ASYMPTOTIC BEHAVIOR OF POISSON KERNELS ON NA GROUPS

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Abstract. On a Lie group $S = NA$, that is a split extension of a nilpotent Lie group N by a one-parameter groups of automorphisms A, a probability measure μ is considered and treated as a distribution according to which transformations $s \in S$ acting on $N = S/A$ are sampled. Under natural conditions, formulated some over thirty years ago, there is a μ-invariant measure m on N. Properties of m have been intensively studied by a number of authors. The present paper deals with the situation when $\mu(A) = \mathrm{P}(s_t \in A)$, where $\mathbb{R}^+ \ni t \to s_t \in S$ is the diffusion on S generated by a second order subelliptic, hypoelliptic, left-invariant operator on S. This paper deals with the most general operators of this kind. Precise asymptotic for m at infinity and for the Green function of the operator are given. To achieve this goal a pseudo-differential calculus for operators with coefficients of finite smoothness is formulated and applied.

1. Introduction

The present paper is an outgrowth of the study of the Poisson kernels on boundaries of symmetric spaces, on one hand side, and invariant measures for Markov processes obtained by random sampling of affine transformations of $\mathbb{R}^d$, on the other. The fact that the Poisson kernel for a symmetric space S, identified with the solvable group NA in the Iwasawa decomposition, is the invariant probability measure m for a Markov process on the boundary N of S, was discovered by H.Furstenberg over forty years ago. In the present paper, we study the tail behavior of m in the particular case when m is invariant for a diffusion process $t \to s_t$ on $S = NA$, S acting on S/N on the left: $s : N \ni x \to s \cdot x \in N$. That is

$$\check{\mu}_t * m = m, \quad \text{or} \quad \int \mathbb{E}_e f(s_t \cdot x_0) dm(x) = \int f(x)\ dm(x), \tag{1.1}$$

where $\{\mu_t\}_{t>0}$ is the semigroup generated by a left-invariant second order subelliptic differential operator $\mathcal{L}$ on S. Then m has density $m(x)$ and we study the asymptotic of $m(x)$ as x goes to infinity along a ray. When $\mathcal{L}$ admits non trivial bounded harmonic functions then m reproduces them [DH1]via

$$F(xa) = \int_N f(x\Phi_a(x))\ dm(x).$$

If $N = \mathbb{R}$ and the transformations $\mathbb{R} \ni x \to ax + b \in \mathbb{R}$ are sampled according to a probability distribution μ, the tail asymptotic of the invariant measure is quite well understood, see e.g. [DF] and [GM] for a fairly recent account. For a general measure μ the following conditions are sufficient for the existence and uniqueness of the invariant measure m.

Research partially financed by the *European Commission* IHP Network 2002-2006 *Harmonic Analysis and Related Problems* (Contract Number: HPRN-CT-2001-00273 - HARP) and by KBN grant 1 P03A 018 26. D. Buraczewski was also supported by Foundation for Polish Science (annual stipend for young scientists, 2004) and by the Polish Academy of Science, being on a special position at the Institute of Mathematics there. The final version was prepared while the first and the second named authors were visiting at Department of Mathematics, Université de Rennes and the third named author was visiting at Department of Mathematics, Cornell University. They would like to express their gratitude to the hosts for hospitality.

$$\int \left(|\log a| + \log^+ |x|\right)^{2+\delta} \, d\mu(xa) < \infty, \tag{1.2}$$

$$\int \log a \; d\mu(xa) \leq 0. \tag{1.3}$$

Under these assumptions the measure m is a probability measure, if, and only if,

$$\int \log a \; d\mu(xa) < 0 \tag{1.4}$$

One also assumes that

$$\mu \text{ is not supported by a line} \tag{1.5}$$

Suppose that conditions (1.5) and (1.2) hold and, moreover, that

$$\int a^\lambda \; d\mu(xa) > 1 \quad \text{for some } \lambda. \tag{1.6}$$

Then it is easy to verify that there exists the unique $\alpha > 0$ such that

$$\int a^\alpha \; d\mu(xa) = 1. \tag{1.7}$$

This α describes the asymptotic behavior of m.

THEOREM [K, Go]. We have

$$\lim_{t\to+\infty} t^\alpha [t, +\infty) = c_+, \quad \lim_{t\to-\infty} t^\alpha (-\infty, t] = c_+ \quad \text{and} \quad c_+ + c_- > 0.$$

In the case when m is unbounded we have

THEOREM [BBE]. Under conditions (1.5) and (1.2),

$$\int \log a \; d\mu(xa) = 0 \quad \text{implies} \quad m[tr_1, tr_2] = L(t) \log \frac{r_2}{r_1},$$

where L is a slowly varying function. So m is a Radon measure with logarithmic tail.

However, our knowledge about the asymptotic behavior of m, for a general measure μ, even in the case $N = \mathbb{R}^d$, $d > 1$, is very limited. In the present paper we describe it for the measures m invariant under $\breve{\mu}_t$, i.e. such that (1.1) holds. Our particular interest has been caused by questions concerning the Martin boundary for harmonic functions on homogenous manifolds with negative curvature [D1] [DHU]. It is also the only case when asymptotic for the invariant measure in multidimensional case is known. Of course our estimates agree with those quoted above for the case $N = \mathbb{R}$.

Our earlier work [DH2] does not allow to consider semi-groups of measures μ_t generated by operators with an independent drift in the direction on N and these are the only cases when, for $N = \mathbb{R}$, one of the constants c_+ or c_- can vanish. To understand this phenomenon for the measures m invariant under $\breve{\mu}_t$ and to formulate appropriate conjectures in the multidimensional case for general measures μ, it is necessary to study the operators $\mathcal{L}$ on S in full generality.

2. Preliminaries and the Main Theorem

The setting of the present paper is as follows.

Let $S = NA$ be a split extension of a nilpotent Lie group N by a one dimensional group $A = \mathbb{R}^+$ of dilating automorphisms Φ_a:

$$(x, a)(y, b) = (x\Phi_a(y), ab).$$

Let $\mathcal{N}$ be the Lie algebra of N. "Dilating" means that in the Lie algebra $\mathcal{A}$ of A there is H_0 such that

$$\text{the real parts of all the eigenvalues of } ad_{H_0} : \mathcal{N} \to \mathcal{N} \text{ are positive.} \tag{2.1}$$

This includes, in particular, semi-direct products of a homogeneous group N (in the sense of E.M.Stein, cf e.g [FS]) with A acting by dilations. In this case,

$$\Phi_a(x) = (a^{d_1}x_1, ..., a^{d_n}x_n) \quad \text{with} \quad d_j > 0. \tag{2.2}$$

There are three important occurrences of Lie groups $S = NA$:

(i) the groups of of affine transformations $\mathbb{R}^d \ni x \to ax + b \in \mathbb{R}^d$,

(ii) rank one symmetric spaces G/K identified with the NA part of the Iwasawa decomposition $G = NAK$, or more generally

(iii) homogenous simply connected Riemannian manifolds of negative curvature. In this case S is the simply transitive solvable Lie group of isometries of the manifold.

(i) and (ii) are special cases of (2.2), as (iii) is not.

We consider a second order left-invariant operator

$$\mathcal{L} = \sum_{j=0}^{m} Y_j^2 + Y$$

on S and we assume that $Y, Y_0, ..., Y_m$ satisfy the weak Hörmander condition, i.e. Lie$\{Y, Y_0, ..., Y_m\}$ is equal to $\mathcal{S}$, the Lie algebra of S.

Under the canonical homomorphism of S onto $A = \mathbb{R}^+$ the image of $\mathcal{L}$, up to a constant, is equal to

$$(a\partial_a)^2 - \alpha a\partial_a \quad \text{or} \quad -\alpha a\partial_a.$$

Here we make an additional assumption: the term $(a\partial_a)^2$ does appear. If it does not, that is the image of $\mathcal{L}$ is only $-\alpha a\partial_a$, then the measures μ_t violate either (1.3) or (1.6), depending on the sign of α. Moreover, if $\alpha > 0$, then we expect very fast decay of m, since in the case of

$$-a\partial_a + (a^d\partial_x)^2 \quad \text{on} \quad \mathbb{R} \times \mathbb{R}^+$$

the invariant measure has density $m(x) = c\exp[-\frac{dx^2}{2}]$ on $\mathbb{R}$. In the case we consider i.e. $(a\partial_a)^2$ does not vanish, the decay is only polynomial and we attempt to find its is precise asymptotic.

We write the operator $\mathcal{L}$ as

$$\mathcal{L} = \sum_{i,j=1}^{n+1} \alpha_{i,j} W_i W_j + \sum_{i=1}^{n+1} \alpha_i W_i,$$

where $W_1, \ldots, W_n$ is a basis of $\mathcal{N}$, $W_{n+1} = H_0$, the matrix $[\alpha_{i,j}]$ is semi-positive definite and $\alpha_{n+1,n+1} > 0$. It follows from elementary linear algebra (the standard procedure of diagonalizing quadratic forms) that $\mathcal{L}$ can be written as

$$\mathcal{L} = \beta_1(H_0 + \widetilde{Y})^2 + \sum_{i=1}^{m} \widetilde{Y}_j^2 + \widetilde{Y}_0 + \beta_2(H_0 + \widetilde{Y}) = H^2 + \sum_{i=1}^{m} \widetilde{Y}_j^2 + \widetilde{Y}_0 + \beta H \tag{2.3}$$

with $H = \sqrt{\beta_1}(H_0 + \widetilde{Y})$, $\widetilde{Y}, \widetilde{Y}_0, \widetilde{Y}_1, \dots, \widetilde{Y}_m \in \mathcal{N}$ acting on functions on S as

$$\widetilde{Y}_j f(xa) = \frac{d}{dt} f(xa \exp t\widetilde{Y}_j)|_{t=0}.$$

Thus our assumptions are

$$\beta \leq 0 \text{ and } H, \widetilde{Y}_1, \dots, \widetilde{Y}_m, \widetilde{Y}_0 + \beta H \text{ generate } \mathcal{S} \text{ as the Lie algebra.} \tag{2.4}$$

Now we write $\mathcal{L}$ in convenient coordinates. We modify the operator multiplying it by c^2:

$$c^2\mathcal{L} = (cH)^2 + \sum_{i=1}^{m} (c\widetilde{Y}_j)^2 + c^2\widetilde{Y}_0 + c^2\beta H.$$

Clearly, $\mathcal{L}$ and $c^2\mathcal{L}$ have the same harmonic functions, the semi-group for $c^2\mathcal{L}$ is rescaled $\mu_{c^2 t}$ and the Green function is multiplied by c^{-2}. This modification does not change the sign of β.

The decomposition

$$\mathcal{S} = \mathcal{N} \oplus \mathcal{A}$$

is not unique i.e. there is no canonical choice of A. We choose A as

$$A = \exp\{tcH : t > 0\},$$

and assume, without no loss of generality, that the real parts of the eigenvalues of $ad_H|\mathcal{N}$ are strictly positive.

Decomposing $s \in S$ as

$$s = xa = x \exp(\log a) cH, \quad x \in N, a \in A, \tag{2.5}$$

we obtain

$$c^2\mathcal{L} = \mathcal{L}_{-\alpha} = (a\partial_a)^2 - \alpha(a\partial_a) + \sum_{j=1}^{m} \Phi_a(X_j)^2 + \Phi_a(X_0), \tag{2.6}$$

where $\Phi_a = Ad_{\exp(\log a)H}$, $a\partial_a = cH$ and $X_0, X_1, \dots, X_m$ are left-invariant vector fields on N.

Remark. Notice that this way we can make $\alpha = c\beta$ a nonnegative integer and the real parts of the eigenvalues of ad_{cH} as large as we wish. In particular, if α and the eigenvalues are rational they can be made large enough natural numbers.

Assumptions. For the rest of the paper, we assume that

$$\begin{gathered} a\partial_a, \Phi_a(X_j),\ j = 0, ..., m \text{ generate } \mathcal{S}, \\ \alpha \text{ is a nonnegative integer, if } \alpha = 0,\ \Re\lambda_j \geq \lfloor \tfrac{n+2}{2} \rfloor + 3 \text{ and if } \alpha \neq 0,\ \Re\lambda_j > 4. \end{gathered} \tag{2.7}$$

The main result is

Let Σ be a sphere around $e \in N$. For every $x \in \Sigma$ the limit

$$\lim_{a\to\infty} a^{Q+\alpha} m_\alpha(\Phi_a(x)) = c(x)$$

exists and the function $\Sigma \ni x \mapsto c(x)$ is continuous. If $a\partial_a, \Phi_a(X_1), \dots, \Phi_a(X_m)$ generate $\mathcal{S}$ then $c(x) > 0$.

In a number of papers [D1, D2, DH1, DH2, DHZ, DHU, U] we dealt with measures m on N invariant for the semi-groups of operators like $\mathcal{L}$ only the assumptions on $\mathcal{L}$ were stronger. Indeed, instead of (2.4) we assumed that

$$X_1, ..., X_m \quad \text{generate the Lie algebra of } N. \tag{2.8}$$

Under this stronger assumptions the Main Theorem 2.15 was proved in [DH2]. This, however, excluded operators like

$$(a\partial_a)^2 + a^8\partial_x^2 + a^4\partial_y + a^6\partial_z.$$

Here we go beyond this restriction developing a new method that not only solves the problem, but also shows an interesting phenomenon: subelliptic estimates uniform when $a \to 0$.

The plan for the proof follows the pattern of [DHU] and [DH2].

We change $\mathcal{L}_{-\alpha}$ into $\mathcal{L}_\alpha = a^{-\alpha}\mathcal{L}_{-\alpha}(a^\alpha \cdot)$ and we divide it by a^2 to introduce the operator

$$L_\alpha = a^{-2}\mathcal{L}_\alpha = \partial_a^2 + \frac{1+\alpha}{a}\partial_a + a^{-2}\Big(\sum_{j=1}^m \Phi_a(X_j)^2 + \Phi_a(X_0)\Big).$$

Then we observe that if $G^*(x,a;y,b)$ is the Green function of L_α^* with respect to the reference measure $dxa^{\alpha+1}da$ on $N \oplus \mathbb{R}^+$ and the limit

$$\lim_{b\to 0}\lim_{a\to 0} G^*(x,a;y,b) \quad \text{exists for} \quad x \neq y,$$

then

$$m_\alpha(x) = G^*(e,0;x,1)$$

and an easy homogeneity argument yields the result. Also

$$\lim_{b\to 0} G^*(x,0;y,b) = c(y) \tag{2.9}$$

exists, see Section 6.

