BOUNDED PLURIHARMONIC FUNCTIONS ON SYMMETRIC IRREDUCIBLE SIEGEL DOMAINS

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ABSTRACT. Let $\mathcal{D}$ be an irreducible, symmetric Siegel domain and let S be a solvable group which acts simply transitively on $\mathcal{D}$. We exhibit three S-invariant, real, second order, degenerate elliptic operators $\mathbf{L}$, $\mathcal{L}$, $\mathbf{H}$ such that a bounded function F on $\mathcal{D}$ is pluriharmonic, if and only if $\mathbf{L}F = \mathcal{L}F = \mathbf{H}F = 0$. The three operators are the same as in our former paper [DHMP], however there we needed a considerably stronger condition on F to derive the same conclusion.

1. Introduction

Let $\mathcal{D}$ be a symmetric Siegel domain and let S be a solvable Lie group acting simply transitively on $\mathcal{D}$. Our aim is to study bounded pluriharmonic functions by means of S-invariant operators. More precisely, these are real S-invariant second order, elliptic degenerate operators L annihilating holomorphic functions F and, consequently, $\Re F$ and $\Im F$. Such operators will be called admissible.

The particular interest in restricting our attention to the second order, degenerate elliptic operators is caused by the fact that for such an operator there is a very well understood potential theory. To wit, theory of bounded functions harmonic with respect to left-invariant operators L satisfying Hörmander condition was studied in [D], [DH], [DHP]. The origin of this research goes back to H.Furstenberg [F], Y.Guivarc'h [G], and A.Raugi [Ra] who developed a probabilistic approach to bounded functions on groups, harmonic with respect to a probability measure or a second order degenerate elliptic invariant operator. The fundamental result of the theory says that bounded L-harmonic functions are precisely the integrals of their boundary values on a nilpotent subgroup $N(L)$ of S against an appropriate Poisson kernel.

This observation changes considerably our way of looking at functions on symmetric domains. We study S-invariant objects instead of studying the ones invariant under the whole group of isometries G. It turns out that they are more suitable as far as characterization of bounded pluriharmonic functions is the goal, though clearly the space of bounded pluriharmonic functions is invariant under G.

For an admissible L satisfying the Hörmander condition the boundary $N(L)$ always contains the group $N(\Phi)$ that acts simply transitively on the Bergmann-Shilov boundary. In this paper we are mostly interested in operators for which $N(L) = N(\Phi)$. Each of the operators gives rise to a new Poisson kernel. We exploit intensively those Poisson kernels on $N(\Phi)$ to prove the following theorem:

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Let $\mathcal{D}$ be an irreducible symmetric Siegel domain. There are three admissible operators $\mathbf{L}$, $\mathcal{L}$, $\mathbf{H}$ on $\mathcal{D}$ such that if a real valued bounded function F on $\mathcal{D}$ satisfies $\mathbf{L}F = \mathcal{L}F = \mathbf{H}F = 0$, then F is pluriharmonic. If $\mathcal{D}$ is a tube domain, $\mathbf{L}$ and $\mathbf{H}$ are sufficient and if $\mathcal{D}$ is biholomorphically equivalent to the complex ball, $\mathcal{L}$ and $\mathbf{H}$ are sufficient.

This theorem is an improvement of the result obtained in [DHMP], where a more restrictive, not G invariant type of condition was considered. Namely, the functions studied there had the property

$$(H^2) \qquad \sup_{z\in\mathcal{D}} \int_{N(\Phi)} |F(v\cdot z)|^2 \, dv < \infty.$$

As it is proved in [DHMP] the conjugate function $\widetilde{F}$ i.e. the function such that $F + i\widetilde{F}$ is holomorphic, also satisfies (H^2). But already for the upper half-plane the conjugate function $\widetilde{F}$ of a bounded real harmonic (= pluriharmonic) function F is not bounded, in general. Consequently, the same is true for an arbitrary Siegel domain. Except for the last remark, the "bounded" result implies the H^2 result.

In any case of the above theorem, the sum of operators needed to yield the conclusion is an elliptic operator. The Poisson integral characterization says that the space of bounded functions harmonic with respect to one elliptic operator L is always bigger then the space of bounded holomorphic functions. But due to our theorem three of them, in general, or two in special cases, while appropriately chosen are sufficient to obtain the smallest possible space of common zeros - the pluriharmonic functions. For other global characterizations of pluriharmonic functions on symmetric Siegel domains we refer to [L1], [L2], [BBG], [DH1], [J].

Let Ω be the underlying symmetric cone in the Jordan algebra V. Suppose V is the center of the step two nilpotent group $N(\Phi) = \mathcal{Z} \times V$. Let $f(\zeta, u) = f_\zeta(u)$ be the boundary value of F. If $f \in L^2(N(\Phi))$, the basic strategy of the proof , used in [DHMP], is to show that for every $\zeta \in \mathcal{Z}$,

$$(1.1) \qquad \mathrm{supp}\hat{f}_\zeta \subset \bar{\Omega} \cup -\bar{\Omega}$$

and, in the nontube case, that the integrated representation U_f^λ, $\lambda \in \bar{\Omega} \cup -\bar{\Omega}$ is zero on the invariant subspaces which do not contain the vacuum state. The rest is then pretty standard. If $f \in L^\infty(N(\Phi))$, new difficulties arise. One is caused by the fact that 0 may belong to the support of $\hat{f}_\zeta$. To deal with that we produce a sequence of functions F_n with boundary values f_n such that

$$(1.2) \qquad \begin{gathered} \mathbf{L}F_n = \mathcal{L}F_n = \mathbf{H}F_n = 0 \ , \\ \mathrm{supp} f_n(\cdot, \hat{\cdot}) \cap \mathcal{Z} \times \{0\} = \emptyset \ , \\ (F - \lim_{n\to\infty} F_n) \ \text{ is a constant} \end{gathered}$$

(Section 4). Hence we reduce to the functions satisfying (1.2). To prove that f_n satisfies (1.1) we use the operator $\mathbf{H}$ which is basically the Laplace-Beltrami operator on the product of upper half planes (Section 5). The basic idea is the same as in [DHMP], but we have to take into account every point in the complement of $\bar{\Omega} \cup -\bar{\Omega}$ not only almost every one as for L^2-functions. All this requires more delicate algebraic consideration (Lemma 5.7) using the structure of Ω.

Finally, we cannot use the operator U_f^λ directly and we have to replace this by a "weak" argument. We do this to prove our theorem for the Siegel upper half-plane ($N(\Phi)$ being the Heisenberg group, Section 6). Then, for the general case we use Siegel upper-half planes imbedded in S in various directions. This final step simplifies considerably the whole argument, also the one in [DHMP].

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2. Preliminaries

2.1. **Jordan algebras and irreducible cones.** Let Ω be an irreducible symmetric cone in a Euclidean space $(V, \langle \cdot, \cdot \rangle)$. V has structure of a simple Euclidean Jordan algebra [FK], and

$$\Omega = \text{ int}\,\{x^2 : \ x \in V\}$$

([FK], Theorem III.2.1).

We are going to use the language of Jordan algebras to describe solvable groups acting simply transitively on Ω, and so we recall briefly some basic facts which will be needed later. The reader is referred to [FK] for more details.

We fix a Jordan frame $\{c_1, \dots, c_r\}$ in V. This is a complete system of orthogonal primitive idempotents:

$$\begin{aligned} &c_i^2 = c_i, \ c_i c_j = 0 \quad \text{if } i \neq j, \\ &c_1 + \cdots + c_r = e\,, \end{aligned}$$

such that none of the $c_1, ..., c_r$ is a sum of two nonzero idempotents. The length r is independent of the choice of Jordan frame an it is called the rank of V.

Let $L(x)$ be the self - adjoint endomorphism of V given by multiplication by x, i.e.

$$L(x)y = xy.$$

Let

$$V = \bigoplus_{1 \le i \le j \le r} V_{ij} \tag{2.1}$$

be the common diagonalization of the commuting family of self-adjoint endomorphisms $L(c_1), ..., L(c_r)$. It is called be the Peirce decomposition of V ([FK], Theorem IV.2.1). This means that V is the orthogonal direct sum (2.1) and $L(c_k)$ restricted to V_{ij} has the eigenvalue $\frac{1}{2}(\delta_{ik} + \delta_{jk})$. Moreover,

$$\begin{aligned} &V_{ij} \cdot V_{ij} \subset V_{ii} + V_{jj}\,, \\ &V_{ij} \cdot V_{jk} \subset V_{ik}, \quad \text{if} \quad i \neq k\,, \\ &V_{jk} \cdot V_{jl} \subset V_{kl}, \ \text{if } k < l, \\ &V_{ij} \cdot V_{kl} = \{0\}, \ \text{if } \{i, j\} \cap \{k, l\} = \emptyset\,. \end{aligned} \tag{2.2}$$

All V_{ij}, $i < j$ have the same dimension d and $V_{jj} = \mathbf{R} c_j$, $j = 1, ..., r$.

