ON TAILS OF FIXED POINTS OF THE SMOOTHING TRANSFORM IN THE BOUNDARY CASE

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ABSTRACT. Let $\{A_i\}$ be a sequence of random positive numbers, such that only N first of them are strictly positive, where N is a finite a.s. random number. In this paper we investigate nonnegative solutions of the distributional equation $Z =_d \sum_{i=1}^N A_i Z_i$, where $Z, Z_1, Z_2, \ldots$ are independent and identically distributed random variables, independent of $N, A_1, A_2, \ldots$. We assume $\mathbb{E}\big[\sum_{i=1}^N A_i\big] = 1$ and $\mathbb{E}\big[\sum_{i=1}^N A_i \log A_i\big] = 0$ (the boundary case), then it is known that all nonzero solutions have infinite mean. We obtain new result concerning behavior of their tails.

1. INTRODUCTION

Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of random positive numbers. We assume that only first N of them are nonzero, where N is some random number, finite almost surely. For any random variable Z, let $\{Z_n\}_{n\in\mathbb{N}}$ be a sequence of i.i.d. copies of Z independent both on N and $\{A_n\}_{n\in\mathbb{N}}$. Then we define new random variable $Z^* = \sum_{i=1}^N A_i Z_i$ and the map $Z \to Z^*$ is called the smoothing transform. A random variable Z is said to be fixed point of the smoothing transform if Z^* has the same distribution as Z, i.e.

$$Z =_d \sum_{i=1}^N A_i Z_i. \tag{1.1}$$

There exists an extensive literature, where the problems of existence, uniqueness and asymptotic behavior of solutions of (1.1) were studied. It turns out that the answer depends heavily on properties of the function

$$v(\theta) = \log \mathbb{E}\Big[\sum_{i=1}^N A_i^\theta\Big].$$

One assumes usually $v(1) = 1$ and $v'(1) < 0$, i.e. $\mathbb{E}\big[\sum_{i=1}^N A_i\big] = 1$ and $\mathbb{E}\big[\sum_{i=1}^N A_i \log A_i\big] < 0$. Then if N is nonrandom Durrett and Liggett [6] proved existence of solutions of (1.1). Their results were later extended by Liu [10] to the case where N is random and $v(0) = \log \mathbb{E}[N] > 0$ (this value could be infinite). Let us mention that in this case all nonzero solutions of (1.1) have finite mean. Fixed points of the smoothing transform were characterized by Biggins and Kyprianou [3]. Also their asymptotic properties are well described. Durrett and Liggett [6] studied behavior of the Laplace transform of Z close to 0. The tail of Z was described by Guivarc'h [8] (for nonrandom N) and Liu [11, 12] (for random N). They proved that if $v(\chi) = 1$ for some $\chi > 1$ and some further hypotheses are satisfied then the limit $\lim_{x\to\infty} x^\chi \mathbb{P}(Z > x)$ exists, is strictly positive and finite.

In this paper we study 'the boundary case', when $v(1) = 0$ and $v'(1) = 0$. Existence of fixed points of (1.1) was proved by Durrett and Liggett [6] and Liu [10]. Uniqueness was studied by Biggins and Kyprianou [2]. It is known that all the solutions have infinite mean. To our knowledge, up to now,

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no estimates of tails of fixed points in the boundary case have been obtained. All known results were stated in terms of the Laplace transform. Let Z be a solution of (1.1) and let $\phi(\lambda) = \mathbb{E}\big[e^{-\lambda Z}\big]$ be its Laplace transform. Then, under some further hypothesis it was proved by Durrett and Liggett [6], Liu [12], Biggins and Kyprianou [2] that

$$\lim_{\lambda\to 0^+} \frac{1-\phi(\lambda)}{\lambda|\log\lambda|} = C \tag{1.2}$$

for some positive constant C. Finiteness of C was discussed in [2].