The main point of this paper is to show 2.9

A special case. Suppose that the group is as in (2.2) and that our assumptions (2.7) are satisfied. Moreover, assume that

$$\begin{aligned} &(i) \quad d_1, ..., d_n \quad \text{are positive rational numbers,} \\ &(ii) \quad \alpha \quad \text{is a positive rational number.} \end{aligned}$$

Of course, (i) concerns the group, (ii) the operator. By a change of variable $a \to a^r$ (see 2.5) we can make α a positive integer and d_j's positive even integers greater or equal 4. We then take

$$L_\alpha^* = \partial_a^2 + \frac{\alpha+1}{a}\partial_a + \sum_j \alpha_j a^{2d_j-2}X_j^2 + \sum_l \beta_l a^{d_l-2}W_l,$$

where the summation in the first order term is over distinct eigenvalues i.e. $l_1 \neq l_2$ implies $d_{l_1} \neq d_{l_2}$. In particular on $\mathbb{R}^3 \times \mathbb{R}^+$ it could be

$$\partial_a^2 + \frac{\alpha+1}{a}\partial_a + a^6\partial_x^2 + a^2\partial_y + a^4\partial_z$$

Let

$$u = (u_1, ..., u_s), \quad s = \alpha + 2 \quad \text{and} \quad a = \sqrt{u_1^2 + ... + u_s^2} = |u|.$$

Then, if $\Delta_u = \partial_{u_1}^2 + ... + \partial_{u_s}^2$, we have

$$\begin{aligned} \widetilde{L}_\alpha^* =& \Delta_u + \sum_j \alpha_j |u|^{2(d_j-1)}X_j^2 + \sum_j \beta_j |u|^{d_j-2}X_j \\ =& \Delta_u + \sum_j \alpha_j |u|^{2\cdot\delta_j}X_j^2 + \sum_j \beta_j |u|^{2\cdot\gamma_j}X_j \ , \end{aligned}$$

where

$$\delta_j = d_j - 1, \quad \gamma_j = \frac{d_j-2}{2} \ .$$

For our particular case we have

$$\Delta_u + |u|^6\partial_x^2 + |u|^2\partial y + |u|^4\partial_z,$$

which clearly satisfies the Hörmander condition on $\mathbb{R}^{3+s}$. Now, putting

$$\mathbf{k} = (k_1, ..., k_s),\ k_j \geq 0,\ |\mathbf{k}| = k_1 + ... + k_s,$$

we obtain

$$|u|^{2\cdot\delta_j} = (u_1^2 + ... + u_s^2)^{\delta_j} = \sum_{|\mathbf{k}|=\delta_j} \binom{\delta_j}{\mathbf{k}} (u_1^{k_1} ... u_s^{k_s})^2,$$

and similarly,

$$|u|^{2\cdot\gamma_j} = (u_1^2 + ... + u_s^2)^{\gamma_j} = \sum_{|\mathbf{k}|=\gamma_j} \binom{\gamma_j}{\mathbf{k}} (u_1^{k_1} ... u_s^{k_s})^2.$$

Therefore,

$$\widetilde{L}_\alpha^* = \Delta_u + \sum_j \alpha_j \sum_{|\mathbf{k}|=\delta_j} \binom{\delta_j}{\mathbf{k}} (u_1^{k_1} ... u_s^{k_s} X_j)^2 + \sum_l \beta_l \sum_{|\mathbf{k}|=\gamma_l} \binom{\gamma_l}{\mathbf{k}} (u_1^{k_1} ... u_s^{k_s})^2 W_l.$$

Thus we see that the condition $\mathrm{Lie}(H, Y_1, ..., Y_m, Y_0) = \mathcal{S}$ implies that the Lie algebra generated by $\alpha_j X_j$'s and $\beta_l W_l$'s is $\mathcal{N}$ i.e $\widetilde{L}_\alpha^*$ satisfies the weak Hörmander condition. Then $\widetilde{L}_a^*$ is hypo-elliptic as an operator on $\mathbb{R}^s \times N$ an so its Green function $\widetilde{G}$ is C^∞ off the diagonal,[Bo]. Going back to the original operator, before the change of variable, we see that

$$\widetilde{G}(e, 0; \cdot) = G^*(e, 0; \cdot).$$

Hence 2.9.

It is tempting to prove the result in the case when Φ_a have only eigenvalues with positive real parts (not necessarily rational) and are not diagonal, based on the argument above. However, we have been unable to use any procedure of approximation arbitrary real d_j by rationals and replace the diagonal action of Φ_a's by the general. To prove the main theorem, as it is formulated, we need a uniform subelliptic estimate when $a \to 0$. Therefore, we go deeper into the proof of the Hörmender's theorem and use subelliptic estimates for operators with coefficients of finite smoothness. The proof of the subelliptic estimates 5.1 and 5.3 is long and technical. A big part of it is rather standard so we put it into Appendix. Section 5 contains only what is specific to our operators, Theorem 5.6 being the crucial step.

In a series of papers cf. [BGGK] and [BBG] the Green function for operators like

$$\Delta_u + |u|^k \Delta_t \quad \text{on} \quad \mathbb{R}^n \oplus \mathbb{R}^m$$

has been studied. The authors give explicit formulae for the Green functions G, if k is an even integer and the change of variable $|u| \to |u|^\gamma$ yields immediately continuity of G for arbitrary positive rational k. However, continuity of the Green function of a slightly more general operator

$$\Delta_u + |u|^k \Delta_t + |u|^l \Delta_s \quad \text{on} \quad \mathbb{R}^n \oplus \mathbb{R}^m \oplus \mathbb{R}^j, \quad k, l \text{ being independent over } \mathbb{Q} \tag{2.10}$$

cannot be obtained along these lines and, to our embarrassment, we have to go via tedious estimates to obtain the result for operators like (2.10).

Now we are going to introduce the rest of notation and to formulate the main theorems. In N we define a "homogeneous" norm $|\cdot|$. Let $(\cdot,\cdot)$ be an arbitrary fixed inner product in $\mathcal{N}$ and let

$$\langle X, Y\rangle = \int_0^1 (\Phi_a(X), \Phi_a(Y)) \frac{da}{a}, \quad \|X\| = \sqrt{\langle X, X\rangle}.$$

We put

$$|\exp X| = |X| = (\inf\{a > 0 : \|\Phi_a(X)\| \geq 1\})^{-1}.$$

Since for $X \neq 0$ we have

$$\lim_{a\to 0} \|\Phi_a(X)\| = 0,$$

$$\lim_{a\to\infty} \|\Phi_a(X)\| = \infty,$$
$$\text{and } a \mapsto \|\Phi_a(X)\| \text{ is increasing,}$$

if follows that for every $Y \neq 0$ there is precisely one a such that

$$Y = \Phi_a(X), \quad |X| = 1, \quad |Y| = a.$$

If the action of A on N is diagonal, $|\cdot|$ is the usual homogeneous norm on N.

Let $\mathcal{N}^{\mathbb{C}}$ be the complexification of $\mathcal{N}$ and let $E = \{\lambda_1, \dots, \lambda_D\}$ denote the set of all eigenvalues of $ad_H|\mathcal{N}$. For $\lambda \in E$ let $\mathcal{N}^{\mathbb{C}}_\lambda$ be the coresponding eigenspace. We may assume $\Re\lambda_1 \leq \dots \leq \Re\lambda_D$. Then there exists a basis Z^λ_j of $\mathcal{N}^{\mathbb{C}}$, $Z^\lambda_j \in \mathcal{N}^{\mathbb{C}}_\lambda$, for $\lambda \in \Lambda, j = 1, \dots, d(\lambda)$, in which $ad_H|\mathcal{N}$ can be written in the Jordan canonical form, i.e.

$$\begin{aligned} &ad_H(Z^\lambda_1) = \lambda Z^\lambda_1, \\ &ad_H(Z^\lambda_j) = \lambda Z^\lambda_j + \delta Z^\lambda_{j-1}, \quad \text{for } j = 2, \dots, d(\lambda), \delta = 0, 1. \end{aligned} \tag{2.11}$$

In the chosen basis one can compute explicitly the action of Φ_a. Namely we have

$$\Phi_a(Z^\lambda_j) = a^\lambda \Big(\sum_{k=k_j}^{j} \frac{1}{(j-k)!} (\log a)^{j-k} Z^\lambda_k \Big). \tag{2.12}$$

The above formula implies that the are $m_1 \geq m_2 > 4$ and $C > 0$ such that

$$\|\Phi_a\|_{\mathcal{N}\to\mathcal{N}} \leq C(a^{m_1} + a^{m_2}). \tag{2.13}$$

We choose a basis $X_1, \dots, X_n$ of left invariant vector fields coresponding to the decomposition

$$\mathcal{N} = \oplus_{\Im\lambda \geq 0} V_\lambda,$$

where

$$V_\lambda = \mathcal{N} \cap (\mathcal{N}^{\mathbb{C}}_\lambda \oplus \mathcal{N}^{\mathbb{C}}_{\bar\lambda})$$

i.e.

$$X_j \in V_{\lambda(X_j)}. \tag{2.14}$$

For a multiindex I we write $X^I = X_1^{i_1} \dots X_n^{i_n}$ and $|I| = i_1 \Re\lambda(X_1) + \dots + i_n \Re\lambda(X_n)$. Let

$$Q = \sum_{\Im\lambda \geq 0} \Re\lambda \cdot \dim V_\lambda.$$

In the case of diagonal action 2.2 this means that

$$\begin{aligned} \Phi_a(X_j) =& a^{d_j} X_j \\ |I| =& i_1 d_1 + \dots + i_n d_n \end{aligned}$$

and Q is the homogeneous dimension. Now, given all the necessary definitions, the main results of this paper can be formulated as follows

Main Theorem 2.15. *Let Σ be the unit sphere in $\mathcal{N}$, then for every $x \in \Sigma$*

$$\lim_{a\to\infty} a^{Q+\alpha} m_\alpha(\Phi_a(x)) = c(x)$$
$$\lim_{a\to\infty} a^{Q+\alpha} |(\Phi_a(X^I) m_\alpha)(\Phi_a(x))| = c_I(x)$$

is finite and the function $\Sigma \ni x \mapsto c_I(x)$ is continuous. In particular, for diagonal action 2.2

$$|X^I m_\alpha(x)| \leq c(1 + |x|)^{-Q-\alpha-|I|}$$

and for arbitrary action

$$|X^I m_\alpha(x)| \leq c(1 + |x|)^{-Q-\alpha-|I|} \log(2 + |x|)^{I_0},$$

where

$$I_0 = \sum_{j=1}^{n} i_j(\dim V_{\lambda(X_j)} - 1).$$

If $a\partial_a, \Phi_a(X_1), \dots, \Phi_a(X_m)$ *generate* $\mathcal{S}$ *then* $c(x) > 0$.

Let

$$\begin{aligned} V_1 =& \{(x,a) : |x| \le 2, 0 < a \le \frac{1}{2}\} \\ V_2 =& \{(x,a) : |x| \le a, a \ge 2\} \\ V_3 =& \{(x,a) : |x| \ge a, |x| \ge 2\} \end{aligned}$$

and let

$$h(xa) = \begin{cases} a^{\alpha} & (x,a) \in V_1 \\ a^{-Q} & (x,a) \in V_2 \\ a^{\alpha}|x|^{-\alpha-Q} & (x,a) \in V_3 \end{cases}$$

Main Theorem 2.16. *Let* U *be the density, with respect to the right Haar measure, of* $\int_0^\infty \mu_t$ *. If* $a\partial_a$*,* $\Phi_a(X_j)$*, j=1,...,m generate* $\mathcal{S}$ *then there are* $C_1, C_2 > 0$ *such that*

$$C_1 \le \frac{U(xa)}{h(xa)} \le C_2. \tag{2.17}$$

If all $a\partial_a$*,* $\Phi_a(X_j)$*, j=0,...,m are needed to generate* $\mathcal{S}$*, then there is* $C_2 > 0$ *such that*

$$U(xa) \le C_2 h(xa). \tag{2.18}$$

Notice that the change of variables (2.6) is indeed of the form $(x,a) \to (x,a^c)$ so it only changes the speed of dilations and so both theorems remain valid as far as $\Re\lambda_j > 0$. The subelliptic estimate cannot be transferred by this transformation, but the final result can.

3. Evolution run by the Bessel process

We start by describing the evolution that is used later for the construction of the fundamental solution for our operator. Let $\mathbb{R}^+ \ni t \to \sigma(t)$ denote the Bessel process with a parameter $\alpha \ge 0$, [RY]. This is a continuous Markov process with state space $[0,\infty)$ generated by

$$\Delta_\alpha = \partial_a^2 + \frac{\alpha+1}{a}\partial_a.$$

Of course, if α is a positive integer, then Δ_α is the radial part of the Laplacian on $\mathbb{R}^{\alpha+2}$. Δ_α is an infinitesimal generator of the Bessel semi-group of operators on $L^2(a^{\alpha+1}da)$.

$$P_t f(a) = \int_0^\infty p_t(a,b) f(b) b^{\alpha+1} db.$$

Then for $f \in L^2(a^{\alpha+1}da)$ we have

$$\lim_{t\to 0} \|P_t f - f\|_{L^2(a^{\alpha+1}da)} = 0$$

and for $f \in C_c^\infty$

$$\lim_{t\to 0} \|\frac{P_t f - f}{t} - \Delta_\alpha f\|_{L^2(a^{\alpha+1}da)} = 0.$$

Also,

$$p_t(a,b) \le c_1 t^{-1-\frac{\alpha}{2}} e^{-c_2 \frac{(a-b)^2}{4t}}$$

and so for $\gamma \geq 0$

$$\sup_{t\leq 1, a\leq 1} \mathbb{E}_a \int_0^t \sigma(s)^\gamma \, ds = \sup_{t\leq 1, a\leq 1} \int_{\mathbb{R}^+} b^\gamma p_t(a,b) b^{1+\alpha} db < \infty. \tag{3.1}$$

For a multi-index $I = (i_1, ..., i_n)$ and a basis $X_1, ..., X_n$ of the Lie algebra $\mathcal{N}$ we write

$$X^I = X_1^{i_1} \cdots X_n^{i_n}.$$

For $k = 0, 1, ...$ we define

$$C^k = \{f : \ X^I f \in C(N), \text{ for } \sum i_j \leq k\}$$

and

$$C^k_\infty = \{f \in C^k : \lim_{x\to\infty} X^I f(x) \text{ exists for } \sum i_j \leq k\}.$$

C^k_∞ is a Banach space with the norm

$$\|f\|_{C^k_\infty} = \sum_{\sum i_j \leq k} \|X^I f\|_{C(N)}.$$

For a continuous function $\sigma : [0, \infty) \to (0, \infty)$ let

$$L_{\sigma(t)} = \sigma(t)^{-2} \Big(\sum_{j=1}^m \Phi_{\sigma(t)}(X_j)^2 + \Phi_{\sigma(t)}(X_0) \Big) \tag{3.2}$$

As we will see later on $L_{\sigma(t)}$ is closely related to $a^{-2}\mathcal{L}_\alpha$. A standard argument (see e.g. [T]) shows the existence of a unique family $U^\sigma(s,t)$, $0 \leq s < t$ of bounded operators on $C_\infty(N) = C^0_\infty(N)$ that satisfy

- $U^\sigma(s,t)f = f * p^\sigma(t,s)$, for a probability measure $p^\sigma(t,s)$,
- $p^\sigma(s,s) = \delta_e$,
- $p^\sigma(t,r) * p^\sigma(r,s) = p^\sigma(t,s)$ for $s < r < t$,
- $\lim_{h\to 0} \|f * p^\sigma(s+h,s) - f\|_{C_\infty(N)} = 0$ for $f \in C_\infty(N)$
- $\partial_t(f * p^\sigma(t,s)) = (L_{\sigma(t)} f) * p^\sigma(t,s)$, for $f \in C^2_\infty(N)$,
- $\partial_s(f * p^\sigma(t,s)) = -L_{\sigma(s)}(f * p^\sigma(t,s))$, for $f \in C^2_\infty(N)$.

For the argument [T] to work the following properties of $L_{\sigma(t)}$ are needed.

- For a fixed s, $L_{\sigma(s)}$ generates a semi-group μ^s_t with the property

$$\lim_{h\to 0} \Big\| \frac{f * \mu^s_h - f}{h} - L_{\sigma(s)} f \Big\|_{C_\infty} \to 0$$

  for $f \in C^2_\infty$.
- Let τ be a Riemannian distance on N, $\phi \in C^\infty_c(N)$, $\phi \geq 0$. $C_T = (1 + \sup_{0\leq s\leq T} |\sigma(s)|)^{2\Re\lambda_D}$. There is C such that for every $T > 0$, $s_1, ..., s_n \leq T$, $t_1, ..., t_n > 0$ and all $\beta > 0$

$$\langle \mu^{s_1}_{t_1} * ... * \mu^{s_n}_{t_n}, e^{\beta(\phi*\tau)} \rangle \leq e^{\beta(\phi*\tau)}(0) e^{C(\beta+\beta^2)C_T(t_1+...+t_n)}.$$

The proof uses the same techniques as in Lemma 3.3 below. As a consequence we conclude that for every $k \geq 1$ there is $C_K > 0$ such that

$$\|U^\sigma(s,t)\|_{C^k_\infty(N)\to C^k_\infty(N)} \leq C_k e^{C_k C_T(t-s)}$$

for $0 \leq s, t \leq T$.

Here are some further properties of $p^\sigma(s,t)$.

Lemma 3.3. *Let*

$$A^\sigma(0,t) = \int_0^t \left(\sigma(s)^{2m_1-2} + \sigma(s)^{m_2-2}\right) ds$$

and $r > 0$. *Then there exist* c_1, c_2 *such that*

$$\int_{\tau(z)>r} p^\sigma(0,t) \le c_1 e^{-\frac{c_2}{A^\sigma(0,t)^\gamma}}, \tag{3.4}$$

where $\gamma = \frac{1}{2}$ *when* $A^\sigma(0,t) < 1$ *and* $\gamma = 1$ *when* $A^\sigma(0,t) \ge 1$.