For each $i < j$ we fix once for all an orthonormal basis of V_{ij} which we note e_{ij}^{α} with $1 \leq \alpha \leq d$. To simplify the notation we write $c_i = e_{ii}^1$. Then e_{ij}^{α} is an orthonormal basis of V.

Let G be the connected component of the group $G(\Omega)$ of all transformations in $GL(V)$ which leave Ω invariant. Its Lie algebra will be denoted by $\mathcal{G}$. $\mathcal{G}$ contains $L(x)$ for all $x \in V$.

The choice of the Jordan frame determines a solvable Lie group $S_0 = N_0 A$ acting simply transitively on Ω. $A = \exp \mathcal{A}$, where $\mathcal{A}$ is the Abelian subalgebra of $\mathcal{G}$ consisting of elements

$$H = L(a), \; a = \sum_{j=1}^{r} a_j c_j \in \bigoplus_i V_{ii}.$$

For the nilpotent part $N_0 = \exp \mathcal{N}_0$, we have

$$\mathcal{N}_0 = \bigoplus_{i<j\leq r} \mathcal{N}_{ij},$$

where, (cf. [FK]),

$$\mathcal{N}_{ij} = \{ z \Box c_i : z \in V_{ij} \}.$$

2.2. **Symmetric Siegel domains.** Let $V^{\mathbf{C}} = V + iV$ be the complexification of V. We extend the action of $G(\Omega)$ to $V^{\mathbf{C}}$.

In addition, suppose that we are given a complex vector space $\mathcal{Z}$ and a Hermitian symmetric bilinear mapping

$$\Phi : \; \mathcal{Z} \times \mathcal{Z} \to V^{\mathbf{C}}.$$

We assume that

$$\begin{aligned} &\Phi(\zeta, \zeta) \in \overline{\Omega}, \quad \zeta \in \mathcal{Z}, \\ \text{and} \quad &\Phi(\zeta, \zeta) = 0 \;\text{ implies } \zeta = 0. \end{aligned}$$

The Siegel domain associated with these data is defined as

$$\mathcal{D} = \{ (\zeta, z) \in \mathcal{Z} \times V^{\mathbf{C}} : \; \Im z - \Phi(\zeta, \zeta) \in \Omega \}.$$

There is an representation $\sigma : G(\Omega) \ni g \mapsto \sigma(g) \in Gl(\mathcal{Z})$ such that

$$g\Phi(\zeta, \omega) = \Phi(\sigma(g)\zeta, \sigma(g)\omega)\,, \tag{2.3}$$

and all automorphisms $\sigma(a) \in A$ admit a joint diagonalization. The transformation

$$(\zeta, z) \mapsto (\sigma(g)\zeta, \; gz)$$

is a holomorphic automorphism of $\mathcal{D}$, [KW]. The elements $\zeta \in \mathcal{Z}$, $x \in V$ and $g \in G(\Omega)$ act on $\mathcal{D}$ in the following way:

$$\begin{aligned} \zeta \circ (\omega, z) &= (\zeta + \omega, z + 2i\Phi(\omega, \zeta) + i\Phi(\zeta, \zeta))\,, \\ x \circ (\omega, z) &= (\omega, z + x)\,, \\ g \circ (\omega, z) &= (\sigma(g)\omega, gz)\,. \end{aligned} \tag{2.4}$$

The first two actions generate a two-step nilpotent (*or abelian, if* $\mathcal{Z} = 0$) group $N(\Phi)$ of biholomorphic automorphisms of $\mathcal{D}$:

$$(\zeta, x)(\omega, y) = (\zeta + \omega, x + y + 2\Im\, \Phi(\zeta, \omega)). \tag{2.5}$$

All three actions (g restricted to S_0) generate a solvable Lie group $S = N(\Phi)S_0$, the group $N(\Phi)$ being a normal subgroup of S.

For $X \in \mathcal{S}_0$, (2.3) implies that

$$X\Phi(\zeta,\omega) = \Phi(\sigma(X)\zeta,\omega) + \Phi(\zeta,\sigma(X)\omega)\,, \tag{2.6}$$

where σ denotes both representation of the group $G(\Omega)$ and the Lie algebra $\mathcal{G}$. An easy consequence of (2.6) is that the only possible eigenvalues for $\sigma(H)$, $H \in \mathcal{A}$ are $\frac{\lambda_j(H)}{2}$, $j = 1, ..., r$. So we may write

$$\mathcal{Z} = \bigoplus_{j=1}^{r} \mathcal{Z}_j$$

with the property that

$$\sigma(H)\zeta = \frac{\lambda_j(H)}{2}\zeta \quad \text{for } \zeta \in \mathcal{Z}_j.$$

(A standard argument is contained e.g. in [DHMP]).

Finally, the Lie algebra $\mathcal{S}$ of S has the following decomposition

$$\mathcal{S} = \mathcal{N}(\Phi) \oplus \mathcal{S}_0 = \left(\bigoplus_{j=1}^{r} \mathcal{Z}_j\right) \oplus \left(\bigoplus_{i\le j} V_{ij}\right) \oplus \left(\bigoplus_{i<j} \mathcal{N}_{ij}\right) \oplus \mathcal{A}.$$

that corresponds to a diagonalization of the adjoint action of $\mathcal{A}$:

$$\begin{aligned}
[H,X] &= \frac{\lambda_j(H)}{2}X && \text{for } X \in \mathcal{Z}_j,\\
[H,X] &= \frac{\lambda_i(H)+\lambda_j(H)}{2}X && \text{for } X \in V_{ij},\\
[H,X] &= \frac{\lambda_j(H)-\lambda_i(H)}{2}X && \text{for } X \in \mathcal{N}_{ij},
\end{aligned} \tag{2.7}$$

where $\lambda_j(H) = \lambda_j(\sum_{k=1}^{r} a_k c_k) = a_j$. If $\mathcal{Z} = 0$ then $N(\Phi) = V$ and

$$\mathcal{D}_T = V + i\Omega$$

is a tube domain. To distinguish those two cases the solvable group for the tube will be denoted $S_T = VS_0$. Clearly

$$\mathcal{S}_T = \left(\bigoplus_{i\le j} V_{ij}\right) \oplus \left(\bigoplus_{i<j} \mathcal{N}_{ij}\right) \oplus \mathcal{A}.$$

In virtue of (2.4) we identity $\mathcal{D}$ with S. More precisely let $\mathbf{e} = ie \in \mathcal{D}$ and let

$$\theta : S \ni s \mapsto \theta(s) = s \cdot ie \in \mathcal{D} \tag{2.8}$$

Then θ is a diffeomorphism of S and $\mathcal{D}$ (or S_T and $\mathcal{D}_T$ in the particular case of a tube domain). θ identifies also the spaces of smooth functions on S and $\mathcal{D}$.

The Lie algebra S is then identified with the tangent space $T_{\mathbf{e}}$ of $\mathcal{D}$ at $\mathbf{e}$ by the differential $d\theta_{\mathbf{e}}$. We then transport the Bergmann metric g and the complex structure $\mathcal{J}$ from $\mathcal{D}$ to S, where they become left-invariant tensors on S. Moreover, the complexified tangent space $T_{\mathbf{e}}^{\mathbf{C}}$ is identified with $\mathcal{S}^{\mathbf{C}}$ and the decomposition $T_{\mathbf{e}}^{\mathbf{C}} = T_{\mathbf{e}}^{1,0} \oplus T_{\mathbf{e}}^{0,1}$ is transported into

$$\mathcal{S}^{\mathbf{C}} = \mathcal{Q} \oplus \mathcal{P}.$$

Now we pick up a g-orthonormal basis of $\mathcal{S}$ and the corresponding basis of $\mathcal{Q}$. For $j < k$ and $1 \leq \alpha \leq d$, we define $X_{jk}^{\alpha} \in V_{jk}, Y_{jk}^{\alpha} \in \mathcal{N}_{jk}$ as the left-invariant vector fields on S corresponding to e_{jk}^{α} and $2e_{jk}^{\alpha}\Box c_j$, respectively. For each j we define X_j and H_j as left-invariant vector fields on S corresponding to c_j and $L(c_j)$ respectively. Finally, we choose an orthonormal basis $\{e_{j\alpha}\}$, $\alpha = 1, ..., n$ related to $4\langle c_j, \Phi(\cdot,\cdot)\rangle$. For $\zeta_{j\alpha} = x_{j\alpha} + iy_{j\alpha}$ the corresponding coordinates, we define $\mathcal{X}_j^{\alpha}, \mathcal{Y}_j^{\alpha}$ as the left-invariant vector fields on S which coincide with $\partial_{x_{j\alpha}}$ and $\partial_{y_{j\alpha}}$ at $\mathbf{e}$. Then $X_j, X_{jk}^{\alpha}, H_j, Y_{jk}^{\alpha}, \mathcal{X}_j^{\alpha}, \mathcal{Y}_j^{\alpha}$ form a g-orthonormal basis of $\mathcal{S}$ and $Z_j = X_j - iH_j$, $Z_{jk}^{\alpha} = X_{jk}^{\alpha} - iY_{jk}^{\alpha}$, $\mathcal{Z}_j^{\alpha} = \mathcal{X}_j^{\alpha} - i\mathcal{Y}_j^{\alpha}$ form an orthonormal basis of $\mathcal{Q}$ with respect to the Hermitian scalar product $(\cdot,\cdot) = \frac{1}{2}g(\cdot,\bar{\cdot})$. For the detailed calculation of $d\theta_{\mathbf{e}}$ and $\mathcal{J}$ see [BBDHPT]. In the coordinates

$$(\zeta, z) = \Big(\sum_{j,\alpha} \zeta_{j\alpha} e_{j\alpha}, \sum_{\substack{i\leq j\\ \alpha}} z_{ij}^{\alpha} e_{ij}^{\alpha}\Big),$$

Z_j, Z_{jk}^{α}, $\mathcal{Z}_j^{\alpha}$ are left-invariant vector fields corresponding to ∂_{z_j}, $\partial_{z_{jk}^{\alpha}}$, $\partial_{z_{j\alpha}}$ at $\mathbf{e}$.