Our main result is the following

Theorem 1.3. *Assume*

$$\mathbb{E}\left[\sum_{i=1}^{N} A_i\right] = 1, \tag{1.4}$$

$$\mathbb{E}\left[\sum_{i=1}^{N} A_i \log A_i\right] = 0, \tag{1.5}$$

$$\mathbb{E}\left[\sum_{i=1}^{N} A_i^{1-\delta_1}\right] < \infty, \text{ for some } \delta_1 > 0, \tag{1.6}$$

$$\mathbb{E}\left[\left(\sum_{i=1}^{N} A_i\right)^{1+\delta_2}\right] < \infty, \text{ for some } \delta_2 > 0, \tag{1.7}$$

$$\mathbb{E}[N] > 1 \text{ (it could be infinite).} \tag{1.8}$$

Let Z be nonnegative and nonzero solution of (1.1) then if A_i are aperiodic

$$\lim_{x\to\infty} x\mathbb{P}\big[Z > x\big] = C_0,$$

for some finite and strictly positive constant C_0.

If A_i are periodic, then there exist two positive constants C_1 and C_2 such that

$$C_1 = \liminf_{x\to\infty} x\mathbb{P}\big[Z > x\big] \le \limsup_{x\to\infty} x\mathbb{P}\big[Z > x\big] = C_2.$$

To prove the theorem, following ideas of Guivarc'h [8] and Liu [12], we reduce the problem to study tails of solutions of the random difference equation $R =_d AR + B$, where (A, B) and R are independent. In Section 2 we describe all information on the random difference equation, that are needed for our purpose and in Section 3 we present complete proof of Theorem 1.3.

2. Random difference equation in the critical case

Given a probability measure μ on $\mathbb{R}^+ \times \mathbb{R}$ we define the Markov chain on $\mathbb{R}$

$$\begin{aligned} X_0 &= 0, \\ X_n &= A_n X_{n-1} + B_n, \end{aligned}$$

where the random pairs $\{(A_n, B_n)\}$ are independent and identically distributed according to the measure μ. This process is usually considered under assumption $\mathbb{E}\big[\log A_1\big] < 0$. Then, if additionally $\mathbb{E}\big[\log^+ |B|\big] < \infty$, there exists a unique stationary measure ν of $\{X_n\}$. The tail of ν is well understood, namely it was proved by Kesten [9] (see also Goldie [7] for much simpler argument) that $\lim_{t\to\infty} \nu(|x| > t) = C_+$ for some positive constant C_+, where α is the unique positive number such that $\mathbb{E}A_1^\alpha = 1$. Exactly this result was used by Guivarc'h [8] and Liu [12] to study solutions of (1.1) with finite mean.

However, in this paper we will refer to the 'critical case', when $\mathbb{E}\big[\log A_1\big] = 0$. Then there is no finite stationary measure, but Babbilot, Bougerol, Elie [1] proved that if

- $\mathbb{P}[A_1 = 1] < 1$ and $\mathbb{P}[A_1 x + B_1 = x] < 1$ for all $x \in \mathbb{R}^d$,
- $\mathbb{E}\left[(|\log A_1| + \log^+ |B_1|)^{2+\varepsilon}\right] < \infty$, for some $\varepsilon > 0$

then there exists a unique (up to a constant factor) invariant Radon measure ν of $\{X_n\}$, i.e. the measure ν on $\mathbb{R}^d$ satisfying

$$\mu * \nu(f) = \nu(f), \tag{2.1}$$

for any positive measurable function f, where

$$\mu * \nu(f) = \int_{\mathbb{R}^+ \times \mathbb{R}} \int_{\mathbb{R}^d} f(ax + b)\nu(dx)\mu(da\, db).$$

Recently precisely behavior of the measure ν at infinity has been described:

Lemma 2.2 ([4, 5])**.** *Assume that hypotheses above are satisfied and moreover*

$$\mathbb{E}\big[A^{-\delta} + A^{\delta} + |B|^{\delta}\big] < \infty$$

for some $\delta > 0$. If A_1 is aperiodic, then there exists a strictly positive and finite constant C_+ such that

$$\lim_{t\to\infty} \nu(x :\ \alpha t < |x| \le \beta t) = C_+ \log \frac{\beta}{\alpha},$$

for any pair $0 < \alpha < \beta$.

Furthermore, if A_1 is periodic and the group generated by the support of A_1 is $\{e^{np} :\ n \in \mathbb{Z}\}$ for some $p > 0$, then

$$\lim_{t\to\infty} \nu(x :\ t < |x| \le e^{np} t) = nC_+,$$

for every $n \ge 1$ and some positive constant C_+.

