ON THE RIESZ TRANSFORM ASSOCIATED WITH THE ULTRASPHERICAL POLYNOMIALS

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Abstract. We define and investigate the Riesz transform associated to the differential operator $L_\lambda f(\theta) = -f''(\theta) - 2\lambda \cot\theta f'(\theta) + \lambda^2 f(\theta)$. We prove that it can be defined as a principal value, and that it is bounded on $L^p([0,\pi], dm_\lambda(\theta))$, $dm_\lambda(\theta) = \sin^{2\lambda}\theta d\theta$, for every $1 < p < \infty$ and of weak type (1,1). The same boundedness properties hold for the maximal operator of the truncated operators. The speed of convergence of the truncated operators is measured in terms of the boundedness in $L^p(dm_\lambda)$, $1 < p < \infty$ and weak type $(1,1)$ of the oscillation and ρ-variation associated to them. Also, a multiplier theorem is proved to get the boundedness of the conjugate function studied by Muckenhoupt and Stein for $1 < p < \infty$ as a corollary of the results for the Riesz transform. Moreover, we find a condition on the weight v which is necessary and sufficient for the existence of a weight u such that the Riesz transform is bounded from $L^p(v\, dm_\lambda)$ into $L^p(u\, dm_\lambda)$.

1. Introduction

We consider the ultraspherical polynomials of type λ, with any $\lambda > 0$, $P_n^\lambda(x)$ defined as the coefficients in the expansion of the generating function $(1 - 2x\omega + \omega^2)^{-\lambda} = \sum_{n=0}^\infty \omega^n P_n^\lambda(x)$ (see [17] for further details). It is known that the set $\{P_n^\lambda(\cos\theta) : n \in \mathbb{N}\}$ is orthogonal and complete in $L^2[0,\pi]$ with respect to the measure $dm_\lambda(\theta) := \sin^{2\lambda}\theta d\theta$. More precisely, we have

$$(1.1) \qquad \int_0^\pi P_n^\lambda(\cos\theta) P_m^\lambda(\cos\theta) dm_\lambda(\theta) = \delta_{n,m} 2^{1-2\lambda}\pi\Gamma(\lambda)^{-2}\frac{\Gamma(n+2\lambda)}{(n+\lambda)n!} = \delta_{n,m}/\gamma_n.$$

It is also known that the functions $P_n^\lambda(\cos\theta)$ are eigenfunctions of the operator L_λ

$$(1.2) \qquad L_\lambda f(\theta) = -f''(\theta) - 2\lambda\cot\theta f'(\theta) + \lambda^2 f(\theta),$$

with $L_\lambda P_n^\lambda(\cos\theta) = (n+\lambda)^2 P_n^\lambda(\cos\theta)$.

In this paper we develop a general method for studying the properties of certain operators associated to the differential operator L_λ, that appear similarly to the way that classical operators do when considering the Laplacian operator ∂_θ^2 in $[0, 2\pi]$. Given an operator T we produce a partition of T into its "local" and its "global" parts according to two regions: the "local region", where roughly speaking we can use that the Lebesgue measure $d\phi$ and $dm_\lambda(\phi)$ are equivalent, and its complementary, that we call "global region" (see (3.2) and (3.3)). This method was widely used by many authors in various contexts (see e.g. [4, 5, 9, 11, 14]). The procedure for the local part of the operators

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consists in comparing the kernel of the operator with the kernel of the classical analogous operator in the torus. We show that the difference between them is bounded by a "good" kernel, i.e., a kernel which defines a bounded operator on $L^p(d\phi)$ for every p in the range $1 \le p \le \infty$. By a "heritage" theorem (Theorem 3.8), we prove that the local part of the classical operator in the torus inherits the boundedness in L^p with respect to the Lebesgue measure. Next, for operators with kernels supported in the local part ("local operators"), we prove (see Lemma 3.9) that the boundedness with respect to Lebesgue measure is equivalent to the boundedness with respect to $dm_\lambda(\phi)$. Therefore we conclude that the local part of T is bounded in $L^p(dm_\lambda)$. In the global part we just use that the kernel of the operator T is bounded by a positive and nice kernel.