2.3. **Admissible operators.** Under identification (2.8), holomorphic functions on $\mathcal{D}$ are called holomorphic functions on S. The left-invariant differential operators on S that annihilate holomorphic functions and are real second order and elliptic degenerate are called admissible (see [DHP], [DHMP] and [BBDHPT] for more details about admissible operators). In particular, the second order left-invariant operators Δ_j, Δ_{jk}^{α}, $\mathcal{L}_j^{\alpha}$ with the property

$$\begin{aligned}
\Delta_j(\mathbf{e}) &= \partial_{z_j}\partial_{\bar{z}_j}(\mathbf{e}),\\
\Delta_{jk}^{\alpha}(\mathbf{e}) &= \partial_{z_{jk}^{\alpha}}\partial_{\bar{z}_{jk}^{\alpha}}(\mathbf{e}),\\
\mathcal{L}_j^{\alpha}(\mathbf{e}) &= \partial_{z_j^{\alpha}}\partial_{\bar{z}_j^{\alpha}}(\mathbf{e}).
\end{aligned}$$

are such. More explicitly, (see e.g. [DHMP]):

$$\begin{aligned}
\Delta_j &= X_j^2 + H_j^2 - H_j,\\
\mathcal{L}_j^{\alpha} &= (\mathcal{X}_j^{\alpha})^2 + (\mathcal{Y}_j^{\alpha})^2 - H_j,\\
\Delta_{ij}^{\alpha} &= (X_{ij}^{\alpha})^2 + (Y_{ij}^{\alpha})^2 - H_j.
\end{aligned} \tag{2.9}$$

We will use also left-invariant vector fields on $N(\Phi)$ that coincide with ∂_{x_j}, $\partial_{x_{ij}}^{\alpha}$, $\partial_{x_j^{\alpha}}$, $\partial_{y_j^{\alpha}}$ at $\mathbf{e}$. They will be denoted by

$$\widetilde{X}_j, \widetilde{X}_{ij}^{\alpha}, \widetilde{\mathcal{X}}_j^{\alpha}, \widetilde{\mathcal{Y}}_j^{\alpha}$$

respectively. Notice that

$$XF(\zeta, u, y, a) = (\mathrm{Ad}_{ya}\widetilde{X})F_{ya}(\zeta, u). \tag{2.10}$$

Then

$$\begin{aligned}
\Delta_j F(\zeta, u, y, a) &= a_j^2\left((\mathrm{Ad}_y\widetilde{X}_j)^2 + \partial_{a_j}^2\right)F(\zeta, u, y, a),\\
\mathcal{L}_j^{\alpha} F(\zeta, u, y, a) &= a_j\left((\mathrm{Ad}_y\widetilde{\mathcal{X}_j^{\alpha}})^2 + (\mathrm{Ad}_y\widetilde{\mathcal{Y}_j^{\alpha}})^2 - \partial_{a_j}\right)F(\zeta, u, y, a).
\end{aligned}$$

2.4. **Poisson integrals.** In this section, we recall some general results about bounded harmonic functions on S. Let L be a second order left-invariant operator on S satisfying Hörmander condition. Denote by $\pi_A(L)$ its image on A under the homomorphism $S \to A = S/N$. The first order part Z in $\pi_A(L)$ determines the Poisson boundary for L.

More precisely, let $W = \{\lambda_1, ..., \lambda_r, \frac{\lambda_1}{2}, ..., \frac{\lambda_r}{2}, \frac{\lambda_i+\lambda_j}{2}, \frac{\lambda j-\lambda_i}{2}, i < j\}$ and let

$$W_1 = \{\eta \in W : \ \eta(Z) < 0\},$$

$$\begin{aligned} \mathcal{N}(L) &= \oplus_{\eta\in W_1}\mathcal{N}_\eta, \qquad \mathcal{N}^+(L) = \oplus_{\eta\in W\setminus W_1}\mathcal{N}_\eta,\\ N(L) &= \exp \mathcal{N}(L), \qquad N^+(L) = \exp \mathcal{N}^+(L). \end{aligned}$$

Then the bounded L-harmonic functions on S are in one-one correspondence with L^∞ functions on $N(L)$ via the following Poisson integral

$$F(s) = \int_{N(L)} f(s \circ x) P_L(x) \ dx, \tag{2.11}$$

where $x \to s \circ x$ denotes the action on $N(L)$ as on the quotient space $N/N^+(L)$.

We will need this representation for the case when $N(L) = N(\Phi)$ (the particular case $N(\Phi) = V$ included). Then (2.11) becomes

$$F(s) = F(xs_0) = F_{s_0}(x) = \int_{N(\Phi)} f(xs_0us_0^{-1}) P_L(u) \ du \tag{2.12}$$

and f is *weak limit of F_{ya_t} when $\lambda_j(\log a_t) \to 0$ for $j = 1, ..., r$. The function f is called the boundary value of F.

In what follows we will consider functions $F(\zeta, u, y, a)$ harmonic with respect to an admissible operator L with the additional property that for every ζ, $F_\zeta(u, y, a) = F(\zeta, u, y, a)$ is annihilated by an admissible operator $\mathbf{L}$ on S_T. Moreover, we assume $N(L) = N(\Phi)$, $N(\mathbf{L}) = V$. Thus F has a well defined boundary value f on $N(\Phi)$ such that for every $\zeta \in \mathcal{Z}$, $f_\zeta(u) = f(\zeta, u)$ is the boundary value of F_ζ on V. Then on top of (2.12), we have another representation of F. Namely,

$$F(\zeta, u, y, a) = \int_V f_\zeta(u - v) P_{ya}(v) \ dv, \tag{2.13}$$

where $P_{ya}(v) = \det a^{-1} P_{ya}((ya)^{-1} \circ v)$ and $\det a$ is the determinant of the action $V \ni v \mapsto a \circ v \in V$.

3. Main theorem

Let $\mathbf{L}$ be an elliptic admissible operator on S_T. $\mathbf{L}$ may be considered also as a left-invariant operator on S.

Given strictly positive numbers $\alpha_1, ..., \alpha_r$ and $\beta_1, ..., \beta_r$ let

$$\mathbf{H} = \sum_{j=1}^{r} \alpha_j \Delta_j \tag{3.1}$$

$$\mathcal{L} = \sum_{\substack{j=1\\ \gamma}}^{r} \beta_j \mathcal{L}_j^\gamma. \tag{3.2}$$

We are going to prove the following

Main Theorem 3.3. *Let $\mathcal{D}$ be an irreducible symmetric Siegel domain, and let F be a bounded function on $\mathcal{D}$ such that*

$$\begin{aligned} &(i) && \mathbf{L}F = 0, \\ &(ii) && \mathbf{H}F = 0, \\ &(iii) && \mathcal{L}F = 0. \end{aligned}$$

Then F is a pluriharmonic function.

If the domain $\mathcal{D}$ is of type I, then condition (iii) is void. If $\mathcal{D}$ is biholomorphically equivalent to the complex ball then $\mathbf{L}$ may be taken to be a multiple of $\mathbf{H}$, so (i) is void.

Let us make some comments concerning the role of $\mathbf{H}$. It turns out that for a bounded function F on S we have

$$\mathbf{H}f = 0 \quad \Longleftrightarrow \quad \forall_j \quad \Delta_j F = 0\,. \tag{3.4}$$

Indeed, since V is a normal subgroup of S_T, we may write $S_T = N_0 V A$. In the coordinates $y \in N_0,\ x \in V,\ a \in A$, for a smooth function g on $N_0 V A$ we have

$$\Delta_j g(yxa) = a_j^2(\partial_{x_j}^2 + \partial_{a_j}^2) g(yxa).$$

Therefore $\mathbf{H} = \sum_{j=1}^r \alpha_j a_j^2(\partial_{x_j}^2 + \partial_{a_j}^2)$ is the Laplace-Beltrami operator on the product of r-hyperbolic half-planes

$$\mathcal{D} = \mathbf{R}_1 \times \mathbf{R}_1^+ \times ... \times \mathbf{R}_r \times \mathbf{R}_r^+,$$

the metric being scaled by α_j^{-1} on $\mathbf{R}_j \times \mathbf{R}_j^+$. Thus every bounded $\mathbf{H}$-harmonic function on $\mathcal{D}$ is the Poisson integral of a L^∞ function on

$$\partial\mathcal{D} = \mathbf{R}_1 \times ... \times \mathbf{R}_r,$$

the Poisson kernel being the tensor product of the Poisson kernels for the hyperbolic half-plane $\mathbf{R} \times \mathbf{R}^+$, [Ko]. Consequently, $\mathbf{H}g = 0$ implies that g is annihilated by each of the operators Δ_j, $j = 1, ..., r$.

