt{n}}{2\pi}\int\int_{|t|<\delta} e^{-itnm}k(t)^n\pi_t(1)(x,s)\widehat{h}(t)dt\widetilde{\nu}(dx\,ds) \\
&\quad+\frac{\sqrt{n}}{2\pi}\int\int_{|t|<\delta} e^{-itnm}Q^n_t(1)(x,s)\widehat{h}(t)dt\widetilde{\nu}(dx\,ds) \\
&\quad+\frac{\sqrt{n}}{2\pi}\int\int_{\delta<|t|<M} e^{-itnm}P^n_t(1)(x,s)\widehat{h}(t)dt\widetilde{\nu}(dx\,ds)
\end{aligned}$$

since the function $t\to\limsup_{n\to\infty}\|P^n_t1\|^{\frac{1}{n}}$ is upper semicontinuous, $r(P_t)<1$ (Lemma 2.7) and by Proposition 2.6, $r(Q_t)<1$ for $t<\delta$, the second and the third expressions converge to 0 as n tends to infinity. Therefore, since $h(0)=l(h)$, by the Lebesgue theorem and (3.1) we have

$$\begin{aligned}
\lim_{n\to\infty}\sqrt{n}\mathbb{E}\big[h(S_n-nm)\big] &= \lim_{n\to\infty}\frac{\sqrt{n}}{2\pi}\int\int_{|t|<\delta} e^{-itnm}k(t)^n\pi_t(1)(x,s)\widehat{h}(t)dt\widetilde{\nu}(dx\,ds) \\
&= \lim_{n\to\infty}\frac{1}{2\pi}\int\int_{|t|<\delta\sqrt{n}}\Big(e^{-i\frac{tm}{\sqrt{n}}}k(t/\sqrt{n})\Big)^n\pi_{\frac{t}{\sqrt{n}}}(1)(x,s)\widehat{h}(t/\sqrt{n})dt\widetilde{\nu}(dx\,ds) \\
&= \frac{l(h)}{2\pi}\int_{\mathbb{R}} e^{-\frac{t^2}{2}\sigma^2}dt.
\end{aligned}$$

□

Proof of Theorem 1.9. It is sufficient to prove

$$\lim_{y\to\infty} U(h_y) = \frac{l(h)}{m}$$

where $h \in L^1$, the Fourier transform of h is differentiable and compactly supported and $h_y(x) = h(x-y)$. We assume that the support of $\widehat{h}$ is contained in the interval $[-M, M]$ for some $M > 0$. Since $\widehat{h}_y(t) = e^{-ity}\widehat{h}(t)$, also $\mathrm{supp}\widehat{h}_y \subset [-M, M]$. For $\eta < 1$ define

$$U_\eta(h) = \sum_{n\geq 1} \eta^n \mathbb{E}[h(S_n)],$$

then by the inverse Fourier formula and Proposition 2.6 for $\delta < t_0$ we have

$$\begin{aligned}
U_\eta(h_y) &= \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{h}_y(t) \sum_{n\geq 1} \eta^n \mathbb{E}[e^{itS_n}]dt \\
&= \frac{1}{2\pi}\int\int_{\mathbb{R}} \widehat{h}_y(t) \sum_{n\geq 1} \eta^n P_t^n(1)(x,s)dt\widetilde{\nu}(dx\,ds) \\
&= \frac{1}{2\pi}\int\int_{|t|<\delta} \widehat{h}_y(t) \sum_{n\geq 1} \eta^n k^n(t)\pi_t(1)(x,s)dt\widetilde{\nu}(dx\,ds) \\
&\quad + \frac{1}{2\pi}\int\int_{|t|<\delta} \widehat{h}_y(t) \sum_{n\geq 1} \eta^n Q_t^n(1)(x,s)dt\widetilde{\nu}(dx\,ds) \\
&\quad + \frac{1}{2\pi}\int\int_{\delta<|t|<M} \widehat{h}_y(t) \sum_{n\geq 1} \eta^n P_t^n(1)(x,s)dt\widetilde{\nu}(dx\,ds)
\end{aligned}$$