3. Proof of Theorem 1.3

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which random variables $\{A_i\}_{i\in\mathbb{N}}$, N, $\{Z_i\}_{i\in\mathbb{N}}$ are supported. We denote by $\mathbb{E}$ the expected value with respect to $\mathbb{P}$. Let η be the law of Z. We define the measure ν on $\mathbb{R}^+$ putting $\nu(dx) = x\eta(dx)$, then $\nu(f) = \mathbb{E}\big[f(Z)Z\big]$ for any bounded and compactly supported function f. Measure ν is unbounded on $\mathbb{R}^+$, however it is a Radon measure. Using ideas of Guivarc'h [8] and Liu [12] we will prove that ν satisfies (2.1) for some appropriately chosen probability measure μ on $\mathbb{R}^+ \times \mathbb{R}$. We cannot use directly their proofs. Guivarc'h assumed A_i to be independent of each other, N to be a constant and obtained the random recurrence equation just by a simple algebraic transformation of measures. Whereas Liu introduced the Peyriere's measure, which cannot be defined here. However we follow the approach of Liu [12], p. 276.

Define $\widetilde{\Omega} = \Omega \times \mathbb{N}$, and let $\widetilde{\mathcal{F}}$ be the σ-field on $\widetilde{\Omega}$ being the direct product of $\mathcal{F}$ and $\mathcal{B}$, where $\mathcal{B}$ is the Borel σ-field on $\mathbb{N}$. We denote by ω an element of Ω and by (ω, i) an element of $\widetilde{\Omega}$. Let δ_i be the Dirac measure on $\mathbb{N}$, i.e. $\delta_i(k) = 0$ if $k \ne i$ and $\delta_i(i) = 1$. For every $U \in \widetilde{\mathcal{F}}$ we define

$$\widetilde{\mathbb{P}}(U) = \mathbb{E}\bigg[\sum_{i=1}^{N} A_i(\omega) \int_{\mathbb{N}} \mathbf{1}_U(\omega, j)\delta_i(dj)\bigg],$$

then, in view of (1.4), $\widetilde{\mathbb{P}}$ is a probability measure on $\widetilde{\Omega}$. We write $\mathbb{E}_{\widetilde{\mathbb{P}}}$ for its expected value. Thus, we have defined a new probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$.

Given $(\omega, i) \in \widetilde{\Omega}$ we define

$$\begin{aligned} \widetilde{Z}(\omega, i) &= Z_i(\omega), \\ \widetilde{A}(\omega, i) &= A_i(\omega), \\ \widetilde{B}(\omega, i) &= \sum_{j \neq i} A_j(\omega) Z_j(\omega). \end{aligned}$$

Lemma 3.1. *Random variables $\widetilde{Z}$ and $(\widetilde{A}, \widetilde{B})$ are $\widetilde{\mathbb{P}}$ independent. Moreover for every nonnegative functions h and g on $\mathbb{R}^+ \times \mathbb{R}$ and $\mathbb{R}$ respectively:*

$$\mathbb{E}_{\widetilde{\mathbb{P}}}\big[g(\widetilde{Z})\big] = \mathbb{E}\big[g(Z)\big], \tag{3.2}$$

$$\mathbb{E}_{\widetilde{\mathbb{P}}}\big[h(\widetilde{A}, \widetilde{B})\big] = \mathbb{E}\Big[\sum_{i=1}^{N} A_i h\Big(A_i, \sum_{j \neq i} A_j Z_j\Big)\Big]. \tag{3.3}$$

In particular $\widetilde{Z}$ and Z have the same law η.

Proof. We write

$$\begin{aligned} \mathbb{E}_{\widetilde{\mathbb{P}}}\big[h(\widetilde{A}, \widetilde{B}) g(\widetilde{Z})\big] &= \mathbb{E}\Big[\sum_{i=1}^{N} A_i(\omega) \int_{\mathbb{N}} \Big(h\big(\widetilde{A}(\omega, j), \widetilde{B}(\omega, j)\big) g\big(\widetilde{Z}(\omega, j)\big)\Big) \delta_i(dj)\Big] \\ &= \mathbb{E}\Big[\sum_{i=1}^{N} A_i h\Big(A_i, \sum_{j \neq i} A_j Z_j\Big) g(Z_i)\Big] \\ &= \mathbb{E}\Big[\sum_{i=1}^{N} A_i h\Big(A_i, \sum_{j \neq i} A_j Z_j\Big)\Big] \mathbb{E}\big[g(Z_i)\big]. \end{aligned}$$