This leads us to the following

Proposition 3.5. *Suppose F is a bounded function annihilated by the operators (i)-(iii). Then, adding to $\mathbf{L}$ and $\mathcal{L}$ appropriate linear combination of Δ_j's with non-negative coefficients, by Proposition (2.1) in [DHMP], $\mathbf{L}$ and $\mathcal{L}$ can be replaced by operators such that the maximal boundary of L is V and the maximal boundary for $\mathbf{L} + \mathcal{L}$ is $N(\Phi)$.*

Consequently, from now on we assume that the operators $\mathbf{L}$ and $\mathcal{L}$ in the main theorem have their maximal boundaries V and $N(\Phi)$, correspondingly. We then have

$$\lim_{a\to 0} F(\zeta, x, y, a) = f(\zeta, x) = f_\zeta(x)$$

in the *weak sense both on $N(\Phi)$ and V. Moreover, (2.13) holds.

Convolving (on $N(\Phi)$) from the left by a $\mathcal{S}(N(\Phi))$ function ρ we may assume that

$$f = \rho * \tilde{f}, \quad \rho \in \mathcal{S}(N(\Phi)), \quad f \in L^\infty(N(\Phi)). \tag{3.6}$$

Throughout the rest of the paper we assume that F satisfies the assumptions of the main theorem and (3.6)

Now the proof of the main theorem splits into four parts.

In the first part we deal with the distributional partial Fourier transform of f_ζ along V. We show that there is a sequence ψ_n of functions on V such that the Fourier transform $\hat{\psi}_n$ vanishes in a neighbourhood of 0 in V, $F_n(\zeta,x,y,a) = f_\zeta *_V \psi_n *_V P_{ya}$ satisfy (i)-(iii) and are *-weakly convergent to F+constant (Section 4). For the remaining three parts we deal with functions F which satisfy (i)-(iii) and such that the support of the Fourier transform of the corresponding f_ζ does not contain 0. In the second part we prove the main theorem for the tube domains (Section 5), in the third part for the type II domains which are biholomorphically equivalent to the complex ball (Section 6). Finally in the fourth part the proof for the arbitrary irreducible symmetric domains of type II is reduced to the proof for the cases settled earlier (Section 7).

4. Partial Fourier transform

Let ϕ be a Schwartz function on V such that

$$\hat{\phi}(\lambda) = \begin{cases} 1 & \text{for} \quad |\lambda| \le 1 \\ 0 & \text{for} \quad |\lambda| \ge 2. \end{cases}$$

For a given sequence $\{k_n\}_{n=1,2,...}$ of natural numbers tending to infinity, let

$$\psi_n(x) = \frac{1}{k_n^d}\phi\left(\frac{x}{k_n}\right). \tag{4.1}$$

Given a bounded function g on V there exists $\{k_n\}_{n=1,2,...}$, $k_n \to \infty$ such that

$$\lim_{n\to+\infty}\langle g, \psi_n\rangle \quad \text{exists.}$$

We need a bit more here: we are going to select a sequence $\{k_n\}_{n=1,2,...}$ such that for all $\zeta \in \mathcal{Z}$

$$\lim_{n\to+\infty}\langle f_\zeta, \psi_n\rangle \quad \text{exists and is independent of} \quad \zeta.$$

The details are as follows:

Lemma 4.2. *Let $\rho \in \mathcal{S}(N(\Phi))$ and $f \in L^\infty(N(\Phi))$. Then the mapping*

$$\mathcal{Z} \ni \zeta \to \rho * f(\zeta,\cdot) \in L^\infty(V)$$

is continuous.

Proof. Indeed, we prove that for each compact subset K of $\mathcal{Z}$

$$\sup_{\zeta\in K, x\in V} |\rho * f(\zeta+h,x) - \rho * f(\zeta,x)| \le C_K|h|.$$

By (2.5) we have

$$\begin{aligned}&|\rho * f(\zeta+h,x) - \rho * f(\zeta,x)|\\ &\qquad \le \|f\|_{L^\infty(N(\Phi))}\int_{N(\Phi)} \left|\rho\Big((h,-2\Im\Phi(h,\zeta))(\omega,u)^{-1}\Big) - \rho\Big((\omega,u)^{-1}\Big)\right| d\omega du.\end{aligned}$$

If h and ζ are in a compact set, so are all elements $(h,-2\Im\Phi(h,\zeta)) = \exp[X]$ with $|X| \le C|h|$. Moreover,

$$\rho\Big((h,-2\Im\Phi(h,\zeta))(\omega,u)^{-1}\Big) - \rho\Big((\omega,u)^{-1}\Big) = \int_0^1 \frac{d}{dr}\rho\Big(\exp[rX](\omega,u)^{-1}\Big)dr.$$

Hence

$$\begin{aligned} |\rho * f(\zeta+h,x)-\rho * f(\zeta,x)| &\leq \|f\|_{L^\infty(N(\Phi))}\int_{N(\Phi)}\left|X\rho\Big((\omega,u)^{-1}\Big)\right|d\omega du \\ &\leq C\|f\|_{L^\infty(N(\Phi))}|h|. \end{aligned}$$

□

Lemma 4.3. *Let $g\in L^\infty(V)$ and let ψ_n be as in (4.1) be such that $\lim_{n\to\infty}\langle g,\psi_n\rangle$ exists. Then for every $\gamma\in L^1(V)$ such that $\int_V\gamma\neq 0$ we have*

$$\lim_{n\to+\infty}\langle g *_V \gamma,\psi_n\rangle=\lim_{n\to+\infty}\langle g,\psi_n\rangle\int_V\gamma.$$

Proof. Assume $\int\gamma=1$. We have $\langle\gamma * g,\psi_n\rangle=\langle g,\gamma^* * \psi_n\rangle$ and

$$|\langle g,\gamma^* * \psi_n\rangle-\langle g,\psi_n\rangle|\leq\|g\|_{L^\infty(V)}\|\gamma^* * \psi_n-\psi_n\|_{L^1(V)}.$$

But

$$\gamma^* * \psi_n(x)-\psi_n(x)=\int_V\gamma^*(y)\Big(\psi_n(x-y)-\psi_n(x)\Big)dy.$$

Hence

$$|\langle g,\gamma^* * \psi_n\rangle-\langle g,\psi_n\rangle|\leq\|g\|_{L^\infty(V)}\int_V\int_V|\gamma(y)|\left|\phi\left(x-\frac{y}{k_n}\right)-\phi(x)\right|dxdy\to 0.$$

□

Lemma 4.4. *Let $f=\rho * \widetilde{f}$, with $\widetilde{f}\in L^\infty(N(\Phi))$, $\rho\in\mathcal{S}(N(\Phi))$ and let ψ_n be as in (4.1). There is a subsequence $\{k_n\}_{n=1,2,\ldots}$ such that for every $\zeta\in\mathcal{Z}$*

$$\lim_{n\to+\infty}\langle f_\zeta,\psi_n\rangle=H(\zeta)\quad\textit{exists.}$$

Proof. Let $\zeta_1,\zeta_2,\ldots$ be a dense subset of $\mathcal{Z}$. Using the above lemma and the diagonal method, we select a subsequence ψ_n such that $\lim_{n\to+\infty}\langle f_{\zeta_j},\psi_n\rangle$ exists for all j. Hence, by Lemma 4.2, the limit exists for all $\zeta\in\mathcal{Z}$. □

Lemma 4.5. *Assume that F satisfies the assumptions of Theorem 3.3 and let its boundary value be $f=\rho * \widetilde{f}$, $\widetilde{f}\in L^\infty(N(\Phi))$, $\rho\in\mathcal{S}(N(\Phi))$. Then the function*

$$\mathcal{Z}\ni\zeta\to H(\zeta)=\lim_{n\to\infty}\langle f_\zeta,\psi_n\rangle$$

is bounded and harmonic with respect to the Euclidean Laplacian on $\mathcal{Z}$, so it is a constant function.