Proof. Fix r. Let $\phi \in C_c^\infty(N)$ be a positive function, supported in the ball $B_{\frac{r}{4}}(0)$ and $\int \phi = 1$. Let $\eta(x) = \tau * \phi(x)$. There exists a positive constant $C = C(r)$ such that if $X_1, ..., X_n$ is a fixed basis of $\mathcal{N}$ then

$$|X_j \eta(x)| \le C, \qquad |X_i X_j \eta(x)| \le C, \qquad \text{for } i,j = 1, ..., n \tag{3.5}$$

[H]. Moreover,

$$\tau(x) \le \int (\tau(xy^{-1}) + \tau(y))\phi(y)dy \le \eta(x) + \frac{r}{4}, \tag{3.6}$$

and

$$\eta(e) = \int \tau(y^{-1})\phi(y)dy \le \frac{r}{4}. \tag{3.7}$$

For a positive integer m let $\eta_m(x) = \tau_m * \phi(x)$, where

$$\tau_m(x) = \min\{m, \tau(x)\}.$$

τ_m is subadditive and there exists a positive constant C such that for every m, (3.5), (3.6), (3.7) hold with η_m and τ_m instead of η and τ respectively.

Let

$$h(t) = e^{\beta\eta_m} * p^\sigma(0,t)(e).$$

Since $e^{\beta\eta_m} \in C^2_\infty$,

$$\partial_t h(t) = (L_{\sigma(t)} e^{\beta\eta_m}) * p^\sigma(0,t)(e)$$

and by (2.13) there is $C = C(r)$ such that

$$\partial_t h(t) \le C\Big[(\beta+\beta^2)\sigma(s)^{-2}(\sigma^{m_1}(s) + \sigma^{m_2}(s))^2 + \beta\sigma(s)^{-2}(\sigma^{m_1}(s) + \sigma^{m_2}(s))\Big]h(t).$$

Hence

$$\log h(t) - \log h(0) = \int_0^t \frac{\partial_s h(s)}{h(s)} ds \le C(\beta+\beta^2)A^\sigma(0,t)$$

and

$$h(t) \le e^{\beta\eta_m(e)} e^{C(\beta+\beta^2)A^\sigma(0,t)}.$$

Letting $m \to \infty$, by (3.5) and (3.7) we have

$$(e^{\beta\tau}, p^\sigma(0,t)) \le e^{\frac{\beta r}{2}} e^{C(\beta+\beta^2)A^\sigma(0,t)}.$$

Therefore,

$$\int_{\tau(z)>r} p^\sigma(0,t) \le e^{-\frac{\beta r}{2} + C(\beta+\beta^2)A^\sigma(0,t)}.$$

We may assume that $C \ge 1$. If $A^\sigma(0,t) \ge 1$, we substitute $\beta = \frac{r}{4CA^\sigma(0,t)}$, if $A^\sigma(0,t) \le 1$, we substitute $\beta = \frac{1}{C\sqrt{A^\sigma(0,t)}}$ and the Lemma follows. □

4. Heat equation on $N \times \mathbb{R}^+$ and $N \times \mathbb{R}^{\alpha+2}$

Now we change $\mathcal{L}_{-\alpha}$ into $\mathcal{L}_\alpha = a^{-\alpha}\mathcal{L}_{-\alpha}(a^\alpha \cdot)$ and we divide it by a^2. We describe the fundamental solution for $a^{-2}\mathcal{L}_\alpha - \partial_t$ as well as for its extension to $N \times \mathbb{R}^{\alpha+2}$. Let

$$L_\alpha = a^{-2}\mathcal{L}_\alpha = \partial_a^2 + \frac{1+\alpha}{a}\partial_a + a^{-2}\Big(\sum_{j=1}^m \Phi_a(X_j)^2 + \Phi_a(X_0)\Big).$$

The diffusion generated by L_α on $N \times \mathbb{R}^+$ is decomposed into its vertical and horizontal components. Namely, for $f \in C_c^\infty(N \times \mathbb{R}^+)$ and $a \geq 0$ we define

$$\begin{aligned} F(x,a,t) = T_t f(x,a) &= \int U^\sigma(0,t) f(x,\sigma(t)) dB_a(\sigma) \\ &= \mathbb{E}_a U^\sigma(0,t) f(x,\sigma(t)), \end{aligned} \tag{4.1}$$

where dB_a is the probability measure on the space $C([0,\infty),\mathbb{R}^+)$ of trajectories of the Bessel process $\sigma(t)$. Then F is a solution of the heat equation, i.e.

$$L_\alpha F(x,a,t) = \partial_t F(x,a,t), \tag{4.2}$$

on $N \times \mathbb{R}^+ \times \mathbb{R}^+$, F is continuous on $N \times \mathbb{R}^+ \times [0,\infty)$ and

$$\lim_{t\to 0} F(x,a,t) = f(x,a). \tag{4.3}$$

Writing an appropriate maximum principle we prove that a bounded F satisfying (4.2) and (4.3) is unique. Furthermore, the Markov property implies that T_t is a contractions semigroup on $L^2(dxa^{\alpha+1}da)$ [DHU]. Finally, for $f \in C_c^\infty(NA)$

$$\lim_{t\to 0} \Big\| \frac{T_t f - f}{t} - L_\alpha f \Big\|_{L^2(a^{1+\alpha}dadx)} = 0 \tag{4.4}$$

[DH2]. Now let

$$\widetilde{L}_\alpha = \Delta_u + |u|^{-2}\Big(\sum_{j=1}^m \Phi_{|u|}(X_j)^2 + \Phi_{|u|}(X_0)\Big), \tag{4.5}$$

where $\Delta_u = \sum_{j=1}^d \partial_{u_j}^2$, $d = \alpha+2$ and if $u = 0$, then $\widetilde{L}_\alpha = \Delta_u$. For $\widetilde{L}_\alpha - \partial_t$ we write a convenient formula for the heat semi-group. Let dW_u be the Wiener measure on the space $C([0,\infty)),\mathbb{R}^d$ of trajectories for the Brownian motion generated by Δ_u. For $\widetilde{f} \in C_c^\infty(N \times \mathbb{R}^d)$ let

$$\begin{aligned} \widetilde{F}(x,u,t) = \widetilde{T}_t \widetilde{f}(x,u) &= \int \widetilde{U}^{|b|}(0,t)\widetilde{f}(x,b(t)) dW_u(b) \\ &= \mathbb{E}_u\big[\widetilde{U}^{|b|}(0,t)\widetilde{f}(x,b(t))\big], \end{aligned}$$

Then $\widetilde{F}$ is the unique solution of the heat equation

$$(\widetilde{L}_\alpha - \partial_t)\widetilde{F} = 0$$

with the boundary data $\widetilde{f}$ as described in Theorem 4.7 below. Moreover, if

$$\widetilde{f}(x,u) = f(x,|u|).$$

for $f \in C_c^\infty(NA)$ then

$$\widetilde{T}_t \widetilde{f}(x,u) = T_t f(x,|u|), \tag{4.6}$$

Indeed, if $\widetilde{f}$ is u-radial then so is $\widetilde{F}$ and (4.6) follows from uniqueness. In particular, by the next theorem, $T_t f$ is continuous on $N \times [0,\infty) \times [0,\infty)$.

Theorem 4.7. *Let $\eta = \lfloor \Re\lambda_1 \rfloor - 2$. If $\widetilde{f} \in C_c^\infty(N \times \mathbb{R}^d)$, then $\widetilde{F} = \widetilde{T}_t \widetilde{f} \in C^{\infty,\eta,1}(N \times \mathbb{R}^d \times \mathbb{R}^+)$. Furthermore,*

$$\widetilde{L}_\alpha \widetilde{F}(x,u,t) = \partial_t \widetilde{F}(x,u,t) \text{ on } N \times \mathbb{R}^d \times \mathbb{R}^+ \tag{4.8}$$

and

$$\lim_{t\to 0} \widetilde{F}(x,u,t) = \widetilde{f}(x,u), \tag{4.9}$$

Writing an appropriate maximum principle we can prove that a bounded $\widetilde{F}$ satisfying (4.8) and (4.9) is unique. Notice that here $\widetilde{f}$ is not necessarily radial and we cannot obtain this regularity of $\widetilde{F}$ when $u \to 0$ from subelliptic estimate (5.4). More derivatives with respect to u would be needed. First we prove that $\widetilde{F}$ is a solution of an integral equation

Lemma 4.10. *For fixed (x,u,t), the function $\widetilde{F}$ satisfies the equation*

$$\widetilde{F}(x,u,t) = \mathbb{E}_u \widetilde{f}(x,b(t)) + \int_0^t \mathbb{E}_u \big[L_{|b(t-s)|} \widetilde{F}(x,b(t-s),s) \big] ds, \tag{4.11}$$

where $L_{|b(t-s)|}$ is given by formula (3.2).

Proof. Observe that (4.11) is well defined. Namely, $\widetilde{F}$ is a bounded function and smooth with respect to x. Indeed, writing X^I as a combination of right-invariant vector fields we have

$$X^I \widetilde{F}(x,u,t) = \sum_{|J|\le |I|} W_J(x) \widetilde{T}_t (\widetilde{X}^J \widetilde{f})(x,u), \tag{4.12}$$

where for a multi-index $J = (j_1, ..., j_p)$, $\widetilde{X}^J = \widetilde{X}^{j_1} \cdot \ldots \cdot \widetilde{X}^{j_p}$ is a right-invariant differential operator and W_J's are polynomials on N.

Furthermore

$$|L_{|b(t-s)|} \widetilde{F}(x,b(t-s),s)| \le C_f (1+|b(t-s)|)^{2Q} (1+|x|)^M, \tag{4.13}$$

for an appropriate large constant M. Hence

$$\begin{aligned} \Big| \int_0^t \mathbb{E}_u \big[L_{|b(t-s)|} \widetilde{F}(x,b(t-s),s) \big] ds \Big| &\le C_f (1+|x|)^M \int_0^t \mathbb{E}_u (1+|b(t-s)|)^{2Q} ds \\ &\le C_f (1+|x|)^M \int_0^t \int_{\mathbb{R}^d} (1+|w|)^{2Q} p^G_{t-s}(u-w)\, dw ds, \end{aligned}$$

where p^G_t denotes the classical gaussian kernel on $\mathbb{R}^d$. So (4.11) is well defined.

Now, by the Markov property, we have

$$\begin{aligned} \mathbb{E}_u \big[L_{|b(t-s)|} \widetilde{F}(x,b(t-s),s) \big] &= \mathbb{E}_u \big[L_{|b(t-s)|} \mathbb{E}_{b(t-s)} [\widetilde{U}^{|\sigma|}(0,s) \widetilde{f}(x,\sigma(s))] \big] \\ &= \mathbb{E}_u \big[L_{|b(t-s)|} \widetilde{U}^{|b|}(t-s,t) \widetilde{f}(x,b(t)) \big] \\ &= \mathbb{E}_u \big[\partial_s \widetilde{U}^{|b|}(t-s,t) \widetilde{f}(x,b(t)) \big]. \end{aligned}$$

Whence

$$\begin{aligned} \int_0^t \mathbb{E}_u \big[L_{|b(t-s)|} \widetilde{F}(x,b(t-s),s) \big] ds &= \mathbb{E}_u \big[\widetilde{U}^{|b|}(0,t) \widetilde{f}(x,b(t)) \big] - \mathbb{E}_u \widetilde{f}(x,b(t)) \\ &= \widetilde{F}(x,u,t) - \mathbb{E}_u \widetilde{f}(x,b(t)), \end{aligned}$$

that finishes the proof. □

In the proof of (4.8) some more regularity of $\widetilde{F}$ is needed.

Lemma 4.14. *The function $\widetilde{F}$ is continuous and for fixed t, $\widetilde{F}(\cdot,\cdot,t) \in C^{\infty,\eta}(N \times \mathbb{R}^d)$. Moreover, for every I and $|J| \le \eta$ there are constants $C = C(I)$, $M = M(I)$ such that*

$$|X^I \partial_u^J \widetilde{F}(x,u,t)| \le C(1+|x|+|u|+t)^M.$$

Proof. First, observe that because of (4.12), $\widetilde{F}$ is smooth w.r. to x. Next, we shall prove that for all x and t, $\widetilde{F}(x,\cdot,t) \in C^\eta(\mathbb{R}^d)$. To do so we use Lemma 4.10. Notice, that the function

$$x \to \mathbb{E}_u \widetilde{f}(x,b(t)) = p_t^G *_{\mathbb{R}^d} \widetilde{f}(x,u),$$

is smooth. Hence it is enough to show that

$$\int_0^t \mathbb{E}_u \big[L_{|b(t-s)|}\widetilde{F}(x,b(t-s),s)\big]ds \tag{4.15}$$

has η derivatives. Using that $L_u\widetilde{F}(x,u,s)$ grows polynomially as a function of x and u, we may compute

$$\begin{aligned}\partial_{u_i}\int_0^t \mathbb{E}_u \big[L_{|b(t-s)|}\widetilde{F}(x,b(t-s),s)\big]ds &= \partial_{u_i}\int_0^t p_{t-s}^G *_{\mathbb{R}^d} L_{|u|}\widetilde{F}(x,u,s)ds \\ &= \int_0^t\int_{\mathbb{R}_d}(\partial_{u_i}p_{t-s}^G)(u-w)L_{|w|}\widetilde{F}(x,w,s)dwds\end{aligned}$$

The above expression is well defined, because by (4.13) the inner integral can be dominated by $\frac{C(t,x,u)}{\sqrt{t-s}}$ and $C(t,x,u)$ growing polynomially with respect to t,x,u. Therefore, by (4.11) there is M such that

$$|\partial_{u_j}\widetilde{F}(x,u,t)| \le C(1+|x|+|u|+t)^M$$

and so

$$|\partial_{w_j}L_{|w|}\widetilde{F}(x,w,s)| \le C(1+|x|+|w|+s)^M$$

Using that we compute the second derivative

$$\partial_{u_i}\partial_{u_j}\int_0^t \mathbb{E}_u \big[L_{|b(t-s)|}\widetilde{F}(x,b(t-s),s)\big]ds = \int_0^t\int_{\mathbb{R}_d}(\partial_{u_i}p_{t-s}^G)(u-w)(\partial_{w_j}L_{|w|}\widetilde{F})(x,w,s)dwds$$

Applying the same argument η times we prove C^η smoothness with respect to u. Notice that at every step we dominate the derivatives by a polynomial and we can put more and more derivatives on $L_{|w|}\widetilde{F}(x,w,s)$.

To prove continuity of $\widetilde{F}$ with respect to x, u, t together, we use again (4.11), elementary properties of p_t^G and the fact that $L_u\widetilde{F}(x,u,s)$ grows polynomially. □

Now, we show that Lemmas 4.10 and 4.14 imply Theorem 4.7. First we prove continuity of F, then we apply ∂_t and conclude (4.8). Here are the details.

Proof of Theorem 4.7. By Lemma 4.10 we have

$$\begin{aligned}\frac{\widetilde{F}(x,u,t+h)-\widetilde{F}(x,u,t)}{h} &= \frac{\mathbb{E}_u\widetilde{f}(x,b(t+h))-\mathbb{E}_u\widetilde{f}(x,b(t))}{h} \\ &+ \frac{1}{h}\int_0^t\Big(\mathbb{E}_u\big[L_{|b(t+h-s)|}\widetilde{F}(x,b(t+h-s),s)\big]-\mathbb{E}_u\big[L_{|b(t-s)|}\widetilde{F}(x,b(t-s),s)\big]\Big)ds \\ &+ \frac{1}{h}\int_t^{t+h}\mathbb{E}_u\big[L_{|b(t+h-s)|}\widetilde{F}(x,b(t+h-s),s)\big]ds.\end{aligned}$$

Letting h go to 0, by Lemma 4.14, we get a pointwise equality

$$\partial_t \widetilde{F}(x,u,t) = \Delta_u \mathbb{E}_u \widetilde{f}(x,b(t)) + \Delta_u \int_0^t \mathbb{E}_u\big[L_{|b(t-s)|}\widetilde{F}(x,b(t-s),s)\big]ds + L_{|u|}\widetilde{F}(x,u,t)$$

On the other hand, by Lemma 4.10, again,

$$\begin{aligned}\widetilde{L}_\alpha \widetilde{F}(x,u,t) &= L_{|u|}\widetilde{F}(x,u,t) + \Delta_u \widetilde{F}(x,u,t)\\ &= L_{|u|}\widetilde{F}(x,u,t) + \Delta_u\big(\mathbb{E}_u \widetilde{f}(x,b(t)) + \int_0^t \mathbb{E}_u\big[L_{|b(t-s)|}\widetilde{F}(x,b(t-s),s)\big]ds\big).\end{aligned}$$

□

5. Subelliptic estimates

In this chapter we deal with subelliptic estimates for $\widetilde{L}_\alpha - \partial_t$ harmonic functions. It can be easily seen that the operator satisfies the Hörmander condition on $N \times (\mathbb{R}^d \setminus \{0\}) \times \mathbb{R}^+$ and so, if the vector fields $|u|^{-1}\Phi_{|u|}(X_j)$, $|u|^{-2}\Phi_{|u|}(X_0)$ happen to be smooth, this also is true on the whole of $N \times \mathbb{R}^d \times \mathbb{R}^+$. In general they are not smooth but at least they are C^2. So we still can get a subelliptic estimate for u-radial functions.