The second and the third factors converge to 0 as η tends to 1 and y to infinity. Indeed, notice first that since

$$\int \left|\widehat{h}(t) \sum_{n\geq 1} \eta^n Q_t^n(1)(x,s)\right| \widetilde{\nu}(dx\,ds) \leq \left|\widehat{h}(t)\cdot \frac{\rho\eta}{1-\rho\eta}\right|,$$

for ρ being as in Proposition 2.6, the function on the left hand side is integrable on $[-\delta, \delta]$. Therefore, by the Riemann-Lebesgue theorem

$$\lim_{y\to\infty}\lim_{\eta\to 1}\left|\int\int_{|t|<\delta} e^{-iyt}\widehat{h}(t) \sum_{n\geq 1} \eta^n Q_t^n(1)(x,s)dt\widetilde{\nu}(dx\,ds)\right| = 0.$$

Similarly we deal with the third term. To estimate the first factor we decompose it further into three factors

$$\begin{aligned}
&\frac{1}{2\pi}\int\int_{|t|<\delta} e^{-ity}\frac{\eta k(t)\widehat{h}(t)}{1-\eta k(t)}\pi_t(1)(x,s)dt\widetilde{\nu}(dx\,ds) \\
&\qquad = \frac{\eta}{2\pi}\int\int_{|t|<\delta} e^{-ity}\left(\frac{\widehat{h}(t)k(t)}{1-\eta k(t)} - \frac{\widehat{h}(0)}{1-\eta(1+imt)}\right)\pi_t(1)(x,s)dt\widetilde{\nu}(dx\,ds) \\
&\quad + \frac{\eta}{2\pi}\int\int_{|t|<\delta} e^{-ity}\widehat{h}(0)\frac{\pi_t(1)(x,s)-1}{1-\eta(1+imt)}dt\widetilde{\nu}(dx\,ds) + \frac{\eta\widehat{h}(0)}{2\pi}\int_{|t|<\delta}\frac{e^{-ity}}{1-\eta(1+imt)}dt \\
&\qquad = I(\eta,y) + II(\eta,y) + III(\eta,y).
\end{aligned}$$

We will prove that I and II converge to 0 as η tends to 1 and $y \to \infty$. Indeed notice that for t sufficiently small and η close to 1:

$$|1-\eta k(t)| \geq m|t|/2 \qquad \text{and} \qquad |1-\eta(1+imt)| \geq m|t|/2.$$

Since both k and $\widehat{h}$ are differentiable there exists a continuous function ψ such that $\widehat{h}(t)k(t) = \widehat{h}(0) + t\psi(t)$. Therefore, writing

$$\begin{aligned} F(t,\eta) &= \frac{\widehat{h}(t)k(t)}{1-\eta k(t)} - \frac{\widehat{h}(0)}{1-\eta(1+imt)} \\ &= \frac{t\psi(t)}{1-\eta k(t)} + \widehat{h}(0)\cdot\frac{k(t)-1-imt}{(1-\eta k(t))(1-\eta(1+imt))}, \end{aligned}$$

we obtain that $\widetilde{F}(t) = \lim_{\eta\to 1} F(\eta,t)$ is an integrable function on the interval $[-\delta,\delta]$. Then by the Riemann-Lebesgue theorem

$$\lim_{y\to\infty}\lim_{\eta\to 1} I(y,\eta) = \lim_{y\to\infty}\int\int e^{-iyt}1_{\{|t|\leq\delta\}}\widetilde{F}(t)\pi_t(1)(x,s)dt\widetilde{\nu}(dx\,ds) = 0.$$

Similarly we prove that II converges to 0. Indeed, from Proposition 2.6 follows integrability of the function $t \to \frac{\int(\pi_t(1)(x,s)-1)\widetilde{\nu}(dxds)}{imt}$. Finally

$$\begin{aligned} \lim_{y\to\infty} U(h_y) &= \lim_{y\to\infty}\lim_{\eta\to 1} U_\eta(h_y) = \lim_{y\to\infty}\lim_{\eta\to 1}\frac{1}{2\pi}\int_{|t|<\delta}\frac{e^{-ity}}{1-\eta(1+imt)}dt \\ &= \lim_{y\to\infty}\frac{1}{2\pi m}\left(\pi + \int_{|t|<\delta y}\frac{\sin t}{dt}\right)\cdot l(h) \\ &= \frac{l(h)}{m}, \end{aligned}$$

where the last but one equality was proved in [9], page 47. □

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