Proof. For positive t let $p_t(\zeta,u)$ be the fundamental solution of

$$\widetilde{\mathcal{L}}-\partial_t=\sum_{j,\alpha}(\widetilde{\mathcal{X}_j^\alpha})^2+(\widetilde{\mathcal{Y}_j^\alpha})^2-\partial_t.$$

We restrict our function F to the submanifold

$$\{(\zeta,x)\exp[tH]:(\zeta,x)\in N(\Phi),\ H=\sum_{j=1}^r H_j,\ t\in\mathbf{R}\}.$$

More precisely, let $a_0=\exp H$ and

$$\widetilde{F}(\zeta,u,t)=F(\zeta,u,e,ta_0),\quad t>0.$$

Then

$$(\mathcal{L}F)(\zeta, u, e, ta_0) = t(\widetilde{\mathcal{L}} - \partial_t)\widetilde{F}(\zeta, u, t).$$

Hence

$$\widetilde{F}(\zeta, u, t) = F(\zeta, u, e, ta_0) = f * p_t(\zeta, u).$$

On the other hand,

$$F(\zeta, u, e, ta_0) = f_\zeta *_V P_{ta_0}(u)$$

and so by Lemmas 4.4 and 4.3

$$\begin{aligned} H(\zeta) &= \lim_{n\to\infty} \langle f_\zeta, \psi_n\rangle = \lim_{n\to\infty} \langle f_\zeta *_V P_{ta_0}, \psi_n\rangle \\ &= \lim_{n\to\infty} \langle f * p_t(\zeta, \cdot), \psi_n\rangle. \end{aligned} \tag{4.6}$$

But

$$\begin{aligned} \langle f * p_t(\zeta, \cdot), \psi_n\rangle &= \int_{\mathcal{Z}}\int_V\int_V f(\zeta - \omega, u - v - 2\Im\Phi(\zeta, \omega))p_t(\omega, v)dv\psi_n(u)dud\omega \\ &= \int_{\mathcal{Z}} \langle f_{\zeta-\omega} *_V p_t(\omega, u + 2\Im\Phi(\zeta, \omega)), \psi_n\rangle d\omega \end{aligned}$$

Now

$$\lim_{n\to+\infty} \langle f_{\zeta-\omega} *_V p_t(\omega, u + 2\Im\Phi(\zeta, \omega)), \psi_n\rangle = H(\zeta - \omega)\int_V p_t(\omega, u)du.$$

Therefore, by (4.6)

$$H(\zeta) = H *_{\mathcal{Z}} \overline{p}_t(\zeta), \tag{4.7}$$

where $\overline{p}_t(\zeta) = \int_V p_t(\zeta, u)\ du$ and $*_{\mathcal{Z}}$ denotes the Abelian convolution on $\mathcal{Z}$. Since $\overline{p}_t$ is the usual heat kernel corresponding to the Laplace operator on $\mathcal{Z}$, (4.7) says that $H(\zeta)$ is harmonic on $\mathcal{Z}$, so it is constant. □

To summarize, the above lemmas allow to formulate the main result of this section:

Theorem 4.8. *Assume that F satisfies the assumptions of the main theorem and let its boundary value be $f = \rho * \widetilde{f}$, $\widetilde{f} \in L^\infty(N(\Phi))$, $\rho \in \mathcal{S}(N(\Phi))$. Denote by $f(\zeta, \hat{\lambda})$ the distributional partial Fourier transform of f along V. Let*

$$\begin{aligned} \eta_n(x) &= k_n^d\phi(k_n x) - k_n^{-d}\phi(k_n^{-1}x), \\ f_n &= \eta_n *_V f, \\ F_n(\zeta, x, y, a) &= (f_n)_\zeta *_V P_{ya}(x). \end{aligned}$$

Then

$$supp f_n(\cdot, \hat{\cdot}) \subset \mathcal{Z} \times \{\lambda \in V :\ 2k_n \geq |\lambda| \geq k_n^{-1}\},$$

F_n is annihilated by **L**, **H** *and $\mathcal{L}$ and there is a constant c such that the sequence F_n tends to $F + c$.*

Proof. We select $\{k_n\}_{n=1,2...}$ as in Lemma 4.4. Then by Lemmas 4.3 and 4.5

$$\lim_{n\to+\infty} \psi_n *_V f_\zeta *_V P_{ya}(x) = \lim_{n\to+\infty} \langle f_\zeta, \psi_n\rangle = c.$$

Since $\{k_n^d\phi(k_n\cdot)\}_{n=1,2,...}$ is an approximate identity, the conclusion follows. □

5. Tube domains

In this section we are going to prove the main theorem for the tube domains.

Theorem 5.1. *Let F be a real bounded function on a tube domain $\mathcal{D}_T$ such that $\mathbf{L}F = 0$ and $\mathbf{H}F = 0$. Then F is pluriharmonic.*

Let f be the boundary value of F. In view of the previous section we may assume that for an $\varepsilon > 0$

$$\mathrm{supp}\hat{f} \subset \{\varepsilon < |\lambda| < \varepsilon^{-1}\} \tag{5.2}$$

So we restrict ourselves to this class of functions. We are going to show that

$$\mathrm{supp}\hat{f} \subset \overline{\Omega} \cup -\overline{\Omega}. \tag{5.3}$$

Assume (5.3) has been proved. From (5.2) and (5.3) it follows that

$$f = f_1 + f_2 \quad \text{with} \quad \hat{f}_1 = \hat{f}|_{\bar{\Omega}}, \quad \hat{f}_2(-\lambda) = \overline{\hat{f}_1(\lambda)} \quad f_1, f_2 \in L^\infty(V).$$

(Notice that $f_j = \phi_j *_V f$ for appropriate Schwartz functions ϕ_j on V. $\hat{\phi}_1$ is 1 on a neighbourhood of $\mathrm{supp}\hat{f} \cap \bar{\Omega}$, $\mathrm{supp}\hat{\phi}_1 \cap -\bar{\Omega} = \emptyset$ and $\hat{\phi}_2(\lambda) = \hat{\phi}_1(-\lambda)$). Then

$$F = F_1 + F_2 \quad \text{with} \quad F_1(xya) = f_1 *_V P_{ya}(x), \quad F_2(xya) = f_2 *_V P_{ya}(x),$$

F_1 is holomorphic, F_2 is anti-holomorphic and $F_2 = \bar{F}_1$. To see the latter, we write

$$f_n = e^{i\frac{1}{n}\langle x,e\rangle} f_1(x), \quad g_n = e^{-i\frac{1}{n}\langle x,e\rangle} f_2(x),$$

so

$$\mathrm{supp}\hat{f}_n \subset \Omega, \quad \mathrm{supp}\hat{g}_n \subset -\Omega$$

whence

$$\widehat{f_n *_V P_{ya}}(\lambda) = \hat{f}_n(\lambda)e^{-\langle \lambda, ya\cdot e\rangle}, \quad \widehat{g_n *_V P_{ya}}(\lambda) = \hat{g_n}(\lambda)e^{\langle \lambda, ya\cdot e\rangle}$$

which implies that F_1 is a limit of holomorphic functions $f_n *_V P_{ya}$, F_2 is a limit of anti-holomorphic functions $g_n *_V P_{ya}$.

For the proof of (5.3) we need few lemmas.

Lemma 5.4. *For every λ, the function $ya \mapsto \widehat{P}_{ya}(\lambda)$ is smooth.*

Proof. Let $P(xya) = P_{ya}(x)$. Then $\mathbf{L}P = 0$ and so, there is an elliptic operator $\widehat{\mathbf{L}}$ in variables ya such that

$$\widehat{(\mathbf{L}P)}_{ya}(\lambda) = \widehat{\mathbf{L}}\widehat{P}_{ya}(\lambda).$$

□

Lemma 5.5. *For $\lambda \in supp\hat{f}$*

$$\widehat{P}_{ya}(\lambda) = e^{-\sum_{i=1}^{r} a_i |W_i(\lambda,y)|},$$

where $W_i(\lambda, y) = \langle \lambda, Ad_y c_i\rangle$.

Proof. By (3.4), for every j we have

$$0 = \Delta_j F(xya) = \Delta_j(f * P_{ya}(x)) = f * (\Delta_j P)_{ya}(x),$$

where $P(xya) = P_{ya}(x)$ and $(\Delta_j P)_{ya}(x) = \Delta_j P(xya)$. Now we use the following theorem of Wiener [Ru]: Let $f \in L^\infty(V)$, $g \in L^1(V)$ and $f *_V g = 0$. Then $\operatorname{supp}\hat{f} \subset \{\lambda : \hat{g}(\lambda) = 0\}$. Hence if $\lambda \in \operatorname{supp}\hat{f}$ we have

$$0 = \widehat{(\Delta_j P)}_{ya}(\lambda) = a_j^2 \Big(\partial_{a_j}^2 - \langle \lambda, \mathrm{Ad}_y e_j \rangle^2\Big) \widehat{P}_{ya}(\lambda).$$

Since $|\hat{P}_{ya}(\lambda)| \leq 1$ for every $\lambda \in V$, $ya \in S_0$, the uniqueness of the solution of the above system of equations completes the proof of Lemma 5.5. □

For a λ we write λ_{ij}, $1 \leq j \leq i \leq n$ for its coordinates.