Let U be an open set in N, K a compact subset of U and $B_r(u_0)$ the ball in $\mathbb{R}^d$ with the center u_0 and the radius r. For $x_0 \in N$, $u_0 \in \mathbb{R}^d$ and $t_0 < t_2 < t_3 < t_1$ we consider

$$\Omega(x_0,u_0) = x_0 U \times B_{\frac{3}{2}}(u_0) \times (t_0,t_1)$$

and

$$\widetilde{K}(x_0,u_0) = x_0 K \times \overline{B_1(u_0)} \times [t_2,t_3].$$

Theorem 5.1. *If on $\Omega(x_0,u_0)$, $|u_0| \geq \frac{7}{4}$, $(\widetilde{L}_\alpha - \partial_t)F = 0$ in the sense of distributions, then $F \in C^\infty(\Omega(x_0,u_0))$. Moreover, there are $C, M > 0$ such that for every $x_0 \in N$, $u_0 \in \mathbb{R}^d$ and every F satisfying $(\widetilde{L}_\alpha - \partial_t)F = 0$ on $\Omega(x_0,u_0)$ we have*

$$\|F\|_{L^\infty(\widetilde{K})} \leq C(1+|u_0|)^M \|F\|_{L^2(\Omega)} \tag{5.2}$$

with C independent of x_0. If the vector fields, $|u|^{-1}\Phi_{|u|}(X_j)$, $|u|^{-2}\Phi_{|u|}(X_0)$ are smooth, then the conclusion holds for all u_0.

Assume now that $\Re\lambda_j > 4$, $j = 1,...,D$ and so the vector fields in $\widetilde{L}_\alpha - \partial_t$ are C^2. We consider the space

$$W(\Omega(x_0,0))$$

that consists of functions F on $\Omega(x_0,0)$ such that

$$F,\ \partial_t F\ \partial_u^\beta F,\ \partial_x^\beta F,\ \in L^2(\Omega(x_0,0)), |\beta| \leq 2.$$

Clearly, $(\widetilde{L}_\alpha - \partial_t)F \in L^2(\Omega(x_0,0))$ and we have the following theorem

Theorem 5.3. *Let*

- $F \in W(\Omega(x_0,0))$
- $(\widetilde{L}_\alpha - \partial_t)F = 0$ *on* $\Omega(x_0,0)$*;*
- *for fixed x and t the function $F(x,\cdot,t)$ depends only on $|u|$,*

then

$$\|F\|_{L^\infty(\widetilde{K})} \leq C\|F\|_{L^2(\Omega)}, \tag{5.4}$$

with C independent of x_0. If $\Re\lambda_1 > \lfloor\frac{\alpha+n+2}{2}\rfloor + 3$ then (5.4) holds also for F that are non-radial w.r. to u .

Below we present a proof of Theorems 5.1 and 5.3 trying to separate what is specific for our operator from what is a rather standard pseudodifferential calculus. Let

$$\begin{aligned}\widetilde{L}_{\alpha,u_0}-\partial_t &= \sum_{j=1}^{d}\partial_{u_j}^2+|u_0+u|^{-2}\sum_{j=1}^{m}\Phi_{|u_0+u|}(X_j)^2+|u_0+u|^{-2}\Phi_{|u_0+u|}(X_0)-\partial_t\\ &= \sum_{j=1}^{d}\partial_{u_j}^2+\sum_{j=1}^{m}Y_j^2+Y_0-\partial_t,\end{aligned}$$

where

$$Y_j=|u_0+u|^{-1}\Phi_{|u_0+u|}(X_j),\quad j=1,\dots m$$

and

$$Y_0=|u_0+u|^{-2}\Phi_{|u_0+u|}(X_0).$$

In this notation $\widetilde{L}_{\alpha,0}-\partial_t=\widetilde{L}_\alpha-\partial_t$. In order to keep the subelliptic constant in (5.2) under control we consider the family of operators $\widetilde{L}_{\alpha,u_0}$ on one set

$$\Omega=\Omega(e,0)=U\times B_{\frac{3}{2}}(0)\times(t_0,t_1)$$

instead of the operator $\widetilde{L}_\alpha-\partial_t$ on the family of sets $\Omega(x_0,u_0)$. Let

$$F_{x_0,u_0}(x,u,t)=F(x_0x,u_0+u,t).$$

Clearly,

$$(\widetilde{L}_{\alpha,u_0}-\partial_t)F_{x_0,u_0}=0 \text{ on } \Omega, \tag{5.5}$$

if, and only if,

$$(\widetilde{L}_\alpha-\partial_t)F=0 \text{ on } \Omega(x_0,u_0).$$

We identify $N\times\mathbb{R}^+$ with $\mathbb{R}^n\times\mathbb{R}^+$ and for $f\in C_c^\infty(N\times\mathbb{R}^d\times\mathbb{R}^+)$ we define a partial Sobolev norm

$$\|f\|_s^2=\int_{\mathbb{R}^d}|f_u|_s^2\,du,$$

where $f_u(x,t)=f(x,u,t)$ and $|f_u|_s$ is the s-Sobolev norm in $\mathbb{R}^n\times\mathbb{R}$. *For $\phi,\psi\in C_c^\infty$ we write $\psi\succ\phi$, if $\psi=1$ on a neighborhood of the support of ϕ.*

Our procedure is going to estimate derivatives of the solutions of of (5.5) and to keep the subelliptic constant under control. Since the operators have more smoothness in the x,t-direction than in the u-direction, taking partial Sobolev norms seems to be very convenient. *The crucial estimate is contained in Theorem 5.6 below.* It requires only C^2 vector fields which is guaranteed by the assumption $\Re\lambda_j>4$. Indeed, then $|u|^{\lambda_j-2}\log^\beta|u|\in C^2$, first and second derivatives being equal to 0 at 0.

For a positive number ε let $\mathcal{P}_\varepsilon$ be the set of smooth vector fields X on Ω with the following property: there is M, that for any $\psi,\phi\in C_c^\infty(\Omega)$ such that $\psi\succ\phi$ there exists a constant C that for any $f\in C^\infty(N\times\mathbb{R}^d\times\mathbb{R}^+)$ and any $u_0\in\mathbb{R}^d$

$$\|X(\phi f)\|_{-1+\varepsilon}^2\le C(1+|u_0|)^M(\|\psi(\widetilde{L}_{\alpha,u_0}-\partial_t)f\|_{L^2}^2+\|\psi f\|_{L^2}^2).$$

Theorem 5.6. *Assume that $\Re\lambda_j>4$, $j=1,...,D$. Then there is ε such that every smooth vector field on Ω is an element of $\mathcal{P}_\varepsilon$.*

Example 5.1. Since the proof is rather technical, we look at an example first. Consider $\mathbb{R}^2\times\mathbb{R}$ with

$$ad_H=\begin{pmatrix}5\frac{1}{2} & 1\\ -1 & 5\frac{1}{2}\end{pmatrix}$$

and the operator

$$\widetilde{L}_\alpha = \Delta_u + |u|^{-2}\Phi_u(\partial_x)^2 + |u|^{-2}\Phi_u(\partial_y)^2.$$

Clearly,

$$\Phi_{|u|} = |u|^{\frac{11}{2}} \begin{pmatrix} \cos\log|u| & \sin\log|u| \\ -\sin\log|u| & \cos\log|u| \end{pmatrix}$$

and so

$$\begin{aligned} |u|^{-1}\Phi_u(\partial_x) =& |u|^{\frac{9}{2}}(\cos\log|u|\partial_x + \sin\log|u|\partial_y) \\ |u|^{-1}\Phi_u(\partial_y) =& |u|^{\frac{9}{2}}(-\sin\log|u|\partial_x + \cos\log|u|\partial_y) \end{aligned}$$

First we prove that

$$\partial_{u_j}, |u|^{-1}\Phi_u(\partial_x), |u|^{-1}\Phi_u(\partial_y) \in \mathcal{P}_1.$$

But since we take only the partial Sobolev norms then also

$$(\sin\log|u|)|u|^{-1}\Phi_u(\partial_x) + (\cos\log|u|)|u|^{-1}\Phi_u(\partial_y) = |u|^{\frac{9}{2}}\partial_y \in \mathcal{P}_1.$$

By the same principle $|u|^{10}\partial_y \in \mathcal{P}_1$. Now taking a few commutators of the kind $[\partial_{u_j}, |u|^{10}\partial_y]$ we arrive to ∂_y. With every step the constant increases and grows polynomially with u.

Proof. To get the above subelliptic estimate we have to deal with two obstacles: the first order term being $Y_0 - \partial_t$ instead of Y_0 and degeneracy of $\widetilde{L}_\alpha$ when $u \to 0$. First we prove that $\partial_{u_j}, Y_j \in \mathcal{P}_1$ (Lemma A.1) and $Y_0 - \partial_t \in \mathcal{P}_{\frac{1}{2}}$ (Lemma A.2). The proof is rather a standard argument that uses two special properties of our operator:

- coefficients as functions of u are elements of C^2,
- multiplication by functions of u commutes with the action of vector fields $Y_0, \dots, Y_m$.

Here and in what follows we think of $u \in B_{\frac{3}{2}}(0)$ as a variable and of u_0 as a parameter. The constant that appears in front of the subelliptic estimate is a power of the C^2-norm of the coefficients $|u_0 + u|^\lambda \log^\beta |u_0 + u|$ (see (2.12)). Hence the conclusion. The only dependence on u_0 is reflected in the factor $(1 + |u_0|)^M$ in front of the subelliptic estimate.

Now we estimate the brackets. Finitely many brackets of the vector fields

$$H, \Phi_a(X_0), ..., \Phi_a(X_m)$$

spans $\mathcal{S}$ as a linear space. Since a bracket of two left-invariant vector fields is left-invariant and $[\mathcal{S}, \mathcal{S}] = \mathcal{N}$, all the brackets to consider are of the form $\Phi_a(W)$, $W \in \mathcal{N}$. Therefore, taking a finite number of brackets we generate vector fields $\Phi_a(W_1), ..., \Phi_a(W_p)$ that together with H form a basis of $\mathcal{S}$. In particular, $W_1, ..., W_p$ is a basis of $\mathcal{N}$.

The first step is to prove that there is ε such that

$$\Phi_{|u+u_0|}(W_j) \in \mathcal{P}_\varepsilon, \quad j = 1, ..., p$$

For that we consider brackets of vector fields

$$\sum_{j=1}^d (u + u_0)_j \partial_{u_j}, \quad \Phi_{|u+u_0|}(X_j), \quad \Phi_{|u+u_0|}(X_0) - |u + u_0|^2\partial_t$$

By Lemmas A.1 and A.2, they belong either to $\mathcal{P}_1$ or to $\mathcal{P}_{\frac{1}{2}}$. Taking brackets of vector fields, we substitute $\sum_{j=1}^d (u + u_0)_j \partial_{u_j}$, in place of H and $\Phi_{|u+u_0|}(X_j)$ in place of $\Phi_a(X_j)$, $j = 1, ..., m$ and also $\Phi_{|u+u_0|}(X_0) - |u + u_0|^2\partial_t$ in place of $\Phi_a(X_0)$. In this way we obtain

$$\Phi_{|u+u_0|}(W_j), \text{ or } \Phi_{|u+u_0|}(W_j) - 2^k|u + u_0|^2\partial_t \tag{5.7}$$

Indeed,

$$[\Phi_{|u+u_0|}(W), |u + u_0|^2\partial_t] = 0$$

and

$$[\Phi_{|u+u_0|}(W), \Phi_{|u+u_0|}(Z)] = \Phi_{|u+u_0|}([W, Z]).$$

Moreover,

$$[\sum_{j=1}^{d}(u+u_0)_j\partial_{u_j}, |u+u_0|^2\partial_t] = 2|u+u_0|^2\partial_t$$

and, if $[a\partial_a, \Phi_a(W)] = \Phi_a(Z)$, then

$$[\sum_{j=1}^{d}(u+u_0)_j\partial_{u_j}, \Phi_{|u+u_0|}(W)] = \Phi_{|u+u_0|}(Z)$$

The second case in (5.7) corresponds only to the bracket with $V_l = \sum_{j=1}^{d}(u+u_0)_j\partial_{u_j}$ for all l's. A subsequent application of Lemmas A.3, A.4 and A.5 shows that there is $\varepsilon > 0$ such that for every element W of $\mathcal{N}$ we have

$$\Phi_{|u+u_0|}(W) \in \mathcal{P}_\varepsilon, \text{ or } \Phi_{|u+u_0|}(W) - c_W|u+u_0|^2\partial_t \in \mathcal{P}_\varepsilon \tag{5.8}$$

Next, for $\lambda \in \Lambda$ and $j = 1, \dots, d(\lambda)$ let W_j^λ and V_j^λ be elements of $\mathcal{N}$ such that

$$W_j^\lambda + iV_j^\lambda = Z_j^\lambda,$$

where Z_j^λ are as in (2.11).

Then, (5.8) shows that for all $\lambda \in \Lambda$ and $j = 1, ..., d(\lambda)$

$$\begin{aligned} &\Phi_{|u+u_0|}(W_j^\lambda) - c_j^\lambda|u+u_0|^2\partial_t \in \mathcal{P}_\varepsilon, \\ &\Phi_{|u+u_0|}(V_j^\lambda) - d_j^\lambda|u+u_0|^2\partial_t \in \mathcal{P}_\varepsilon, \end{aligned} \tag{5.9}$$

for some constants c_j^λ, d_j^λ.

Now applying Lemma A.5, we are going to deduce from (5.9) that there is ε_j^λ such that

$$W_j^\lambda, V_j^\lambda \in \mathcal{P}_{\varepsilon_j^\lambda},$$

which implies that every $W \in \mathcal{N}$ is an element of $\mathcal{P}_{\varepsilon_0}$ for $\varepsilon_0 = \min\{\varepsilon_j^\lambda\}$.

We proceed by induction. Fix $\lambda \in \Lambda$ and assume that $W_1^\lambda, V_1^\lambda, ..., W_{j-1}^\lambda, V_{j-1}^\lambda \in \mathcal{P}_\varepsilon$ for some ε. Then, by (2.12), we have

$$\Phi_{|u+u_0|}(W_j^\lambda) + i\Phi_{|u+u_0|}(V_j^\lambda) = \Phi_{|u+u_0|}(Z_j^\lambda) = |u_0+u|^\lambda\Big(\sum_{k=k_j}^{j}\frac{1}{(j-k)!}(\log|u_0+u|)^{j-k}Z_k^\lambda\Big)$$

$$= |u_0+u|^{\Re\lambda}\Big(\cos(\Im\lambda\cdot\log|u_0+u|) + i\sin(\Im\lambda\cdot\log|u_0+u|)\Big)\cdot\Big(\sum_{k=k_j}^{j}\frac{1}{(j-k)!}(\log|u_0+u|)^{j-k}Z_k^\lambda\Big)$$

$$= |u_0+u|^{\Re\lambda}\sum_{k=k_j}^{j}\Big(\frac{1}{(j-k)!}(\log|u_0+u|)^{j-k}\big(\cos(\Im\lambda\cdot\log|u_0+u|)W_k^\lambda - \sin(\Im\lambda\cdot\log|u_0+u|)V_k^\lambda\big)\Big)$$

$$+ i|u_0+u|^{\Re\lambda}\sum_{k=k_j}^{j}\Big(\frac{1}{(j-k)!}(\log|u_0+u|)^{j-k}\big(\sin(\Im\lambda\cdot\log|u_0+u|)W_k^\lambda + \cos(\Im\lambda\cdot\log|u_0+u|)V_k^\lambda\big)\Big).$$

Therefore,

$$\begin{aligned}\Phi_{|u+u_0|}(W_j^\lambda) &= |u_0+u|^{\Re\lambda}\Big(\sum_{k=k_j}^{j}\frac{1}{(j-k)!}(\log|u_0+u|)^{j-k}Z_k^1\Big),\\ \Phi_{|u+u_0|}(V_j^\lambda) &= |u_0+u|^{\Re\lambda}\Big(\sum_{k=k_j}^{j}\frac{1}{(j-k)!}(\log|u_0+u|)^{j-k}Z_k^2\Big),\end{aligned}$$

where

$$\begin{aligned}Z_k^1 &= C(u)W_k^\lambda - S(u)V_k^\lambda,\\ Z_k^2 &= S(u)W_k^\lambda + C(u)V_k^\lambda,\\ C(u) &= \cos(\Im\lambda\cdot\log|u_0+u|),\\ S(u) &= \sin(\Im\lambda\cdot\log|u_0+u|).\end{aligned}$$

By induction, Z_k^l are elements of $\mathcal{P}_\varepsilon$ for $k=1,...,j-1$ and $l=1,2$, hence by (5.9)

$$\begin{aligned}&|u_0+u|^{\Re\lambda}Z_j^1 - c_j^\lambda|u_0+u|^2\partial_t\in\mathcal{P}_\varepsilon,\\ &|u_0+u|^{\Re\lambda}Z_j^2 - d_j^\lambda|u_0+u|^2\partial_t\in\mathcal{P}_\varepsilon.\end{aligned}\tag{5.10}$$

Now we are going to prove that

$$W_j^\lambda, V_j^\lambda\in\mathcal{P}_{\frac{\varepsilon}{2^{\lceil\Re\lambda\rceil}}},$$

If λ is a real number then $Z_j^1=W_j^\lambda$ and $Z_j^2=V_j^\lambda$, and so, by Lemma A.5, we obtain the conclusion.