In this paper we apply the above method in order to study the Riesz transform (Theorem 2.14), the maximal operator associated to it (Theorem 7.1) and its oscillation and variation (Theorem 8.2). We also prove some multiplier theorem in order to show that the results in [10] can be obtained from ours, see section 6. We would like to thank S. Meda for some helpful suggestions concerning this theorem.

For every "nice" function f defined on $[0,\pi]$ we associate its ultraspherical expansion

$$f(\theta) \sim \sum_{n=0}^{\infty} a_n P_n^\lambda(\cos\theta), \quad \text{where} \quad a_n = \gamma_n \int_0^\pi f(\theta) P_n^\lambda(\cos\theta)\, dm_\lambda(\theta). \tag{1.3}$$

In the paper [10] Muckenhoupt and Stein studied ultraspherical expansions from a harmonic analysis point of view. For any function as in (1.3) they define the "harmonic" extension of f as

$$f(r,\theta) = \sum_{n=0}^{\infty} a_n r^n P_n^\lambda(\cos\theta) \tag{1.4}$$

and the "conjugate harmonic" series as

$$\widetilde{f}(r,\theta) = \sum_{n=1}^{\infty} \frac{2\lambda}{n+2\lambda} a_n r^n \sin\theta P_{n-1}^{\lambda+1}(\cos\theta). \tag{1.5}$$

Then, both functions satisfy Cauchy–Riemann equations:

$$\begin{aligned} \partial_r((r\sin\theta)^{2\lambda}\widetilde{f}) &= -r^{2\lambda-1}(\sin\theta)^{2\lambda}\partial_\theta f, \\ \partial_\theta((r\sin\theta)^{2\lambda}\widetilde{f}) &= r^{2\lambda+1}(\sin\theta)^{2\lambda}\partial_r f. \end{aligned}$$

One of the main results in [10] is that if $f \in L^p(dm_\lambda)$ for $1 < p < \infty$, then there exists the boundary function $\tilde{f}(\theta)$ of $\widetilde{f}(r,\theta)$ such that $\tilde{f}(\theta) = \lim_{r\to 1} \widetilde{f}(r,\theta)$ in $L^p(dm_\lambda)$ norm and $\|\tilde{f}\|_{L^p(dm_\lambda)} \le A_p \|f\|_{L^p(dm_\lambda)}$. $\tilde{f}(\theta)$ is called the conjugate function of f. Concerning the case $p=1$, they show that the "conjugate harmonic" function $f(\theta) \to \tilde{f}(r,\theta)$ satisfies an appropriate substitute relation (see Corollary 2, page 43, in [10]). As far as we understand, this result does not lead in an obvious way to the weak type $(1,1)$ of the conjugate function $\tilde{f}(\theta)$.

The connection between the results in [10] and the ones contained in this paper is as follows. In section 6 we prove a multiplier theorem that allows us to obtain the boundedness properties in L^p for p in the range $1 < p < \infty$, of the conjugate function $\tilde{f}(\theta)$ as a corollary of our results for the Riesz transform defined spectrally in (1.7), and a posteriori as a principal value operator. Although we obtain indeed the weak type $(1,1)$ for the Riesz transform (1.7), it is not clear for us wether the conjugate mapping

$f(\theta) \to \tilde{f}(\theta)$ of Muckhenhoupt and Stein, which is defined only for $f \in L^p$, satisfies also a weak type $(1,1)$ inequality.

It is easy to check that L_λ is formally self-adjoint on the space $L^2(dm_\lambda)$. Our operator L_λ factorizes as $L_\lambda f(\theta) = (-\partial_\theta^*\partial_\theta + \lambda^2)f(\theta)$, see section 2, where $\partial_\theta^* = \partial_\theta + 2\lambda\cot\theta$ is formally adjoint to ∂_θ, i.e., $\langle \partial_\theta^* f, g\rangle_{L^2(m_\lambda)} = -\langle f, \partial_\theta g\rangle_{L^2(m_\lambda)}$. For a function f as in (1.3) we define its Poisson integral associated to our operator L_λ as follows

$$Pf(e^{-t},\theta) = e^{-t\sqrt{L_\lambda}} f(\theta) = \sum_{n=0}^{\infty} a_n e^{-t(n+\lambda)} P_n^\lambda(\cos\theta). \tag{1.6}$$