Lemma 5.6 (DHMP)**.** *Let $y \in N$. Then*

$$Ad_y c_j = c_j + y^j + L(e - c_j)(y^j \cdot y^j)$$

and so

$$W_j(\lambda, y) = \langle \lambda, c_j \rangle + \sum_{k>j} \langle \lambda_{jk}, y_{jk} \rangle + \sum_{j<k\leq l<r} \langle \lambda_{kl}, L(e - c_j)(y_{jk} \cdot y_{jl}) \rangle.$$

In the set of pairs of natural numbers we introduce the lexicographic order:

$$(i,j) < (k,l) \Leftrightarrow (i < k) \vee (i = k \ \wedge \ j < l)$$

Lemma 5.7. *If $\lambda \notin \overline{\Omega} \cup -\overline{\Omega}$, then there exist m and $y_1, y_2 \in N$ such that*

$$W_m(\lambda, y_1) > 0 \quad \textit{and} \quad W_m(\lambda, y_2) < 0.$$

Proof. Let $(k,l) = \max\{(i,j) : \ \lambda_{ij} \neq 0)\}$. Then
Case 1: $k = l = 1$.

This means that only one coordinate of λ is not equal to zero, whence $\lambda \in \overline{\Omega} \cup -\overline{\Omega}$.
Case 2: $k < l$.

Suppose $y_{ij} = 0$ for $(i,j) \neq (k,l)$. Then by Lemma (5.6)

$$W_k(\lambda, y) = \lambda_{kk} + \langle \lambda_{kl}, y_{kl} \rangle,$$

because $L(e - c_k)(y_{kl} \cdot y_{kl}) \in V_{ll}$ and $\lambda_{ll} = 0$. Consequently, we can choose y_{kl} twice in such a way that W_k changes sign.
Case 3: $1 < k = l$.

We may assume $\lambda_{kk} < 0$. From $\lambda \notin -\overline{\Omega}$ it follows that there exists $x \in \Omega$ such that

$$\langle \lambda, x \rangle > 0.$$

Also, by Proposition (VI.3.5) [FK], there is $y \in N$ such that

$$x = \mathrm{Ad}_y(\sum_{j=1}^r \lambda_j c_j),$$

where $\lambda_j > 0$. Consequently,

$$0 < \langle \lambda, x \rangle = \langle \lambda, \mathrm{Ad}_y(\sum_{j=1}^r \lambda_j c_j) \rangle = \sum_{j=1}^r \lambda_j \langle \lambda, \mathrm{Ad}_y c_j \rangle. \tag{5.8}$$

Notice that by Lemma (5.6), $W_p(\lambda, y) = 0$ for $p > k$ and $W_k(\lambda, y) = \lambda_{kk}$. Indeed,

$$L(e - c_p)(y^p \cdot y^p) \in \bigoplus_{p<l\leq s\leq r} V_{ls} \quad \text{so} \quad \langle \lambda, L(e - c_p)(y^p \cdot y^p)\rangle = 0.$$

We have also $\langle \lambda, y^p\rangle = 0$ for $p \geq k$. It follows from (5.8) that there is $m < k$ such that

$$W_m(\lambda, y) = \langle \lambda, \mathrm{Ad}_y c_m\rangle > 0$$

We are going to exhibit a y' such that $W_m(\lambda, y') < 0$. We assume that $y_{ij} = 0$ for $(i, j) \neq (m, k)$. Then

$$\begin{aligned} W_m(\lambda, y) &= \lambda_{mm} + \langle \lambda_{mk}, y_{mk}\rangle + \langle \lambda_{kk}, L(e - c_m)(y_{mk} \cdot y_{mk})\rangle \\ &= \lambda_{mm} + \langle \lambda_{mk}, y_{mk}\rangle + \lambda_{kk}\langle y_{mk}, y_{mk}\rangle. \end{aligned}$$

But $\lambda_{kk} < 0$, so selecting y_{mk} sufficiently large we obtain the y'. □

Corollary 5.9. *Under assumptions of Theorem (5.1)*

$$supp\hat{f} \subset \bar{\Omega} \cup -\bar{\Omega}\,.$$

Proof. If $\lambda \in \mathrm{supp}\hat{f} \cap (\overline{\Omega} \cup -\overline{\Omega})^c$, then on one hand side the function $y \to \widehat{P}_{ya}(\lambda)$ is smooth, and on the other, one of the $W_k(\lambda, y)$'s changes sign, which contradicts smoothness of the function and so (5.3) follows. □

6. The Heisenberg group

In this section we prove the main theorem in the case when $\mathcal{Z} = \mathbf{C}^n$, $V = \mathbf{R}$, $\Omega = \mathbf{R}^+$ and

$$\Phi(\zeta, \omega) = \frac{1}{4}\langle \zeta, \omega\rangle = \frac{1}{4}\sum_{j=1}^{n} \zeta_j\overline{\omega_j}.$$

Then

$$\mathcal{D} = \{(\zeta, z) \in \mathbf{C}^n \times \mathbf{C} : \Im z > \frac{1}{4}|\zeta|^2\}$$

may be identified with $N(\Phi)A$, where $A = \mathbf{R}^+$ and $N(\Phi)$ is the Heisenberg group $\mathcal{Z} \oplus V$:

$$(\zeta, u)(\zeta', u') = (\zeta + \zeta', u + u' + \frac{1}{2}\Im\langle \zeta, \zeta'\rangle).$$

In this case we have only two operators

$$\mathbf{H} = \Delta_1 = X^2 + H^2 - H$$

i.e.

$$\Delta_1 F(\zeta, u, a) = a^2(\partial_u^2 + \partial_a^2)F(\zeta, u, a)\ , \tag{6.1}$$

$$\mathcal{L} = \mathcal{L}_1 = \sum_\alpha \left(\mathcal{X}^\alpha\right)^2 + \left(\mathcal{Y}^\alpha\right)^2 - nH \tag{6.2}$$

i.e.

$$\mathcal{L}F(\zeta, u, a) = a(L_B - n\partial_a)F(\zeta, u, a), \tag{6.3}$$

where

$$L_B = \sum_\alpha \left(\widetilde{\mathcal{X}}^\alpha\right)^2 + \left(\widetilde{\mathcal{Y}}^\alpha\right)^2.$$

Let F be a real valued function on S such that $\Delta_1 F = \mathcal{L}F = 0$. Let f be its boundary value. By (6.1) and (6.2) we have

$$F(\zeta, u, a) = f *_{\mathbf{H}^n} p_a(\zeta, u) = f_\zeta *_{\mathbf{R}} P_a(u). \tag{6.4}$$

where $p_a(\zeta, u)$ is the fundamental solution for $L_B - n\partial_a$ and $P_a(u) = \frac{1}{\pi}\frac{a}{a^2+u^2}$ is the Poisson kernel for the Laplace operator $\partial_u^2 + \partial_a^2$. By Theorem (4.8), we may assume that for a positive ε

$$\mathrm{supp} f(\zeta, \hat{\lambda}) \subset \mathbf{C}^n \times \{\lambda : \varepsilon < |\lambda| < \varepsilon^{-1}\} \ .$$

Now proceeding as at the beginning of section 5 we take

$$f_j = \phi_j *_{\mathbf{R}} f_\zeta(u). \tag{6.5}$$

and

$$F_j(\zeta, u, a) = f_j *_{\mathbf{H}^n} p_a(\zeta, u) = (f_j)_\zeta *_{\mathbf{R}} P_a(u). \tag{6.6}$$

Then

$$F = F_1 + F_2 \quad \text{with} \quad F_2 = \bar{F}_1$$

and it remains to prove

Theorem 6.7. *F_1 is holomorphic.*

The proof is based on the elementary theory of unitary representations of the Heisenberg group for which we refer to [T]. Let U_λ be the Schrödinger representation of $\mathbf{H}^n$, ([T], 1.2.1). In the underlying Hilbert space $\mathcal{H}_\lambda = L^2(\mathbf{R}^n)$ we consider the basis consisting of properly scaled Hermite functions ξ_α^λ (1.4.18 and section 2.1 of [T]). Let

$$\phi_{\alpha,\beta}^\lambda(\zeta, u) = (U_{(\zeta,u)}^\lambda \xi_\alpha^\lambda, \xi_\beta^\lambda).$$

Then

$$\phi_{\alpha,\beta}^\lambda(\zeta, u) = (2\pi)^{n/2} e^{i\lambda u} \Phi_{\alpha,\beta}(\sqrt{|\lambda|}\zeta), \tag{6.8}$$

where $\Phi_{\alpha,\beta}$ are the special Hermite functions, ([T], 1.4.19) These functions belong to the Schwartz class on $\mathbf{C}^n$ and

$$L_B \phi_{\alpha,\beta}^\lambda(\zeta, u) = -(2|\alpha| + n)|\lambda| \phi_{\alpha,\beta}^\lambda(\zeta, u). \tag{6.9}$$

Let

$$e_k^\lambda(\zeta, u) = \sum_{|\alpha|=k} \phi_{\alpha,\alpha}^\lambda(\zeta, u)$$

and

$$\psi_\phi^k(\zeta, u) = \int_{\mathbf{R}} e_k^\lambda(\zeta, u) \phi(\lambda) d\lambda. \tag{6.10}$$

for a $\phi \in C_c^\infty(\mathbf{R} \backslash \{0\})$.
Then, $\psi_\phi^k \in \mathcal{S}(\mathbf{C}^n \times \mathbf{R})$ and by (6.9)

$$L_B \psi_\phi^k = -(2k + n) \psi_{\phi'}^k \ ,$$

where $\phi'(\lambda) = |\lambda| \phi(\lambda)$.