For the case, when λ is not real, notice first that in view of (5.10) and Lemma A.5

$$X_1=[\partial_{u_2},[\partial_{u_1},|u+u_0|^{\Re\lambda}Z_j^1-c_j^\lambda|u+u_0|^2\partial_t]]$$

is an element of $\mathcal{P}_{\varepsilon/4}$. We may compute the brackets explicitly:

$$X_1=[\partial_{u_2},[\partial_{u_1},|u+u_0|^{\Re\lambda}Z_j^1]]=f(u)(\alpha Z_j^1-\beta Z_j^2),$$

where

$$\begin{aligned}f(u) &= (u_0+u)_1(u_0+u)_2|u_0+u|^{\Re\lambda-4},\\ \alpha &= \Re\lambda(\Re\lambda-2)-(\Im\lambda)^2,\\ \beta &= 2\Im\lambda(\Re\lambda-1).\end{aligned}$$

In the same way we show that the vector field

$$X_2=[\partial_{u_2},[\partial_{u_1},|u+u_0|^{\Re\lambda}Z_j^2-d_j^\lambda|u+u_0|^2\partial_t]]=f(u)(\alpha Z_j^2+\beta Z_j^1)$$

belongs to $\mathcal{P}_{\varepsilon/4}$, which implies that

$$\begin{aligned}Y_1 &= C(u)X_1+S(u)X_2=f(u)\big(\alpha(C(u)Z_j^1+S(u)Z_j^2)-\beta(C(u)Z_j^2-S(u)Z_j^1)\big)\\ &= f(u)(\alpha W_j^\lambda-\beta V_j^\lambda)\\ Y_2 &= S(u)X_1-C(u)X_2=f(u)\big(\alpha(S(u)Z_j^1-C(u)Z_j^2)-\beta(S(u)Z_j^2+C(u)Z_j^1)\big)\\ &= f(u)(-\alpha V_j^\lambda-\beta W_j^\lambda)\end{aligned}$$

are elements of $\mathcal{P}_{\frac{\varepsilon}{4}}$. Notice that

$$\begin{aligned}\alpha Y_1-\beta Y_2 &= (\alpha^2+\beta^2)f(u)W_j^\lambda,\\ \beta Y_1+\alpha Y_2 &= -(\alpha^2+\beta^2)f(u)V_j^\lambda.\end{aligned}$$

Consequently,

$$f(u)W_j^\lambda, f(u)V_j^\lambda\in\mathcal{P}_{\frac{\varepsilon}{4}},$$

and applying again Lemma A.5 we conclude

$$W_j^\lambda, V_j^\lambda \in \mathcal{P}_{\frac{\varepsilon}{2\lceil \Re\lambda\rceil}},$$

which finishes the proof.

□

Proof of Theorem 5.1. While Theorem 5.6 is proved and Y_j's are smooth then smoothness of the solution follows in a standard way as described in [Tr]. To get the right constant we prove by induction that for all $\phi, \psi \in C_c^\infty(\Omega)$, all $f \in C^\infty(\Omega)$ and all u_0

$$\|\phi f\|^2_{k\frac{\varepsilon}{2}} + \sum_{j=1}^m \|Y_j(\phi f)\|^2_{(k-1)\frac{\varepsilon}{2}} + \sum_{j=1}^m \|\partial_{u_j}(\phi f)\|^2_{(k-1)\frac{\varepsilon}{2}}$$
$$\leq C(1+|u_0|)^M(\|\psi(\widetilde{L}_{\alpha,u_0} - \partial_t)f\|_{(k-2)\frac{\varepsilon}{2}} + \|\psi f\|_{(k-2)\frac{\varepsilon}{2}}.$$

Then we add partial derivatives with respect to u and for harmonic F, i.e. $(\widetilde{L}_{\alpha,u_0} - \partial_t)F = 0$, we have

$$\|\phi\partial_u^\beta F\|_s^2 \leq C(1+|u_0|)^{M(\beta,s)}\|\psi F\|_{L^2}.$$

for every multiindex β. Arguing as in the proof of the next theorem (see the Appendix). We observe that $(1+|u_0|)^M$ is the constant at the right side in front of the L^2 norm of the function. □

Proof of Theorem 5.3. Now we pass to the "nonsmooth" case i.e. we consider $\widetilde{L}_\alpha - \partial_t$ on Ω while some of the Y_j's, $j = 0, ..., m$ are not necessarily smooth.

Theorem 5.11. *For every partial derivative $D = \partial_x^\gamma \partial_t^\delta$, every $|\beta| \leq 2$, every $\phi \in C_c^\infty(\Omega)$, there is C and $\psi \in C_c^\infty(\Omega)$, $\psi \succ \phi$ such that*

$$\|\phi\partial_u^\beta DF\|_{L^2} \leq C\|\psi F\|^2_{L^2}, \tag{5.12}$$

whenever $(\widetilde{L}_\alpha - \partial_t)F = 0$ on Ω. If $\Re\lambda_1 > p+2$, $p \geq 2$, then (5.12) holds for $|\beta| \leq p+1$.

The main ingredient in the proof of the above theorem is Theorem 5.6. The rest is contained in the Appendix (Lemmas A.6, A.9 and the argument at the end). Observe that as a corollary of the Theorem above, by the Sobolev Lemma we obtain the second part of Theorem 5.3.

Now we are going to consider the case $\Re\lambda_1 \leq \lfloor\frac{\alpha+n+2}{2}\rfloor + 3$. (5.4) follows from the next three lemmas:

Lemma 5.13. *Let h be a function on $(0,1)$ satisfying*

$$a\partial_a h(a) + \alpha h(a) = v(a), \tag{5.14}$$

for $\alpha > 0$. Furthermore, assume $|h(a)| \leq Ca^{-M}$ and $|v(a)| \leq Ca^{-N}$, for $N, M < \alpha$. Then there exists a constant C_0 such that

$$|h(a)| \leq C_0 a^{-N}.$$

Proof. We have

$$\partial_a(a^\alpha h(a)) = \alpha a^{\alpha-1}h(a) + a^\alpha\partial_a h(a) = a^{\alpha-1}(\alpha h(a) + a\partial_a h(a)) = a^{\alpha-1}v(a).$$

Hence

$$\begin{aligned} |a^\alpha h(a) - b^\alpha h(b)| &= \Big|\int_b^a \partial_u(u^\alpha h(u))du\Big| \leq \int_b^a u^{\alpha-1}|v(u)|du \\ &\leq \int_b^a u^{\alpha-N-1}du = C(a^{\alpha-N} - b^{\alpha-N}). \end{aligned}$$

Passing with b to zero, we obtain

$$|a^\alpha h(a)| \le Ca^{\alpha-N}$$

□

Let F be a function radial w.r. to u on Ω satisfying assumptions of Theorem 5.3. Define a function on $U \times \mathbb{R}^+ \times (t_0, t_1)$ by

$$F_{rad}(x,a,t) = F(x,u,t),$$

then $(L_\alpha - \partial_t)F_{rad}(x,a,t) = 0$.

Lemma 5.15. *For every $(x,a,t) \in \widetilde{K} = K \times (0,1] \times (t_2,t_3)$ we have*

$$\begin{aligned} |\widetilde{X}^I F_{rad}(x,a,t)| &\le C_I a^{-\frac{\alpha}{2}} \|F\|_{L^2}, \\ |\partial_a \widetilde{X}^I F_{rad}(x,a,t)| &\le C_I a^{-(\frac{\alpha}{2}+1)} \|F\|_{L^2}, \end{aligned}$$

where $\widetilde{X}^I = \partial_t^{i_0} \widetilde{X}_1^{i_1} ... \widetilde{X}_n^{i_n}$, and $\widetilde{X}_n$ are right-invariant vector fields on N.

Proof. Let

$$\begin{aligned} G_{I,rad}(x,a,t) &= \widetilde{X}^I F_{rad}(x,a,t) \\ G_I(x,u,t) &= G_{I,rad}(x,|u|,t). \end{aligned}$$

Notice that $(\widetilde{L}_\alpha - \partial_t)G_I = 0$. Let $\phi_{rad} \in C_c^\infty$ be u radial and equal 1 on $\widetilde{K}$. Then, by the Sobolev inequality and Theorem 5.11, we have

$$\begin{aligned} |G_{I,rad}(x,a,t)| &= \Big| \int_a^{\frac{3}{2}} \partial_b \big(\phi_{rad}(x,b,t) G_{I,rad}(x,b,t)\big) db \Big| \\ &\le \Big(\int_a^{\frac{3}{2}} |\partial_b \big(\phi_{rad}(x,b,t) G_{I,rad}(x,b,t)\big)|^2 b^{1+\alpha} db \Big)^{\frac{1}{2}} \Big(\int_a^{\frac{1}{2}} b^{-(1+\alpha)} db \Big)^{\frac{1}{2}} \\ &\le Ca^{-\frac{\alpha}{2}} \Big(\int_{a<|u|<\frac{3}{2}} |\sum \frac{u_j}{|u|} \partial_{u_j} [\phi(x,u,t) G_I(x,u,t)]|^2 du \Big)^{\frac{1}{2}} \\ &\le Ca^{-\frac{\alpha}{2}} \sum_j \sum_{|I|+i \le \frac{n+3+\alpha}{2}} \Big(\int_{B_{\frac{3}{2}}(0)} \int_U \int_{t_0}^{t_1} |\partial_y^I \partial_s^i \partial_{u_j} [\phi(y,u,s) G_I(y,u,s)]|^2 ds dy du \Big)^{\frac{1}{2}} \\ &\le Ca^{-\frac{\alpha}{2}} \|F\|_{L^2}. \end{aligned}$$

The proof of the second case is exactly the same, but the second derivatives in u appear, for this reason we obtain a different exponent. □

Theorem 5.3 is a consequence of the following lemma

Lemma 5.16. *There is C such that for $k = 0, 1, ..., N = \lceil \alpha/4 \rceil$ and $(x,a,t) \in \widetilde{K}$ we have*

$$|\widetilde{X}^I F_{rad}(x,a,t)| \le C(a^{-\frac{\alpha}{2}+2k} \vee 1) \|F\|_{L^2}, \tag{5.17}$$

where $|I| = 2(N-k)$, or $|I| = 2(N-k)-1$. In particular, taking $k = N$, we obtain

$$|F_{rad}(x,a,t)| \le C\|F\|_{L^2}.$$

Proof. For $k = 0$ this is exactly (5.15). Assume (5.17) for a k. Put

$$G_{rad}(x,a,t) = \widetilde{X}^I F_{rad}(x,a,t),$$

for $|I| = 2(N-k-1)$ or $|I| = 2(N-k-1)-1$. Then $(\mathcal{L}_\alpha - a^2\partial_t)G_{rad} = 0$. Define

$$\begin{aligned} h(a) &= a\partial_a G_{rad}(x,a,t), \\ v(a) &= \big(\sum_{j=1}^m \Phi_a(X_j)^2 + \Phi_a(X_0) - a^2\partial_t\big)G_{rad}(x,a,t), \end{aligned}$$

then both functions satisfy (5.14), and so by Lemma 5.15, $|h(a)| \le Ca^{-M}$, for $M < \alpha$. Furthermore, by the induction hypothesis and the observation that left-invariant vector fields are linear combinations of right-invariant vector fields, the coefficients being smooth, bounded functions on K, we have

$$|v(a)| \le Ca^{-\frac{\alpha}{2}+2(k+1)},$$

In view of Lemma 5.13 we get

$$|a\partial_a G_{rad}(x,a,t)| \le Ca^{-\frac{\alpha}{2}+2(k+1)}.$$

Finally

$$\begin{aligned} |G_{rad}(x,a,t)| &\le \Big|\int_1^a \partial_b G_{rad}(x,b,t)db\Big| + |G_{rad}(x,1,t)| \le C\Big|\int_1^a b^{-\frac{\alpha}{2}+2(k+1)-1}db\Big| + |G_{rad}(x,1,t)| \\ &\le C(a^{-\frac{\alpha}{2}+2(k+1)} \vee 1) \end{aligned}$$

□

6. The Green function for L_α^*

In this section we are going to use Theorem 5.3 to estimate the Green function for L_α^*, where $L_\alpha = a^{-2}\mathcal{L}_\alpha$. We start with an observation that if

$$D_r(x,a) = (\Phi_r(x), ra),\ r > 0,$$

then

$$L_\alpha(f \circ D_r) = r^2(L_\alpha f) \circ D_r. \tag{6.1}$$

Moreover, L_α commutes with the natural action of N on $N \times \mathbb{R}^+$ on the left. Let

$$L_\alpha^* = a^{-2}\mathcal{L}_\alpha = \partial_a^2 + \frac{1+\alpha}{a}\partial_a + a^{-2}\sum_{j=1}^m \Phi_a(X_j)^2 - a^{-2}\Phi_a(X_0)$$

be the formal adjoint of L_α on $L^2(dxa^{\alpha+1}da)$. For $f \in C_c^\infty(N \times \mathbb{R}^+)$, $x \in N$ and $a \ge 0$, define $T_t^* f(x,a)$ according to (4.1). Then

$$\int_0^\infty |T_t^* f(x,a)|\ dt < \infty.$$

Indeed, as is proved below, $|T_t^* f(x,a)| \le |f|_{L^\infty}$ and for $t > 1$

$$|T^* f(x,a)| \le t^{-\frac{Q}{2}-\frac{\alpha}{2}-1}. \tag{6.2}$$

Therefore,

$$f \to \int_0^\infty T_t^* f(x,a)\ dt$$

is a positive functional on $C_c(S)$ and so there is a non-negative Radon measure $G^*(x,a;dydb)$ such that

$$\int_0^\infty T_t^* f(x,a)\ dt = \int_S f(y,b)G^*(x,a;dydb).$$

First observe that for $t \in I = (t_1,t_2)$, by (4.6) and Theorems 5.1 and 5.3, we have

$$|T_t^* f(x,a)| \le C_I(1+a)^M \|T_t^* f\|_{L^2(dyb^{\alpha+1}db)} \le C_I(1+a)^M \|f\|_{L^2(dyb^{\alpha+1}db)},\ a \ge 0 \tag{6.3}$$

for a constant M. Therefore there exists a kernel $K_t^*(x,a;\cdot,\cdot)\in L^2(dyb^{\alpha+1}db)$ such that

$$T_t^* f(x,a)=\int_{N\times\mathbb{R}^+} K_t^*(x,a;y,b)f(y,b)dyb^{\alpha+1}db$$

and

$$\|K_t^*(x,a;\cdot,\cdot)\|_{L^2(dyb^{\alpha+1}db)}\le C_I(1+a)^M, a\ge 0 \tag{6.4}$$