Putting $g = 1$ we obtain (3.3) and next taking $h = 1$ we have (3.2). Therefore

$$\mathbb{E}_{\widetilde{\mathbb{P}}}\big[h(\widetilde{A}, \widetilde{B}) g(\widetilde{Z})\big] = \mathbb{E}_{\widetilde{\mathbb{P}}}\big[h(\widetilde{A}, \widetilde{B})\big] \mathbb{E}_{\widetilde{\mathbb{P}}}\big[g(\widetilde{Z})\big],$$

that proves independence of $\widetilde{Z}$ and $(\widetilde{A}, \widetilde{B})$. □

Next we define a probability measure μ on $\mathbb{R}^+ \times \mathbb{R}$:

$$\mu(U) = \mathbb{E}_{\widetilde{\mathbb{P}}}\Big[\mathbf{1}_U(\widetilde{A}, \widetilde{B})\Big],$$

for every Borel set $U \subset \mathbb{R}^+ \times \mathbb{R}$.

Lemma 3.4. *Measures ν and μ satisfy* (2.1).

Proof. Take arbitrary compactly supported function f on $\mathbb{R}$ and let $h\big((a, b), x\big) = f(ax + b)x$ be a function on $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$. In Lemma 3.1 we proved that $(\widetilde{A}, \widetilde{B})$ and $\widetilde{Z}$ are independent and moreover

that $\widetilde{Z}$ and Z have the same distribution, applying these observations we have

$$\begin{aligned}
\mu * \nu(f) &= \int_{\mathbb{R}^+\times\mathbb{R}} \int_{\mathbb{R}^+} f(ax+b)\nu(dx)\mu(da\,db) \\
&= \int_{\mathbb{R}^+\times\mathbb{R}} \int_{\mathbb{R}^+} h\big((a,b),x\big)\eta(dx)\mu(da\,db) \\
&= \mathbb{E}_{\widetilde{\mathbb{P}}}\Big[h\big((\widetilde{A},\widetilde{B}),\widetilde{Z}\big)\Big] = \mathbb{E}_{\widetilde{\mathbb{P}}}\Big[f\big(\widetilde{A}\widetilde{Z}+\widetilde{B}\big)\widetilde{Z}\Big] \\
&= \mathbb{E}\Big[\sum_{i=1}^{N} A_i f\Big(A_i Z_i + \sum_{j\neq i} A_j Z_j\Big) Z_i\Big] \\
&= \mathbb{E}\Big[f\Big(\sum_{i=1}^{N} A_i Z_i\Big)\sum_{i=1}^{N} A_i Z_i\Big] \\
&= \mathbb{E}\big[f(Z)Z\big] = \nu(f)
\end{aligned}$$

□

The next step is to prove that the measure μ satisfies hypotheses of Lemma 2.2, but first we will prove that the random variable Z has moments smaller than 1. The proof is classical, however we give here all the details for the reader's convenience.

Lemma 3.5. *For every $\alpha < 1$:*

$$\mathbb{E}\big[Z^\alpha\big] < \infty.$$

Proof. First we will show that there exist constants C and M such that

$$\mathbb{P}\big[Z > t\big] \leq \frac{C\log t}{t}, \quad \text{for } t > M. \tag{3.6}$$

For this purpose we will use the asymptotic of the Laplace transform of Z given in (1.2). Fix ε and δ such that

$$\left|\frac{1-e^{-x}}{x} - 1\right| < \delta, \qquad \text{for } |x| \leq \varepsilon$$

and

$$1 - \phi(\lambda) \leq (C+\delta)\lambda|\log\lambda| \qquad \text{for } \lambda < \varepsilon.$$

Then for $\lambda < \varepsilon$ and $t > 1$ we have

$$(C+\delta)|\log\lambda| \geq \frac{1-\phi(\lambda)}{\lambda} = \mathbb{E}\left[\frac{1-e^{-\lambda Z}}{\lambda}\right] \geq t\mathbb{E}\left[\frac{1-e^{-\lambda t}}{\lambda t}\mathbf{1}_{[t,\infty)}(Z)\right].$$

Putting $\lambda = \frac{\varepsilon}{t}$ in the inequality above we obtain

$$(C+\delta)(\log t - \log\varepsilon) \geq t\mathbb{E}\left[\frac{1-e^{-\varepsilon}}{\varepsilon}\mathbf{1}_{[t,\infty)}(Z)\right] \geq t(1-\delta)\mathbb{P}\big[Z > t\big].$$

Hence

$$\mathbb{P}\big[Z > t\big] \leq \frac{(C+\delta)(\log t - \log\varepsilon)}{(1-\delta)t}.$$

But for $t > \frac{1}{\varepsilon}$, $\log t - \log\varepsilon < 2\log t$, thus we obtain (3.6), with $M = \frac{1}{\varepsilon}$.