Lemma 6.11. *For every $k \neq 0$ and $\phi \in C_c^\infty(\mathbf{R} \setminus \{0\})$*

$$\int_{\mathbf{H}^n} F_1(\eta, t, a)\overline{\psi_\phi^k(\eta, t)}d\eta dt = 0. \tag{6.12}$$

Proof. By (6.5) and (6.6)

$$F_1(\zeta, \hat{\lambda}, a) = f_1(\zeta, \hat{\lambda})e^{-a\lambda}.$$

Hence

$$(\partial_u + i\partial_a)F_1(\zeta, u, a) = 0$$

and so

$$(L_B - in\partial_u)F_1(\zeta, u, a) = 0. \tag{6.13}$$

Let $\phi \in C_c^\infty(\mathbf{R} \setminus \{0\})$ and $\widetilde{\phi}(\lambda) = \lambda^{-1}\phi(\lambda)$. By (6.13), (6.10) and (6.8),

$$\begin{aligned} 0 &= \int_{\mathbf{H}^n} F_1(\eta, t, a)\overline{(L_B - in\partial_u)\psi_{\widetilde{\phi}}^k(\eta, t)}d\eta dt \\ &= 2k\int_{\mathbf{H}^n} F_1(\eta, t, a)\overline{\psi_\phi^k(\eta, t)}d\eta dt \end{aligned}$$

and (6.12) follows. □

Using Lebesgue dominated convergence theorem, by (6.12), we have

$$\int_{\mathbf{H}^n} f_1(\eta, t)\overline{\psi_\phi^k(\eta, t)}d\eta dt = 0. \tag{6.14}$$

for $k \neq 0$ and $\phi \in C_c^\infty(\mathbf{R} \setminus \{0\})$.
Furthermore, f_1 translated by any element $(\zeta, u) \in \mathbf{H}^n$ on the left is the boundary value of F_1 translated on the left by (ζ, u). Therefore, the same proof give us:

$$f_1 * \psi_\phi^k(\zeta, u) = \int_{\mathbf{H}^n} f_1((\zeta, u)(\eta, t))\overline{\psi_{\widetilde{\phi}}^k(\eta, t)}d\eta dt = 0. \tag{6.15}$$

for $k \neq 0$ and $\phi \in C_c^\infty(\mathbf{R} \setminus \{0\})$.
For an $L^2(\mathbf{H}^n)$ function f_1, (6.15) means that all the spectral projections $f_1 * e_k^\lambda$ vanish for $k \neq 0$. So, clearly (6.15) can be viewed as a weak version of that. For a good function f_1, the next step would be an application of the Fourier inversion formula to F_1. Since we are not in $L^2(\mathbf{H}^n)$ we have to do it in a slightly more delicate way. Namely, we expand the function

$$g(\zeta, u) = \phi_1 *_{\mathbf{R}} p_a(\zeta, u)$$

and we get

$$g(\zeta, u) = (2\pi)^{-n-1}\sum_{k=0}^{\infty}\int_{-\infty}^{\infty} g *_{\mathbf{H}^n} e_k^\lambda(\zeta, u)|\lambda|^n d\lambda \tag{6.16}$$

(see theorem 2.1.1 [T]), where the above series converges in $L^2(\mathbf{H}^n)$ norm.

Proposition 6.17.

$$\int_{-\infty}^{\infty} g *_{\mathbf{H}^n} e_k^\lambda(\zeta, u)|\lambda|^n d\lambda \in \mathcal{S}(\mathbf{H}^n) \tag{6.18}$$

and the series

$$\sum_{k=0}^{\infty}\int_{-\infty}^{\infty} g *_{\mathbf{H}^n} e_k^{\lambda}(\zeta,u)|\lambda|^n d\lambda \tag{6.19}$$

converges in $L^1(\mathbf{H}^n)$.

To prove this proposition we need more information about $\Phi_{\alpha,\alpha}$. Let L_k be the k-th Laguerre polynomial, i.e.

$$L_k(t)e^{-t} = \frac{1}{k!}\left(\frac{d}{dt}\right)^k\left(e^{-t}t^k\right).$$

Given a multiindex $\alpha = (\alpha_1, ..., \alpha_n)$ and $\zeta \in \mathbf{C}^n$ let

$$L_\alpha(\zeta) = L_{\alpha_1}(\frac{1}{2}|\zeta_1|^2) \times ... \times L_{\alpha_n}(\frac{1}{2}|\zeta_n|^2). \tag{6.20}$$

Then

$$\Phi_{\alpha,\alpha}(\zeta) = (2\pi)^{-n/2}L_\alpha(\zeta)e^{-\frac{1}{4}|\zeta|^2}, \tag{6.21}$$

(see [T] 1.4.20)

We will use the following well-known property of the Laguerre functions.

Lemma 6.22. *For every $l, p \in \mathbf{N}$ there exist $c = c(l,p)$ and $M = M(l,p)$ such that*

$$\int_0^{\infty}(1+t)^l|\partial_t^p L_k(t)|^2 e^{-t}dt \le ck^M. \tag{6.23}$$

Proof. To verify (6.23) we recall 5.1.13 and 5.1.14 in [Sz], which imply that

$$\frac{d}{dt}L_k(t) = -\sum_{j=0}^{k-1}L_j(t).$$

Hence, it suffices to have (6.23) for $p = 0$. But, for $p = 0$, (6.23) follows by the orthogonality and the recurrence relations (5.1.1, 5.1.10 in [Sz])

$$\int_0^{\infty}L_j(t)L_k(t)e^{-t}dt = \delta_{j,k},$$

$$tL_k = (2k+1)L_k - (k+1)L_{k+1} - (k-1)L_{k-1}.$$

□

Proof of proposition (6.17). For (6.18) we prove that

$$\int_{-\infty}^{\infty} g *_{\mathbf{H}^n} e_k^{\lambda}(\zeta,u)|\lambda|^n d\lambda = \psi^k_{\eta_k^a}(\zeta,u) \tag{6.24}$$

with

$$\eta_k^a(\lambda) = e^{-(\frac{2k}{n}+1)|\lambda|a}\widehat{\phi}_1(\lambda)|\lambda|^n.$$

Indeed,

$$\begin{aligned} g *_{\mathbf{H}^n} e_k^{\lambda}(\zeta,u) &= \sum_{|\alpha|=k}\int \phi_1 *_{\mathbf{R}} p_a(\eta,t)\phi^{\lambda}_{\alpha,\alpha}((\eta,t)^{-1}(\zeta,u))d\eta dt \\ &= \sum_{|\alpha|=k}(U^{\lambda}_{(\zeta,u)}\xi^{\lambda}_{\alpha}, U^{\lambda}_{\phi_1 *_{\mathbf{R}} p_a}\xi^{\lambda}_{\alpha}). \end{aligned} \tag{6.25}$$

But

$$U^\lambda_{\phi_1 *_{\mathbf{R}} p_a} = \widehat{\phi}_1(\lambda) U^\lambda_{p_a}$$

and

$$U^\lambda_{p_a} \xi^\lambda_\alpha = e^{-(\frac{2|\alpha|}{n}+1)|\lambda| a} \xi^\lambda_\alpha \tag{6.26}$$

(To obtain (6.26) it is enough to solve the equation $(L_B - n\partial_a)p_a(\zeta, u) = 0$ on the Fourier transform side).

Now putting (6.26) into (6.25) we get

$$g *_{\mathbf{H}^n} e^\lambda_k(\zeta, u) = \sum_{|\alpha|=k} e^{-(\frac{2k}{n}+1)|\lambda| a} \widehat{\phi}_1(\lambda) \phi^\lambda_{\alpha,\alpha}(\zeta, u)$$

which implies (6.24).