Similarly, for L_α we get

$$\|K_t(x,a;\cdot,\cdot)\|_{L^2(dyb^{\alpha+1}db)}\le C_I(1+a)^M. \tag{6.5}$$

Moreover,

$$K_t^*(x,a;y,b)=K_t(y,b;x,a),\ \ a,b>0 \tag{6.6}$$

Lemma 6.7. *For every $t>0$, $a\ge 0$, $b>0$ we have*

$$K_t^*(x,a;y,b)=t^{-\frac{Q}{2}-\frac{\alpha}{2}-1}K_1^*(D_{t^{-\frac12}}(x,a);D_{t^{-\frac12}}(y,b))$$

Proof. As stated in Section 4

$$T_t f(x,a)=F(t,x,a)$$

is the unique solution of

$$(L_\alpha-\partial_t)F=0$$

with

$$\lim_{t\to 0}F(t,x,a)=f(x,a)$$

Therefore, by (6.1) we have:

$$T_t^*(f\circ D_r)\circ D_{r^{-1}}(x,a)=T^*_{r^2t}f(x,a),\ \ a>0 \tag{6.8}$$

because both sides of (6.8) are solutions with the same boundary data f. Passing to the limit as a tends to 0, we get (6.8) also for $a=0$. Hence

$$\begin{aligned}
T_t^* f(x,a) &= \int_{N\times\mathbb{R}^+} K_t^*(x,a;y,b)f(y,b)dyb^{\alpha+1}db\\
&= \int_{N\times\mathbb{R}^+} K_1^*(D_{t^{-\frac12}}(x,a);y,b)f(D_{t^{\frac12}}(y,b))dyb^{\alpha+1}db\\
&= t^{-\frac{Q}{2}-\frac{\alpha}{2}-1}\int_{N\times\mathbb{R}^+} K_1^*(D_{t^{-\frac12}}(x,a);D_{t^{-\frac12}}(y,b))f(y,b)dyb^{\alpha+1}db
\end{aligned}$$

□

Next we prove some pointwise estimates for the kernels K_t^*

Lemma 6.9. *There is C such that*

$$|K_1^*(x,a;y,b)|\le C(1+a)^M(1+b)^M, a\ge 0, b>0.$$

Proof. By (6.4), (6.5) and (6.6) we have

$$\begin{aligned}
|K_1^*(x,a;y,b)| &= |\int_{N\times\mathbb{R}^+} K^*_{1/2}(x,a;z,c)K^*_{1/2}(z,c;y,b)dzc^{\alpha+1}dc|\\
&= |\int_{N\times\mathbb{R}^+} K^*_{1/2}(x,a;z,c)K_{1/2}(y,b;z,c)dzc^{\alpha+1}dc|\\
&\le \|K^*_{1/2}(x,a;\cdot,\cdot)\|_{L^2}\|K_{1/2}(y,b;\cdot,\cdot)\|_{L^2}\\
&\le C(1+a)^M(1+b)^M.
\end{aligned}$$

□

Lemma 6.10. *Fix r. There exists $C = C(r)$ such that for $0 \le a \le 1, 0 < b \le 1$, $\tau(y) \ge \frac{r}{2}$, $\tau(x) \le \frac{r}{8}$ and $t \le 1$*

$$K_t^*(x, a; y, b) \le C. \tag{6.11}$$

Moreover, for every multiindex I

$$|\widetilde{X}^I K_t^*(e, a; y, b)| \le C_I. \tag{6.12}$$

where $\widetilde{X}^I$ is a right invariant differential operator applied to y.

Proof. By (4.1) and N-invariance of the operator we have

$$K_t^*(z, a; y, b) = K_t^*(y^{-1}z, a; e, b) = K_t^*(e, a; z^{-1}y, b) \tag{6.13}$$

and by two previous lemmas there are $C > 0$ and $M > 0$ such that

$$K_t^*(x, a; y, b) \le Ct^{-M}(1 + a)^M(1 + b)^M, \quad t \le 1,$$

Therefore we can write

$$\begin{aligned} K_t^*(x, a; y, b) &= \int_{N\times\mathbb{R}^+} K^*_{\frac{t}{2}}(e, a; z, c) K^*_{\frac{t}{2}}(z, c; x^{-1}y, b) dz c^{\alpha+1} dc \\ &= \int_{N\times\mathbb{R}^+} K^*_{\frac{t}{2}}(e, a; z, c) K_{\frac{t}{2}}(e, b; y^{-1}xz, c) dz c^{\alpha+1} dc \\ &\le Ct^{-M}\Bigg(\int_{\tau(z)<\frac{\tau(y^{-1}x)}{2}} K_{\frac{t}{2}}(e, b; y^{-1}xz, c)(1 + c)^M dz c^{\alpha+1} dc \\ &\quad + \int_{\tau(z)>\frac{\tau(y^{-1}x)}{2}} K^*_{\frac{t}{2}}(e, a; z, c)(1 + c)^M dz c^{\alpha+1} dc\Bigg) \end{aligned}$$

It suffices to estimate

$$\int_{\tau(z)>\frac{r}{8}} K_{\frac{t}{2}}(e, b; z, c)(1 + c)^M dz c^{\alpha+1} dc$$

and $K^*_{\frac{t}{2}}$ instead of $K_{\frac{t}{2}}$. Let $f(z, c) = f_1(z)f_2(c) \in C_c(N \times \mathbb{R}^d)$ with $\operatorname{supp} f_1 \subset \{\tau(z) > \frac{r}{8}\}$. Then

$$\begin{aligned} |T_{\frac{t}{2}} f(e, b)| &= \big|\mathbb{E}_b\big[f_1 * p^\sigma(0, t)(e) f_2(\sigma(t))\big]\big| \\ &\le |f_1|_{L^\infty} \big|\mathbb{E}_b\big[\int_{\tau(z)>\frac{r}{8}} p^\sigma(0, t)(z)\ dz f_2(\sigma(t))\big]\big| \\ &\le |f_1|_{L^\infty} \Big(\mathbb{E}_b\Big(\int_{\tau(z)>\frac{r}{8}} p^\sigma(0, t)(z)\ dz\Big)^2\Big)^{\frac{1}{2}} \Big(E_b f_2(\sigma(t))^2\Big)^{\frac{1}{2}} \end{aligned}$$

But, by (3.4),

$$\int_{\tau(z)>\frac{r}{8}} p^\sigma(0, t)(z)\ dz \le c_1 e^{-\frac{c_2}{A^\sigma(0,t)^\gamma}},$$

Also for every M there is $C = C(M)$ such that

$$\begin{aligned}\mathbb{E}_b e^{-\frac{2c_2}{A^\sigma(0,t)^\gamma}} &\le C\mathbb{E}_b A^\sigma(0,t)^{2M+1} = C\mathbb{E}_b\Big(\int_0^t (\sigma(s)^{2m_1-2} + \sigma(s)^{m_2-2})\ ds\Big)^{2M+1}\\ &\le C\mathbb{E}_b\Big(\int_0^t (\sigma(s)^{2m_1-2} + \sigma(s)^{m_2-2}))^{2M+1}\ ds\Big)\Big(\int_0^t 1\ ds\Big)^{(2M+1)(1-\frac{1}{2M+1})}\\ &\le Ct^{2M}\mathbb{E}_b\Big(\int_0^t \sigma(s)^{(2m_1-2)(2M+1)} + \sigma(s)^{(m_2-2)(2M+1)}\ ds\Big) \le Ct^{2M}\end{aligned}$$

The last inequality follows from (3.1). Therefore

$$|T_{\frac{t}{2}} f(e,b)| \le Ct^{2M}\Big(E_b f_2(\sigma(t))^2\Big)^{\frac{1}{2}}.$$

Approximating the function $c \to (1+c)^M$ by C_c^∞ functions, we then have

$$\int_{\tau(z)>\frac{r}{8}} K_{\frac{t}{2}}(e,b;z,c)(1+c)^M dz c^{\alpha+1} dc \le Ct^{2M}\Big(\int_{\mathbb{R}^+}(1+c)^{2M} p_t(a,c)c^{1+\alpha}\ dc\Big)^{\frac{1}{2}}$$

and (6.11) follows.

For (6.12) we notice that by (6.13)

$$X^I T_t^* f(e,a) = \int \widetilde{X}^I K_t^*(e,a;y,b) f(y,b)\ dy b^{1+\alpha} db,$$

where $\widetilde{X}^I$ is the right-invariant differential operator applied to y. On the other hand,

$$|X^I T_t^* f(e,a)| \le C|T_t^* f|_{L^2(U)},$$

with $U = \{(x,a) : \tau(x) < \frac{r}{8}, a < 1\}$. Now assume that $\operatorname{supp} f \subset \{(z,b) : \tau(z) > \frac{r}{2}\}$. By (6.11) for $(x,a) \in U$

$$\begin{aligned}|T_t^* f(x,a)| &\le C|f|_{L^1(b^{\alpha+1}dbdy)} \sup_{|x|<\frac{r}{8},|y|>\frac{r}{2},a,b<1} K_t^*(x,a;y,b)\\ &\le C|f|_{L^1(b^{\alpha+1}dbdy)}.\end{aligned}$$

and (6.12) follows. □

Lemma 6.14. *For every compact set $K \subset N$, $e \notin K$*

$$\sup_{\substack{0<b<1\\ y\in K}} |\widetilde{X}^I G^*(e,0;y,b)| < \infty. \tag{6.15}$$

Moreover,

$$L_\alpha G^*(e,0;\cdot,\cdot) = 0, \tag{6.16}$$

$$G^*(e,0;y,b) = b^{-Q-\alpha} G^*(e,0;\Phi_{b^{-1}}(y),1) \tag{6.17}$$

Proof. For $y \ne e$ we have the pointwise equality

$$G^*(e,0;y,b) = \int_0^\infty K_t^*(e,0;y,b)\ dt.$$

Moreover, in view of Lemmas 6.7 and 6.10,

$$|\widetilde{X}^I K_t(e,0;y,b)| \le \begin{cases} t^{-\frac{Q}{2}-\frac{\alpha}{2}-1} & \text{for } t \ge 1\\ C & \text{for } t \le 1.\end{cases}$$

Hence (6.15) follows.

For any $f \in C_c^\infty(N \times \mathbb{R}^+)$, by (4.2), we have

$$\begin{aligned}\int_{N\times\mathbb{R}^+} G^*(e,0;y,b)L_\alpha^* f(y,b)dyb^{\alpha+1}db &= \int_0^\infty T_t^* L_\alpha^* f(e,0)dt \\ &= \int_0^\infty \partial_t(T_t^* f(e,0))dt = \lim_{\varepsilon\to 0}(T_{\varepsilon^{-1}}^* f(e,0) - T_\varepsilon^* f(e,0)),\end{aligned}$$

but, by (4.9) and Lemma 6.7, the above limit is zero and so (6.16) follows.

Finally, in view of (6.8), we obtain

$$\begin{aligned}\int_{N\times\mathbb{R}^+} G^*(e,0;y,b)f(y,b)dyb^{\alpha+1}db &= \int_0^\infty T_t^* f(e,0)dt = \int_0^\infty T_{r^{-2}t}^*(f\circ D_r)(e,0)dt \\ &= r^2\int_0^\infty T_t^*(f\circ D_r)(e,0)dt = r^2\int_{N\times\mathbb{R}^+} G^*(e,0;y,b)(f\circ D_r)(y,b)dyb^{\alpha+1}db \\ &= r^{-Q-\alpha}\int_{N\times\mathbb{R}^+} G^*(e,0;\Phi_{r^{-1}}(y),r^{-1}b)(f\circ D_r)(y,b)dyb^{\alpha+1}db,\end{aligned}$$

which proves (6.17). □

7. Proofs of the main theorems

Lemma 7.1. *If $x \neq e$, then $\lim_{a\to 0}\widetilde{X}^I G^*(e,0;x,a)$ exists. Also the function $\widetilde{X}^I G^*(e,0;x,a)$ extended this way to $N\setminus\{e\}\times[0,\infty)$ is continuous.*

Proof. For $x \neq 0$ let

$$\begin{aligned}h(a) &= a\partial_a \widetilde{X}^I G^*(e,0;x,a), \\ v(a) &= -a^{-2}\Big(\sum_{j=1}^m \Phi_a(X_j)^2 + \Phi_a(X_0)\Big)\widetilde{X}^I G^*(e,0;x,a).\end{aligned}$$

Then by (6.16)

$$a\partial_a h(a) + \alpha h(a) = v(a)$$

$h(a)$ is bounded, by (6.15) and the Harnack inequality, $v(a) \le a^2$. A simple calculation contained in [DH2] shows that

$$\lim_{a\to 0}\widetilde{X}^I G^*(e,0;x,a) \text{ exists.}$$

Then to prove continuity on $N\setminus\{e\}\times[0,\infty)$ by (6.15) we estimate

$$|\widetilde{X}^I G^*(x_n,a) - \widetilde{X}^I G^*(x,a)| \le C\tau(x_n,x)$$

for $x_n \to x$ and $a \le 1$. □

Proof of Theorem 2.15. We have

$$\check{m}_\alpha(x) = G^*(e,0;x,1).$$

Indeed, by Lemma 6.14,

$$a^\alpha \mathcal{L}_\alpha(a^{-Q-\alpha}G^*(e,0;\Phi_{a^{-1}}(x),1)) = \mathcal{L}_{-\alpha}(a^{-Q}G^*(e,0;\Phi_{a^{-1}}(x),1)) = 0. \tag{7.2}$$

Let $P(xa) = a^{-Q}\check{m}_\alpha(\Phi_{a^{-1}}(x)$. 7.2 implies that for every $f \in C_c(N)$

$$F(xa)\int_N f(xy^{-1})P(ya)\,dy$$

is $\mathcal{L}_{-\alpha}$ harmonic i.e.

$$F * \mu = F. \tag{7.3}$$

Writing 7.3 in terms of m_α, we obtain

$$\check{\mu} * m_\alpha = m_\alpha,$$

which uniquely determines the measure m_α. Now, again by Lemma 6.14,

$$\begin{aligned}\lim_{a\to 0} a^{-Q-\alpha}\big(\Phi_{a^{-1}}(\widetilde{X}^I)\check{m}_\alpha\big)(\Phi_{a^{-1}}(x)) &= \lim_{a\to 0} a^{-Q-\alpha}\widetilde{X}^I(\check{m}_\alpha\circ\Phi_{a^{-1}})(x)\\ &= \lim_{a\to 0}\widetilde{X}^I G^*(e,0;x,a) = c_I(x).\end{aligned}$$

Here $\widetilde{X}_1,...,\widetilde{X}_n$ is a basis of right-invariant vector fields coresponding to the decomposition 2.14. Therefore,

$$\begin{aligned}\big|a^{-Q-\alpha}(\widetilde{X}^I\check{m}_\alpha)(\Phi_{a^{-1}}(x))\big| &= \big|a^{-Q-\alpha}\Phi_a(\widetilde{X}^I)(\check{m}_\alpha\circ\Phi_{a^{-1}})(x)\big|\\ &= \big|\Phi_a(\widetilde{X}^I)G^*(e,0;x,a)\big|\\ &= \begin{cases} a^{|I|} & \text{for diagonal action 2.2}\\ a^{|I|}(1+|\log a|)^{I_0} & \text{for general action,}\end{cases}\end{aligned}$$

where $I_0 = \sum_{j=1}^n i_j(\dim V_{\lambda(X_j)} - 1)$ as explained in [DHZ] (0.23). Now we assume that the vector fields $a\partial_a, \Phi_a(X_j), j = 1,...,m$ generate $\mathcal{S}$ and we prove that $c(x) > 0$. For $\lambda > 0$ let

$$G^{*,\lambda}(x,a;y,b) = \int_0^\infty e^{-\lambda t}K_t^*(x,a;y,b)\ dt.$$

Then the operator

$$f \to G^{*,\lambda}f(xa) = \int_{N\times\mathbb{R}^+} G^{*,\lambda}(x,a;y,b)f(yb)b^{1+\alpha}dyda$$

is bounded on $L^2(b^{1+\alpha}dydb)$ and for $\lambda > \mu$

$$G^{*,\mu} = G^{*,\lambda} + (\lambda-\mu)G^{*,\mu}G^{*,\lambda}.$$

Letting $\mu \to 0$ and applying the monotone convergence theorem, for $f \in C_c^\infty$, $f \geq 0$ we have

$$\lambda G^* G^{*,\lambda} f(e,a) \leq G^* f(e,a) \tag{7.4}$$

Since $G^*(e,a;\cdot)$ is L^*_α harmonic on $N\times\mathbb{R}^+\setminus\{(e,a)\}$,

$$\lim_{a\to 0} G^*(e,a;\cdot) = G^*(e,0;\cdot)$$

not only in the *weak sense but also pointwise. Therefore, (7.4) implies

$$\lambda G^* G^{*,\lambda} f(e,0) \leq G^* f(e,0),$$

i.e. for every (z,c)

$$\lambda\int_{N\times\mathbb{R}^+} G^*(e,0;y,b)G^{*,\lambda}(y,b;z,c)b^{1+\alpha}dydb \leq G^*(e,0;z,c).$$