Finally for $\alpha < 1$ we write

$$\begin{aligned}\mathbb{E}\big[Z^\alpha\big] &= \mathbb{E}\big[Z^\alpha \cdot \mathbf{1}_{(0,1)}\big] + \sum_{n=0}^{\infty} \mathbb{E}\Big[Z^\alpha \cdot \mathbf{1}_{[2^n,2^{n+1})}(Z)\Big] \\ &\leq 1 + \sum_{n=0}^{\infty} 2^{\alpha(n+1)} \mathbb{P}\big[Z \geq 2^n\big] \\ &\leq 1 + \sum_{n=0}^{\lceil \log_2 M\rceil} 2^{\alpha(n+1)} + C \sum_{n=\lceil \log_2 M\rceil+1}^{\infty} \frac{n}{2^{n(1-\alpha)}}\end{aligned}$$

and the expression above is finite. □

Lemma 3.7. *The measure μ satisfies hypotheses of Lemma 2.2.*

Proof. First notice that by (1.5) we have

$$\mathbb{E}_{\widetilde{\mathbb{P}}}\big[\log \widetilde{A}\big] = \mathbb{E}\Big[\sum_{i=1}^{N} A_i \log A_i\Big] = 0.$$

Next if $\widetilde{A}$ would be equal to 1 almost surely, then we had $\mathbb{E}\big[\sum_{i=1}^{N} A_i\big] = \mathbb{E}[N]$, but the left hand side of this equation by (1.4) is equal to 1, whereas the right one, by (1.8) is strictly larger than 1.

Assume now that for some x: $\widetilde{A}x + \widetilde{B} = x$ a.s., then $x = \frac{\widetilde{B}}{1-\widetilde{A}}$, but by definition $\widetilde{B}$ is positive a.s., whereas $1 - \widetilde{A}$ changes the sign, since we already know: $\mathbb{E}[\log \widetilde{A}] = 0$ and $\widetilde{A} \not\equiv 1$. Therefore such a point x cannot exist.

Finally we have to check moments conditions. Take δ_2 as in (1.7), then

$$\mathbb{E}_{\widetilde{\mathbb{P}}}\Big[\widetilde{A}^{\delta_2}\Big] = \mathbb{E}\Big[\sum_{i=1}^{N} A_i \cdot A_i^{\delta_2}\Big] \leq \mathbb{E}\Big[\Big(\sum_{i=1}^{N} A_i\Big)^{1+\delta_2}\Big] < \infty.$$

Next applying (1.6) we obtain

$$\mathbb{E}_{\widetilde{\mathbb{P}}}\Big[\widetilde{A}^{-\delta_1}\Big] = \mathbb{E}\Big[\sum_{i=1}^{N} A_i^{1-\delta_1}\Big] < \infty.$$

To estimate moments of $\widetilde{B}$ we consider the σ-field generated by N and $\{A_i\}$: $\mathcal{F}_0 = \sigma(N, A_1, A_2, \ldots)$. Take $\alpha = 1 - \delta_1$ and ε such that $\frac{\alpha}{\alpha-\varepsilon} = 1 + \delta_2$. We may assume $\frac{\varepsilon}{\alpha} < 1$. We are going to estimate for every i the conditional expectation of $\big|\sum_{j\neq i} A_j Z_j\big|^\varepsilon$ with respect to $\mathcal{F}_0$. For this purpose, we use first concavity of the function $x \mapsto x^{\frac{\varepsilon}{\alpha}}$ and the Jensen inequality, then the inequality $|a+b|^\alpha \leq |a|^\alpha + |b|^\alpha$, which is valid for $\alpha < 1$, independence Z_j of $\mathcal{F}_0$ and finally Lemma 3.5. Thus, we obtain