To estimate $L^1(\mathbf{H}^n)$ norm of $\psi^k_{\eta^a_k}$ we write

$$\psi^k_{\eta^a_k}(\zeta, u) = (2\pi)^{\frac{n}{2}} \int_{\mathbf{R}} e^{i\lambda u} \eta^a_k(\lambda) \sum_{|\alpha|=k} \Phi_{\alpha,\alpha}(\sqrt{\lambda}\zeta) d\lambda.$$

Hence the Schwartz inequality yields

$$\begin{aligned}
&\int_{\mathbf{H}^n} |\psi^k_{\eta^a_k}(\zeta, u)| du d\zeta \le c \int_{\mathbf{H}^n} (1+u^2) |\psi^k_{\eta^a_k}(\zeta, u)|^2 du d\zeta \\
&\le c \left(\int_{\mathbf{C}^n} \int_{\mathbf{R}} |\partial_\lambda (\eta^a_k(\lambda) \sum_{|\alpha|=k} \Phi_{\alpha,\alpha}(\sqrt{\lambda}\zeta))|^2 d\zeta d\lambda + \int_{\mathbf{C}^n} \int_{\mathbf{R}} |\eta^a_k(\lambda) \sum_{|\alpha|=k} \Phi_{\alpha,\alpha}(\sqrt{\lambda}\zeta)|^2 d\zeta d\lambda \right) \\
&\qquad \le c_1 e^{-c_2(\frac{2k}{n}+1)a} \sum_{|\alpha|=k} \left(\int_{\mathbf{C}^n \times [\varepsilon, \varepsilon^{-1}]} \left(|\partial_\lambda (\Phi_{\alpha,\alpha}(\sqrt{\lambda}\zeta))|^2 + |\Phi_{\alpha,\alpha}(\sqrt{\lambda}\zeta)|^2 \right) d\zeta d\lambda \right)
\end{aligned}$$

Now the relation (6.21) and lemma (6.22) imply that the last integral is dominated by ck^M and so (6.19) follows. □

Now we are able to expand F_1. By (6.15) and proposition (6.17) we have

$$\begin{aligned}
F_1(\zeta, u, a) &= f_1 *_{\mathbf{H}^n} (\phi_1 *_{\mathbf{R}} p_a)(\zeta, u) \\
&= \sum_k f_1 *_{\mathbf{H}^n} \psi^k_{\eta^a_k}(\zeta, u) \\
&= f_1 *_{\mathbf{H}^n} \psi^0_{\eta^a_0}(\zeta, u),
\end{aligned}$$

where

$$\psi^0_{\eta^a_0}(\zeta, u) = (2\pi)^{\frac{n}{2}} \int_0^\infty e^{-\lambda a} \phi^\lambda_{0,0}(\zeta, u) \widehat{\phi}_1(\lambda) \lambda^n d\lambda.$$

So it suffices to prove that

$$G(\zeta, u, a) = e^{-\lambda a} \phi^\lambda_{0,0}(\zeta, u)$$

is holomorphic. We have,

$$G(\zeta, u, a) = (2\pi)^{-n/2} e^{-\lambda a} e^{i\lambda x} e^{-\frac{1}{4}\lambda|\zeta|^2} = (2\pi)^{-n/2} e^{i\lambda(x + ia + \frac{1}{4}i|\zeta|^2)} = e^{i\lambda z}$$

which proves theorem (6.7)

7. Type II domains

Let F be a function on S that satisfies (3.6) and the assumptions of Theorem (3.3). In view of Theorem (4.8) and (5.3) we may assume that its boundary value f satisfies

$$\mathrm{supp} f_1 \subset \mathcal{Z} \times (\Omega \cap \{\lambda : \varepsilon^{-1} > |\lambda| > \varepsilon\}).$$

Then

$$F(\zeta, u, y, a) = F_1(\zeta, u, y, a) + F_2(\zeta, u, y, a),$$

where

– $\mathcal{L}F_1 = \mathcal{L}F_2 = 0$,
– $F_2 = \overline{F_1}$,
– for every fixed ζ, F_1 is holomorphic on S_T and F_2 anti-holomorphic on S_T,
– the boundary value f_j of F_j is $f_j = \phi_j *_V f$ for ϕ_j as in Section 5.

We are going to prove the following

Theorem 7.1. *Let F be a bounded function on S such that*

$$(X_j + iH_j)F = 0 \quad \text{for } j = 1, \dots, r, \tag{7.2}$$

$$(X_{ij}^\alpha + iY_{ij}^\alpha)F = 0 \quad \text{for } 1 \le i < j \le r,\ \alpha = 1, ..., d, \tag{7.3}$$

$$\mathcal{L}F = 0. \tag{7.4}$$

Then F is holomorphic. If instead of (7.2), (7.3), (7.4) we have

$$(X_j - iH_j)F = 0 \quad \text{for } j = 1, \dots, r \tag{7.2$'$}$$

$$(X_{ij}^\alpha - iY_{ij}^\alpha)F = 0 \quad \text{for } 1 \le i < j \le r,\ \alpha = 1, ..., d \tag{7.3$'$}$$

$$\mathcal{L}F = 0 \tag{7.4}$$

then F is anti-holomorphic.

Clearly F_1 satisfies (7.2)-(7.4), while F_2 satisfies (7.2'), (7.3') and (7.4). The proof of the first part and the second part are identical, so we show only the first one. We do it in two steps formulated in the following lemmas.

Lemma 7.5. *Let*

$$\mathcal{L}_j = \sum_\alpha \mathcal{L}_j^\alpha.$$

Assume that a bounded function F satisfies (7.2)-(7.4). Then for every j, $\mathcal{L}_j F = 0$.

Lemma 7.6. *Let F be a bounded function on S satisfying (7.2), (7.3) and*

$$\mathcal{L}_j F = 0 \quad \text{for} \quad j = 1, ..., r. \tag{7.7}$$

Then F is holomorphic.

Proof of Lemma (7.5). Let

$$F_{ya}(\zeta, u) = F(\zeta, u, y, a).$$

By (7.2) and (2.10)

$$\partial_{a_j} F(\zeta, u, y, a) = i\mathrm{Ad}_y(\widetilde{X_j}) F_{ya}(\zeta, u).$$

Therefore, (7.4) implies

$$\left(\sum_{j,\alpha} a_j\Big(\mathrm{Ad}_y(\widetilde{\mathcal{X}}_j^\alpha)\Big)^2+\Big(\mathrm{Ad}_y(\widetilde{\mathcal{Y}}_j^\alpha)\Big)^2-i\Big(\mathrm{Ad}_y(\widetilde{X}_j)\Big)^2\right)F_{ya}=0. \tag{7.8}$$

We fix y and j and we take $a_j=t$, $a_k=t^2$ for $k\neq j$. Then dividing (7.8) by t and letting t tend to zero we obtain

$$D_{j,y}f=\left(\sum_{\alpha}\Big(\mathrm{Ad}_y(\widetilde{\mathcal{X}}_j^\alpha)\Big)^2+\Big(\mathrm{Ad}_y(\widetilde{\mathcal{Y}}_j^\alpha)\Big)^2-i\Big(\mathrm{Ad}_y(\widetilde{X}_j)^2\Big)\right)f=0. \tag{7.9}$$

We do this for every j. $D_{j,y}$ is a left-invariant operator on $N(\Phi)$. We shall show that (7.9) implies

$$D_{j,y}F_s=0. \tag{7.10}$$

for every $s_0\in S_0,\ y\in N_0$. Then taking $s=ya$ in (7.9) we obtain (7.5).

Since $F_s(\zeta,u)=f_\zeta * P_s(u)$, it remains to prove the following statement:

Let D be a left-invariant differential operator on $N(\Phi)$ such that $Df=0$. Then for every $s\in S_0$

$$DF_s(\zeta,u)=D(f_\zeta *_V P_s(u))=0. \tag{7.11}$$

We have

$$f_\zeta *_V P_s(x)=\int_V f(\zeta,x-u)P_{ya}(u)du=\int_V f((0,-u)(\zeta,x))P_{ya}(u)du.$$

Hence

$$D(f_\zeta *_V P_s(x))=\int_V (Df)((0,-u)(\zeta,x))P_{ya}(u)du, \tag{7.12}$$

provided we can justify the change of the order of integration and differentiation. But, since $f=\rho*\widetilde{f}$, right-invariant differential operators on $N(\Phi)$ applied to f yield bounded functions on $N(\Phi)$. Therefore, for every left-invariant operator D, $|Df(\zeta,x)|$ is dominated by a polynomial depending only on ζ. This proves (7.12), (7.11) and completes the proof of Lemma (7.5). $\square$

Proof of Lemma (7.6). We fix j and we consider the group

$$S_j=\exp\Big[\mathrm{lin}\{H_j,X_j,\mathcal{X}_j^\alpha,\mathcal{Y}_j^\alpha\ \alpha=1,...,n\}\Big].$$

S_j acts simply transitively on the Siegel upper-half plane described in Section 6. Assume that F satisfies the assumptions of Lemma (7.6) and let $F_j=F|_{S_j}$. Then

$$(X_j+iH_j)F_j=0 \quad \text{and} \quad \mathcal{L}_jF_j=0.$$

Consequently, by Section 6, F_j is holomorphic on S_j, i.e.

$$(\mathcal{X}_j^\alpha+i\mathcal{Y}_j^\alpha)F_j=0 \ \text{ for every } \ \alpha=1,...,n. \tag{7.13}$$

If instead of F we take $F_{x_1}=F(x_1x)$, $x_1,x\in S$, then (7.14) implies

$$(\mathcal{X}_j^\alpha+i\mathcal{Y}_j^\alpha)F=0. \tag{7.14}$$

But (7.14) for every j and α together with (7.2) and (7.3) yield holomorphicity of F. $\square$

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