In particular, if V is a bounded open set in $N\times\mathbb{R}^+$ then

$$\lambda\int_V G^*(e,0;y,b)G^{*,\lambda}(y,b;z,c)b^{1+\alpha}dydb \leq G^*(e,0;z,c).$$

Now we fix z and we would like to pass with c to 0 to get

$$\lambda\int_V G^*(e,0;y,b)G^{*,\lambda}(y,b;z,0)b^{1+\alpha}dydb \leq c(z)$$

For that we choose V in the way that $\inf\{\tau(y^{-1}z) : (y,b)\in V\} > 0$, we prove that

$$G^{*,\lambda}(y,b;z,c) \leq C$$

whenever $(y,b)\in V$ and we apply the bounded convergence theorem. Indeed,

$$\begin{aligned} G^{*,\lambda}(y,b;z,c) =& G^{\lambda}(z,c;y,b) \\ =& G(z,c;y,b) = \int_0^\infty K_t(z,c;y,b)dt \end{aligned}$$

which is bounded by Lemmas 6.7 and 6.10 applied to K_t instead to K_t^*. Assume now that $c(z)=0$. Then

- either there is an open subset $V_1\subset V$ such that $G^*(e,0;\cdot)|_{V_1}=0$
- or there is an open subset $V_2\subset V$ such that $G^{*,\lambda}(\cdot;z,0)|_{V_2}=0$

In the first case we have

$$G^*f(e,0)=0 \text{ when } \operatorname{supp} f\subset V_1,$$

in the second

$$G^{\lambda}f(z,0)=0 \text{ when } \operatorname{supp} f\subset V_2.$$

The first function is $\widetilde{L}_\alpha^*$ harmonic and the second is $\widetilde{L}_\alpha-\lambda$ harmonic which for $f\geq 0$, by the Harnack inequality means $G^*f\equiv 0$ or $G^\lambda f\equiv 0$ i.e. for every t, $K_t^*f\equiv 0$ or $K_tf\equiv 0$ which contradicts (4.3). □

Since $L_\alpha=a^{-2}\mathcal{L}_\alpha$ there is a close relation between the Green operators for L_α and $\mathcal{L}_\alpha$. Let

$$U=\int_0^\infty \mu_t dt$$

be the fundamental solution for $\mathcal{L}_{-\alpha}$. Then

$$\check{U}=\int_0^\infty \check{\mu}_t dt$$

is the fundamental solution for $\mathcal{L}_{-\alpha}^*$ which is a formal conjugate operator with respect to the right Haar measure. Then

$$\mathcal{L}_{-\alpha}^*=(a\partial_a)^2+\alpha(a\partial_a)+\sum_{j=1}^m \Phi_a(X_j)^2-\Phi_a(X_0)$$

and so $a^{-2}\mathcal{L}_{-\alpha}^*=L_\alpha^*$. Therefore,

$$\check{U}(a^2f)=Gf$$

as Green operators. This means

$$\int_S (ab^{-1})^2 f(xa(yb)^{-1})\check{U}(dydb)=\int_S (ab)^2 f(xayb)U(dydb).$$

For the densities of U and G, the first with respect to the right Haar measure and the second with respect to the measure $b^{1+\alpha}dydb$, we have

$$\begin{aligned} \int_{N\times\mathbb{R}^+} G(x,a;y,b)f(y,b)b^{\alpha+1}dbdy =& \int_S (ab)^2 f(xayb)U(yb)\frac{db}{b}dy \\ =& \int_S f(yb)a^{-Q}U(a^{-1}x^{-1}yb)b^{-\alpha}\ b^{\alpha+1}dydb \end{aligned}$$

i.e.

$$a^{-Q}U(a^{-1}x^{-1}yb)b^{-\alpha}=G(x,a;y,b) \tag{7.5}$$

Proof of Theorem 2.16. We are going to prove (2.17), (2.18) goes along the same lines. Since $U(xa) = G(e,1;x,a)a^\alpha$, it is enough to prove

$$C_1 \le \frac{G(e,1;x,a)}{h(xa)a^{-\alpha}} \le C_2. \tag{7.6}$$

Let $V = \{(x,1) : |x| \le 2\} \cup \{(x,a) : 1 \le |x| \le 2, a < 1\}$. Notice that there are $C_1, C_2 > 0$ such that if $|b| \le \frac{1}{2}$ then

$$\begin{aligned} C_1 < G(e,b;x,a) \le C_2 \\ C_1 < G^*(e,b;x,a) \le C_2 \end{aligned} \tag{7.7}$$

Indeed, if $1 \le |x| \le 2$, $a < 1$, (7.7) follows from lemmas 6.7 and 6.10. Moreover, if $|x| \le 1$ then

$$\begin{aligned} G(e,b;x,1) &\le \int_{|x|\le 2, \frac{1}{2}<a<\frac{3}{2}} |G(e,b;x,a)|^2 dx a^{1+\alpha} da \\ &\le \int_0^\infty |K_t(e,b;x,a)|^2 \, dx a^{1+\alpha} da dt \\ &\le \int_1^\infty t^{-\frac{Q}{2}-\frac{\alpha}{2}-1} dt + C(1+b)^M, \end{aligned}$$

which is bounded. Now if $(x,a) \in V_1$

$$G(e,1;x,a) = G(x^{-1},1;e,a) = G^*(e,a;x^{-1},1)$$

If $(x,a) \in V_2$

$$G(e,1;x,a) = a^{-Q-\alpha} G(e,a^{-1};\Phi_{a^{-1}}(x^{-1}),1)$$

and if $(x,a) \in V_3$

$$G(e,1;x,a) = |x|^{-Q-\alpha} G(e,|x|^{-1};\Phi_{|x|^{-1}}(x),|x|^{-1}a)$$

Hence (7.6) follows. □

Notice that in view of (7.5) the change of variables $(x,a) \to (x,a^c)$ with $c > 0$ does not effect the pointwise behavior of the limits of $G(x,a;y,b)$, $G^*(x,a;y,b)$ as $a,b \to 0$. Therefore, the asymptotic of m_α in Theorem 2.15 as well as Theorem 2.16 remain valid also for $\Re\lambda_j > 0$.

Appendix A. Subelliptic estimates, some lemmas

This section contains some estimates used in Section 5. Although the techniques are classical, the proofs are not completely trivial and depend heavily on the special form of the coefficients. Moreover, we need to keep constants under control and so it seems convenient to include here some proofs.

We shall prove the following lemmas

Lemma A.1. *$Y_i, \partial_{u_j} \in \mathcal{P}_1$ for $i = 1,...,m$, and $j = 1,...,d$.*

Lemma A.2. *$Y_0 - \partial_t \in \mathcal{P}_{\frac{1}{2}}$.*

Lemma A.3. *If $\Phi_{|u_0+u|}(Z) \in \mathcal{P}_\varepsilon$ for $Z \in \mathcal{N}$ and $\varepsilon > 0$, then $[\Phi_{|u_0+u|}(Z), \Phi_{|u_0+u|}(X_j)] = \Phi_{|u_0+u|}([Z,X_j]) \in \mathcal{P}_{\frac{\varepsilon}{2}}$, for $j = 1,\dots,m$.*

Lemma A.4. *If $\Phi_{|u_0+u|}(Z) \in \mathcal{P}_\varepsilon$ for $Z \in \mathcal{N}$ and $\varepsilon > 0$, then $\Phi_{|u_0+u|}([Z,X_0]) \in \mathcal{P}_{\frac{\varepsilon}{4}}$.*

Lemma A.5. *Let $Z = \sum \rho_j(u_0+u)Z_j + \rho_0(u_0+u)\partial_t$, where $Z_j \in \mathcal{N}$, $\rho_j \in C^2$ and $\|\rho_j\|_{C^2} \le C(1+|u_0|)^M$. If $Z \in \mathcal{P}_\varepsilon$, then $[\partial_{u_j}, Z] \in \mathcal{P}_{\frac{\varepsilon}{2}}$.*

In all the lemmas above $Z \in \mathcal{N}$ is a vector field extended to $N \times \mathbb{R}^d \times \mathbb{R}^+$ by $Z(x,u,t) = Z(x)$. In the following two lemmas functions ϕ, ψ are of the form $\phi(x,u,t) = \phi_1(x,t)\phi_2(u)$, $\psi(x,u,t) = \psi_1(x,t)\psi_2(u)$, $\phi_1, \psi_1 \in C_c^\infty(U \times (t_0,t_1))$, $\phi_2, \psi_2 \in C_c^\infty(B_{\frac{3}{2}}(0))$.

Lemma A.6. *For every ϕ, ψ, such that $\psi \succ \phi$ there is C such that for every $f \in W(\Omega)$*

$$\|\phi f\|_\varepsilon^2 \le C(\|\psi(\widetilde{L}_\alpha - \partial_t)f\|_{L^2}^2 + \|\psi f\|_{L^2}^2) \tag{A.7}$$

and

$$\sum_{j=1}^m \|Y_j(\phi f)\|_{\frac{\varepsilon}{2}}^2 + \sum_{j=1}^d \|\partial_{u_j}(\phi f)\|_{\frac{\varepsilon}{2}}^2 \le C(\|\psi(\widetilde{L}_\alpha - \partial_t)f\|_{L^2}^2 + \|\psi f\|_{L^2}^2). \tag{A.8}$$

Lemma A.9. *For every $D = \partial_x^\gamma \partial_t^\delta$, every ϕ, ψ, with $\psi \succ \phi$ there is C such that for every $F \in W(\Omega)$*

$$\|\phi DF\|_{L^2(\widetilde{K})}^2 \le C\|\psi F\|_{L^2(\Omega)}^2 \tag{A.10}$$

whenever $(\widetilde{L}_\alpha - \partial_t)F = 0$.

For any real number s let us denote by Ψ^s the space of all pseudodifferential operators on $N \times \mathbb{R}$ of order s, supported in $U \times (t_0,t_1)$. Define $\Lambda^s = (I - \sum \partial_{x_i}^2 - \partial_t^2)^s$. Next fix $g \in C_c^\infty(\Omega)$ such that $\psi \succ g \succ \phi$, then of course $g\Lambda^{\frac{s}{2}} \in \Psi^s$.

The coefficients of $Y_j's$, by (2.12), are of the form $|u_0+u|^\lambda \log|u_0+u|^\beta$, $\Re\lambda > 2$. Therefore all functions ρ_j, η_j of variable u that appear in the proofs, have the same form, they are in C^2 and the norms of Y_j's and the C^2 norms of ρ_j, η_j's can be estimated by $(1+|u_0|)^M$. Clearly, the constant M changes from line to line.

Proof of Lemma A.1. Notice first that because X_j are left-invariant on N, Y_j are skew symmetric on $N \times \mathbb{R}^d \times \mathbb{R}$ i.e. $(Y_j f, g) = -(f, Y_j g)$ and moreover $(Y_j f, f) = 0$ for $f, g \in C_c^\infty$. Therefore, we have

$$\sum_{j=1}^d \|\partial_{u_j}(\phi f)\|_{L^2}^2 + \sum_{j=1}^m \|Y_j(\phi f)\|_{L^2}^2 = -((\widetilde{L}_{\alpha,u_0} - \partial_t)(\phi f), \phi f)$$

$$= -((\widetilde{L}_{\alpha,u_0} - \partial_t)(\phi)f, \phi f) - 2\sum_{j=1}^m (Y_j(\phi)Y_j(f), \phi f) - 2\sum_{j=1}^d (\partial_{u_j}(\phi)\partial_{u_j}(f), \phi f) - (\phi(\widetilde{L}_{\alpha,u_0} - \partial_t)(f), \phi f).$$

It is enough to dominate the second and the third terms in the above sum. For that we observe

$$(Y_j(\phi)Y_j(f), \phi f) = -(f, Y_j(Y_j(\phi)\phi f)) = -(f, Y_j(Y_j(\phi)\phi)f) - (\phi f, Y_j(\phi)Y_j(f)),$$

hence

$$|(Y_j(\phi)Y_j(f), \phi f)| \le \frac{1}{2}|(f, Y_j(Y_j(\phi)\phi)f)| \le C(1+|u_0|)^M \|\psi f\|_{L^2}^2$$

□

Proof of Lemma A.2. Let $A = g\Lambda^{-\frac{1}{2}}(Y_0 - \partial_t)$. Then $A = \sum \rho_j(u_0+u)A_j + A_0$, where $A_j \in \Psi^0$, which means that A is bounded on L^2. Next we write

(A.11)

$$\|(Y_0 - \partial_t)(\phi f)\|_{-\frac{1}{2}} = (\Lambda^{-\frac{1}{4}}(Y_0 - \partial_t)(\phi f), \Lambda^{-\frac{1}{4}}(Y_0 - \partial_t)(\phi f)) = ((Y_0 - \partial_t)(\phi f), A(\phi f))$$

$$= ((\widetilde{L}_{\alpha,u_0} - \partial_t)(\phi f), A(\phi f)) - \sum_{j=1}^m (Y_j^2(\phi f), A(\phi f)) - (\Delta_u(\phi f), A(\phi f))).$$

To estimate the first factor we use Lemma A.1. For the second we notice that $B_j = g[Y_j, A]$ is a bounded operator on L^2, and therefore

$$|(Y_j^2(\phi f), A(\phi f))| \le |(Y_j(\phi f), B_j A(\phi f))| + |(Y_j(\phi f), AY_j(\phi f))| \le C(1+|u_0|)^M(\|Y_j(\phi f)\|_{L^2}^2 + \|\phi f\|_{L^2}^2)$$

and again Lemma A.1 gives the required estimate.

Finally, we have

$$\begin{aligned} |(\partial_{u_i}^2(\phi f), A(\phi f)| &= |(\partial_{u_i}(\phi f), \partial_{u_i}(\sum \rho_j(u_0+u)A_j(\phi f) + A_0(\phi f)))| \\ &\le 2\|\partial_{u_i}(\phi f)\|_{L^2}^2 + \|\sum(\partial_{u_i}\rho_j)(u_0+u)A_j(\phi f)\|_{L^2}^2 \\ &\quad + \|\sum \rho_j(u_0+u)A_j\partial_{u_i}(\phi f)\|_{L^2}^2 + \|A_0\partial_{u_j}(\phi f)\|_{L^2}^2 \\ &\le C(1+|u_0|)^M(\|\psi(\widetilde{L}_{\alpha,u_0} - \partial_t)f\|_{L^2}^2 + \|\psi f\|_{L^2}^2). \end{aligned}$$

□

Proof of Lemma A.3. Define

$$M = g\Phi_{|u_0+u|}([X_j, Z])\Lambda^{-1+\frac{\varepsilon}{2}} = \sum \rho_k(u_0+u)B_k,$$

then, of course, $B_k \in \Psi^{-1+\varepsilon}$. Using the relation

$$A[B,C] = [A,B]C + BAC - [A,C]B - CAB. \tag{A.12}$$

we have

$$\begin{aligned} \|[\Phi_{|u_0+u|}(X_j), \Phi_{|u_0+u|}(Z)](\phi f)\|_{-1+\frac{\varepsilon}{2}} &= -(M[\Phi_{|u_0+u|}(X_j), \Phi_{|u_0+u|}(Z)](\phi f), \phi f) \\ &= -([M, \Phi_{|u_0+u|}(X_j)]\Phi_{|u_0+u|}(Z)(\phi f), \phi f) - (\Phi_{|u_0+u|}(X_j)M\Phi_{|u_0+u|}(Z)(\phi f), \phi f) \\ &\quad + ([M, \Phi_{|u_0+u|}(Z)]\Phi_{|u_0+u|}(X_j)(\phi f), \phi f) + (\Phi_{|u_0+u|}(Z)M\Phi_{|u_0+u|}(X_j)(\phi f), \phi f). \end{aligned}$$

Each term in the above sum can be easily dominated by

$$C(1+|u_0|)^{M'}(\|\psi(\widetilde{L}_{\alpha,u_0} - \partial_t)f\|_{L^2}^2 + \|\psi f\|_{L^2}^2)$$

□

The proof of Lemma A.5 is analogous.