$$\begin{aligned}\mathbb{E}\Big[\Big|\sum_{j\neq i} A_j Z_j\Big|^\varepsilon \Big| \mathcal{F}_0\Big] &\leq \Big(\mathbb{E}\Big[\Big|\sum_{j\neq i} A_j Z_j\Big|^\alpha \Big| \mathcal{F}_0\Big]\Big)^{\frac{\varepsilon}{\alpha}} \leq \Big(\mathbb{E}\Big[\sum_{j\neq i} A_j^\alpha Z_j^\alpha \Big| \mathcal{F}_0\Big]\Big)^{\frac{\varepsilon}{\alpha}} \\ &\leq \Big(\sum_{j\neq i} A_j^\alpha \cdot \mathbb{E}\big[Z_j^\alpha\big]\Big)^{\frac{\varepsilon}{\alpha}} \leq C\Big(\sum_{j=1}^{N} A_j^\alpha\Big)^{\frac{\varepsilon}{\alpha}}.\end{aligned}$$

Next we use the Hölder inequality with parameters $p=\frac{\alpha}{\alpha-\varepsilon}$ and $q=\frac{\alpha}{\varepsilon}$ and we estimate

$$\begin{aligned}\mathbb{E}_{\widetilde{\mathbb{P}}}\big[|\widetilde{B}|^{\varepsilon}\big] &= \mathbb{E}\Big[\sum_{i=1}^{N} A_i\Big|\sum_{j\neq i} A_jZ_j\Big|^{\varepsilon}\Big] = \mathbb{E}\Big[\mathbb{E}\Big[\sum_{i=1}^{N} A_i\Big|\sum_{j\neq i} A_jZ_j\Big|^{\varepsilon}\Big|\mathcal{F}_0\Big]\Big]\\ &= \mathbb{E}\Big[\sum_{i=1}^{N} A_i\mathbb{E}\Big[\Big|\sum_{j\neq i} A_jZ_j\Big|^{\varepsilon}\Big|\mathcal{F}_0\Big]\Big] \le C\mathbb{E}\Big[\sum_{i=1}^{N} A_i\cdot\Big(\sum_{j=1}^{N} A_j^{\alpha}\Big)^{\frac{\varepsilon}{\alpha}}\Big]\\ &\le C\mathbb{E}\Big[\Big(\sum_{i=1}^{N} A_i\Big)^{1+\delta_2}\Big]^{\frac{1}{p}}\cdot\mathbb{E}\Big[\sum_{j=1}^{N} A_j^{1-\delta_1}\Big]^{\frac{1}{q}}\end{aligned}$$

and in view of (1.6) and (1.7) the value above is finite. □

Now we are ready to conclude.

Proof of Theorem 1.3. Assume A_i are aperiodic. Fix $\beta>1$. In view of Lemma 3.4 hypotheses of Lemma 2.2 are fulfilled, therefore for every ε there exists M such that

$$\Big|\nu(t,\beta t)-C_0\log\beta\Big|<\varepsilon$$

for every $t>M$. Next we estimate the tail of Z. We have

$$\begin{aligned}t\mathbb{P}\big[Z>t\big] &= t\cdot\sum_{n=0}^{\infty}\mathbb{P}\big[t\beta^n<Z\le t\beta^{n+1}\big] = t\cdot\sum_{n=0}^{\infty}\int_{t\beta^n}^{t\beta^{n+1}}\eta(dx)\\ &\le \sum_{n=0}^{\infty}\frac{1}{\beta^n}\int_{t\beta^n}^{t\beta^{n+1}}x\eta(dx) = \sum_{n=0}^{\infty}\frac{1}{\beta^n}\nu\big(t\beta^n,t\beta^{n+1}\big]\\ &\le \sum_{n=0}^{\infty}\frac{C_0\log\beta+\varepsilon}{\beta^n} = \frac{\beta(C_0\log\beta+\varepsilon)}{\beta-1}.\end{aligned}$$

Hence passing with ε to 0 and next with β to 1 we obtain

$$\limsup_{t\to\infty} t\mathbb{P}\big[Z>t\big]\le C_0.$$

Analogously we justify

$$\liminf_{t\to\infty} t\mathbb{P}\big[Z>t\big]\ge C_0,$$

that proves the Theorem in the aperiodic case. If A_i are periodic, then the same arguments gives the result □

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