Proof of Lemma A.4. Let

$$M = g[\Phi_{|u_0+u|}(X_0) - |u_0+u|^2\partial_t, \Phi_{|u_0+u|}(Z)]\Lambda^{-1+\frac{\varepsilon}{4}} = g\sum \rho_k(u_0+u)B_k,$$

where $B_k \in \Psi^{-1+\frac{\varepsilon}{2}}$. Applying (A.12) again we obtain four factors to estimate. Two of them can be dominated by similar technique as the previous lemmas. To handle one of the two remaining terms

(the same proof works for the last one) we write

$$
\begin{aligned}
&|(M\Phi_{|u_0+u|}(Z)(\phi f), (\Phi_{|u_0+u|}(X_0) - |u_0+u|^2\partial_t)(\phi f))| \leq \\
&\quad |(M\Phi_{|u_0+u|}(Z)(\phi f), |u_0+u|^2(\widetilde{L}_{\alpha,u_0} - \partial_t)(\phi f))| + |(M\Phi_{|u_0+u|}(Z)(\phi f), |u_0+u|^2\Delta_u(\phi f))| \\
&\qquad + \sum_{j=1}^m |(M\Phi_{|u_0+u|}(Z)(\phi f), |u_0+u|^2 Y_j^2(\phi f))| \\
&\qquad \leq C(1+|u_0|)^2 \Bigg(\|M\Phi_{|u_0+u|}(Z)(\phi f)\|_{L^2}^2 + \|(\widetilde{L}_{\alpha,u_0} - \partial_t)(\phi f)\|_{L^2}^2 \\
&+ \sum_{j=1}^m \big(\|Y_j M\Phi_{|u_0+u|}(Z)(\phi f)\|_{L^2}^2 + \|Y_j(\phi f)\|_{L^2}^2\big) + \sum_{j=1}^d \big(\|\partial_{u_j} M\Phi_{|u_0+u|}(Z)(\phi f)\|_{L^2}^2 + \|\partial_{u_j}(\phi f)\|_{L^2}^2\big)\Bigg).
\end{aligned}
$$

As is easily seen, we have reduced the proof of the lemma to estimating $\|Y_j M\Phi_{|u_0+u|}(Z)(\phi f)\|_{L^2}^2$ (for $1 \leq j \leq m$) and $\|\partial_{u_j} M\Phi_{|u_0+u|}(Z)(\phi f)\|_{L^2}^2$. We have

$$
\begin{aligned}
&\sum_{j=1}^m \|Y_j M\Phi_{|u_0+u|}(Z)(\phi f)\|_{L^2}^2 + \sum_{j=1}^d \|\partial_{u_j} M\Phi_{|u_0+u|}(Z)(\phi f)\|_{L^2}^2 \\
&\qquad = |((\widetilde{L}_{\alpha,u_0} - \partial_t) M\Phi_{|u_0+u|}(Z)(\phi f), M\Phi_{|u_0+u|}(Z)(\phi f))| \\
&\qquad \leq |([M\Phi_{|u_0+u|}(Z), \widetilde{L}_{\alpha,u_0} - \partial_t](\phi f), M\Phi_{|u_0+u|}(Z)(\phi f))| \\
&\qquad\qquad + |((\widetilde{L}_\alpha - \partial_t)(\phi f), \Phi_{|u_0+u|}(Z) M^* M\Phi_{|u_0+u|}(Z)(\phi f))|
\end{aligned}
$$

Let $Z = \sum_k \eta_k(u_0+u) Z_k$. Observe that by (2.12) the operator $\Phi_{|u_0+u|}(Z) M^* M$ is of the form $g \sum_{k,i,j} \eta_k(u_0+u)\rho_i(u_0+u)\rho_j(u_0+u) Z_k B_i^* B_j$, and $Z_k B_i^* B_j \in \Psi^{-1+\varepsilon}$. Therefore we obtain the required estimates for the second term. To cope with the first one, we compute the commutator:

(A.13)

$$
\begin{aligned}
[\widetilde{L}_{\alpha,u_0} - \partial_t, M\Phi_{|u_0+u|}(Z)] &= \sum_{j=1}^m [Y_j^2, M\Phi_{|u_0+u|}(Z)] + [\Delta_u, M\Phi_{|u_0+u|}(Z)] + [Y_0, M\Phi_{|u_0+u|}(Z)] \\
&= \sum_{i,j} \rho_i(u_0+u)[Y_j^2, B_i\Phi_{|u_0+u|}(Z)] + \sum_{i,j} [\partial_{u_j}^2, \rho_i(u_0+u) B_i \Phi_{|u_0+u|}(Z)] \\
&\quad + \sum_i \rho_i(u_0+u)[Y_0, B_i\Phi_{|u_0+u|}(Z)].
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
[\partial_{u_j}^2, \rho_i(u_0+u) B_i \Phi_{|u_0+u|}(Z)] &= \sum_{i,k} [\partial_{u_j}^2, \rho_i(u_0+u)\eta_k(u_0+u) B_i Z_k] \\
&= \sum_k \big((\partial_{u_j}\rho_i\eta_k(u_0+u))\partial_{u_j} B_i Z_k + \partial_{u_i}^2(\rho_i\eta_k(u_0+u)) B_i Z_k\big),
\end{aligned}
$$

Thus we have to estimate the terms of the form

$$
\begin{aligned}
&(1+|u_0|)^M |(B_i Z_k(\phi f), M\Phi_{|u_0+u|}(Z)(\phi f))| + (1+|u_0|)^M |(\partial_{u_j} B_i Z_k(\phi f), M\Phi_{|u_0+u|}(Z)(\phi f))| \\
&= (1+|u_0|)^M |(\phi f, Z_k B_i^* M\Phi_{|u_0+u|}(Z)(\phi f))| + (1+|u_0|)^M |(\partial_{u_j}(\phi f), Z_k B_i^* M\Phi_{|u_0+u|}(Z)(\phi f))| \\
&\qquad \leq C(\|\psi(\widetilde{L}_\alpha - \partial_t) f\|_{L^2}^2 + \|\psi f\|_{L^2}^2).
\end{aligned}
$$

Next observe that

$$\begin{aligned}[Y_j^2, B_i\Phi_{|u_0+u|}(Z)] &= 2[Y_j, B_i\Phi_{|u_0+u|}(Z)]Y_j + [Y_j,[Y_j, B_i\Phi_{|u_0+u|}(Z)]] \\ &= \sum_k \eta_k(u_0+u)K_k^1 Y_j + \sum_k \eta_k(u_0+u)K_k^2,\end{aligned}$$

where $K_k^1, K_k^2 \in \Psi^{\frac{\varepsilon}{2}}$. So it amounts to estimate terms of the form

$$\begin{aligned}&(1+|u_0|)^M|(K_k^1Y_j(\phi f), M\Phi_{|u_0+u|}(Z)(\phi f))| + (1+|u_0|)^M|(K_k^2(\phi f), M\Phi_{|u_0+u|}(Z)(\phi f))| \\ &= (1+|u_0|)^M|(Y_j(\phi f), (K_k^1)^* M\Phi_{|u_0+u|}(Z)(\phi f))| + (1+|u_0|)^M|(\phi f, (K_k^2)^* M\Phi_{|u_0+u|}(Z)(\phi f))| \\ &\qquad \le C(1+|u_0|)^{M'}(\|\psi(\widetilde{L}_{\alpha,u_0}-\partial_t)f\|_{L^2}^2 + \|\psi f\|_{L^2}^2)\end{aligned}$$

The same argument gives estimates of the last term in (A.13). □

Proof of Lemma A.6. While inequalities (A.7) and (A.8) are proved for smooth f's, then approximating $f \in W(\Omega)$ by functions

$$\chi * f(x,u,t) = \int_{N\times\mathbb{R}^d\times R^+} \chi(xy^{-1}, u-w, t-s) f(y,w,s)\ dydwds,$$

$\chi \in C_c^\infty(N\times\mathbb{R}^d\times R^+)$, we obtain (A.7) and (A.8) for f's in $W(\Omega)$. So, in view of Theorem 5.6 it remains to prove (A.8) for $f \in C^\infty(\Omega)$. Let

$$G^{\frac{\varepsilon}{2}} = g\Lambda^{\frac{\varepsilon}{4}},$$

where $g = g_1g_2$, $g_1 \in C_c^\infty(U\times(t_0,t_1))$, $g_2 \in C_c^\infty(B_{\frac{3}{2}}(0))$, and $\psi_j \succ g_j \succ \phi_j$. Then proceeding as in [Tr], Lemma 5.1 we get

$$\begin{aligned}\sum_{j=1}^m \|Y_j(\phi f)\|_{\frac{\varepsilon}{2}}^2 + \sum_{j=1}^d \|\partial_{u_j}(\phi f)\|_{\frac{\varepsilon}{2}}^2 &\le C(1+|u_0|)^M\Big(|(G^{\frac{\varepsilon}{2}}(\widetilde{L}_{\alpha,u_0}-\partial_t)(\phi f), G^{\frac{\varepsilon}{2}}(\phi f))| + \|\phi f\|_\varepsilon^2\Big) \\ &\le C(1+|u_0|)^M(\|(\widetilde{L}_\alpha-\partial_t)(\phi f)\|_{L^2}^2 + \|\phi f\|_\varepsilon^2\Big).\end{aligned}$$

Now Lemma A.1 and (A.7) yield the conclusion. □

Proof of Lemma A.9. For $\chi \in C_c^\infty(U\times(t_0,t_1))$, $f \in W(\Omega)$ we take

$$\chi * f(x,u,t) = \int_{N\times\mathbb{R}^+} \chi(xy^{-1}, t-s) f(y,u,s)\ dxds$$

and we prove that

$$\forall k\ \forall \psi \succ \phi\ \exists C$$

$$\begin{aligned}\|\phi(\chi*f)\|_{k\frac{\varepsilon}{2}}^2 + \sum_j \|Y_j(\phi(\chi*f)\|_{(k-1)\frac{\varepsilon}{2}}^2 + \sum_j \|\partial_{u_j}(\phi(\chi*f))\|_{(k-1)\frac{\varepsilon}{2}}^2 \\ \le C\Big(\|\psi(\widetilde{L}_\alpha-\partial_t)(\chi*f)\|_{(k-2)\frac{\varepsilon}{2}} + \|\psi(\chi*f)\|_{(k-2)\frac{\varepsilon}{2}}\Big)\end{aligned}$$

If, in addition, $(\widetilde{L}_\alpha-\partial_t)f = 0$ then $(\widetilde{L}_\alpha-\partial_t)(\chi*f) = \chi*(\widetilde{L}_\alpha-\partial_t)f = 0$ and so (A.10) follows.

For the above estimate we proceed by induction in a standard way using the operator G^s, $g = g_1g_2$ satisfying $\psi_j \succ g_j \succ \phi_j$. We have to commute $(\widetilde{L}_\alpha-\partial_t)$ with G^s. Noice that $\chi*f$, $\partial_u^\beta(\chi*f) = \chi*\partial_u^\beta f$, $|\beta| \le 2$ is smooth with respect to x,t. Therefore,

$$\partial_u^\beta G^s(\chi*f) = G^s\partial_u^\beta(\chi*f)$$

and we may write

$$(\widetilde{L}_\alpha - \partial_t)G^s(\chi * f) = G^s(\widetilde{L}_\alpha - \partial_t)(\chi * f) + \sum_{j=1}^m T_j^s Y_j + T_0^s,$$

where

$$T_j^s = \sum_k \rho_{jk}(u) B_{jk}^s$$
$$T_0^s = \sum_k \rho_{0k}(u) B_{0k}^s,$$

B_{jk}^s being properly localized pseudodifferential operators of order s on $U \times (t_0, t_1)$, $\rho_{jk}(u)$ complex valued continuous functions. Hence the induction follows. □

Proof of Theorem 5.11. We put $D = \widetilde{X}^I \partial_t^\delta$, $\widetilde{X}^I$ to be a right invariant derivative on N. Let $B_\varepsilon(0)$ be a ball in $\mathbb{R}^d$ with the center 0 and the radius ε. For $\chi \in C_c^\infty(B_\varepsilon(0))$ we consider

$$\chi * F(x, u, w) = \int_{\mathbb{R}^d} \chi(u - w) F(x, w, t)\, dw.$$

The previous lemma implies that $\chi * F \in C^\infty(\Omega)$ and by Lemma A.1, we have

$$\|\phi \partial_u D(\chi * F)\|_{L^2}^2 \le C(\|\psi' D(\chi * F)\|_{L^2}^2 + \|\psi'(\widetilde{L}_\alpha - \partial_t) D(\chi * F)\|_{L^2}^2.$$

Let χ_k be a smooth approximate identity and assume that we can prove $(\widetilde{L}_\alpha - \partial_t)D(\chi_k * F) \to 0$ in L^2 as $k \to \infty$. Then we have

$$\|\phi \partial_u DF\|_{L^2}^2 \le C\|\psi' DF\|_{L^2}^2 \le C\|\psi F\|_{L^2}^2. \tag{A.14}$$

Clearly $(\widetilde{L}_\alpha - \partial_t)D(\chi_k * F) = D(\widetilde{L}_\alpha - \partial_t)(\chi_k * F)$ tends to 0 weakly. Moreover,

$$(\widetilde{L}_\alpha - \partial_t)D(\chi_k * F) = D(\chi_k * \Delta F) + \sum_{j=1}^m Y_j^2 D(\chi_k * F) + Y_0 D(\chi_k * F) - \partial_t D(\chi_k * F).$$

In view of Lemma A.9 the last three terms converge in L^2. For the first one we have

$$D(\chi_k * \Delta F) = -\chi_k * (\sum_{j=1}^m Y_j^2 DF + Y_0 DF - \partial_t DF)$$

and so Lemma A.9 implies convergence in L^2. Thus (A.14) is proved. For the second derivatives we have

$$\begin{aligned}
\|\phi \partial_{u_i} \partial_{u_j} D(\chi_k * F)\|_{L^2}^2 &\le C\Big(\|\psi'(\widetilde{L}_\alpha - \partial_t)\partial_{u_j} D(\chi_k * F)\|_{L^2}^2 + \|\psi' \partial_{u_j} D(\chi_k * F)\|_{L^2}^2\Big) \\
&\le C\Big(\|\psi' \partial_{u_j}(\widetilde{L}_\alpha - \partial_t) D(\chi_k * F)\|_{L^2}^2 \\
&\quad + \|\psi'[\widetilde{L}_\alpha - \partial_t, \partial_{u_j}] D(\chi_k * F)\|_{L^2}^2 + \|\psi' \partial_{u_j} D(\chi_k * F)\|_{L^2}^2\Big).
\end{aligned}$$

Since $[\widetilde{L}_\alpha - \partial_t, \partial_{u_j}]$ is a combination of derivatives in the direction of N multiplied by continuous functions of u, $\psi'[\widetilde{L}_\alpha - \partial_t, \partial_{u_j}] D(\chi_k * F)$ converges in L^2 and the limit is dominated by $C\|\psi F\|_{L^2}^2$.

By (A.14) the same holds for $\psi'\partial_{u_j}D(\chi_k * F)$. So it remains to prove that the first term tends to 0 in L^2. Clearly, the limit exists in the weak sense and we also have

$$\partial_{u_j}(\widetilde{L}_\alpha - \partial_t)D(\chi_k * F)$$
$$= \partial_{u_j}\Delta D(\chi_k * F) + \partial_{u_j}\Big(\sum_{j=1}^m Y_j^2 D(\chi_k * F) + Y_0 D(\chi_k * F) - \partial_t D(\chi_k * F)\Big)$$

To conclude it is enough to prove that $\partial_{u_j}\Delta D(\chi_k * F)$ converges in L^2. For that we proceed as before. We write

$$\partial_{u_j}\Delta D(\chi_k * F) = -\partial_{u_j}\chi_k * (\sum_{j=1}^m Y_j^2 DF + Y_0 DF - \partial_t DF)$$
$$= -\chi_k * (\sum_{j=1}^m \partial_{u_j} Y_j^2 DF + \partial_{u_j} Y_0 DF - \partial_{u_j}\partial_t DF)$$

and we use (A.14). Finally, letting $k \to \infty$, we obtain the conclusion. To get (5.12) with $p \geq 3$, we need to be able to differentiate $p-1$ times so if $\Re\lambda_1 > p+1$ we can continue this way until $|\beta| = p$. Indeed, if $\Re\lambda > p+1$ then the function $|u|^{\lambda-1}\log^\beta|u| \in C^{p-1}$ with all derivatives being 0 at 0. □

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