We proceed now to define one of the main objects we are going to investigate in this paper, the Riesz transform. Let us call *polynomial function* to any finite linear combination of ultraspherical polynomials, that is, any f of the form $f = \sum_{n=0}^N a_n P_n^\lambda$. We define the Riesz transform of any polynomial function f by spectral techniques, as it is suggested in [15], as follows:

$$\begin{aligned} R_\lambda f(\theta) &= \partial_\theta (L_\lambda)^{-1/2} f(\theta) = \partial_\theta \left(\sum_{n=0}^{N} \frac{a_n}{n+\lambda} P_n^\lambda(\cos\theta)\right) \\ &= -2\lambda \sum_{n=1}^{N} \frac{a_n}{n+\lambda} \sin\theta P_{n-1}^{\lambda+1}(\cos\theta). \end{aligned} \tag{1.7}$$

The above equation follows from the formula $\frac{\partial}{\partial t} P_n^\lambda(t) = 2\lambda P_{n-1}^{\lambda+1}(t)$, see [17].

It turns out that if for a given function f and its Poisson integral Pf, we define analogously to (1.5) the conjugate Poisson integral

$$R_\lambda f(r,\theta) = -2\lambda \sum_{n=1}^{N} \frac{a_n}{n+\lambda} r^{n+\lambda} \sin\theta P_{n-1}^{\lambda+1}(\cos\theta).$$

The following Cauchy–Riemann equations are satisfied

$$\begin{aligned} \partial_\theta Pf(e^{-t},\theta) &= -\,\partial_t R_\lambda f(e^{-t},\theta), \\ \partial_t Pf(e^{-t},\theta) + \lambda^2 \int_t^\infty Pf(e^{-s},\theta)ds &= \partial_\theta^* R_\lambda f(e^{-t},\theta). \end{aligned} \tag{1.8}$$

Our first main result, Theorem 2.13, says that in fact our Riesz transform on polynomial functions is given by a principal value, i.e., for any f a polynomial function,

$$R_\lambda f(\theta) = p.v. \int_0^\pi R_\lambda(\theta,\phi) f(\phi) dm_\lambda(\phi) = \lim_{\varepsilon\to 0} \int_{|\theta-\phi|>\varepsilon} R_\lambda(\theta,\phi) f(\phi) dm_\lambda(\phi) \text{ a.e.}$$

Also, we prove that the Riesz transform R_λ is bounded on $L^p(dm_\lambda)$ and of weak type (1,1) (see Theorem 2.14). As a corollary we obtain that $R_\lambda f(e^{-t},\theta)$ tends to $R_\lambda f$ in L^p norm for $1 < p < \infty$, as t goes to 0.

The second object we are interested in is a maximal function

$$R_\lambda^* f(\theta) := \sup_{\varepsilon>0} \left| \int_{|\theta-\phi|>\varepsilon} R_\lambda(\theta,\phi) f(\phi) dm_\lambda(\phi) \right|.$$

Theorem 7.1 says that R_λ^* is bounded on $L^p(dm_\lambda)$ for $1 < p < \infty$, and of weak type (1,1). As a consequence of this result, we get that the Riesz transform is a principal value also for functions in $L^1(dm_\lambda)$ (Theorem 7.2). Given this, our next aim is measuring the speed of convergence, something which has classically been done by means of certain operators involving sums of the differences of the truncated operators. Namely, we will use the so called oscillation operator and ρ-variation operator, and obtain that they both are bounded in $L^p(dm_\lambda)$ for $1 < p < \infty$, and that they satisfy a weak type $(1,1)$ inequality with respect to dm_λ (see section 8 for the precise definition of the operators and statement of the results). As an intermediate step in the proof of this results we obtain the boundedness in $L^p(d\phi)$ for $1 < p < \infty$ and the weak type $(1,1)$ for the oscillation and ρ-variation associated to the classical conjugate function in the torus. This result, that we believe of intrinsic interest, is obtained by comparison with the same operators for the Hilbert transform in the line.

The last part of this paper is devoted to the following problem. Given a fixed operator T (in our case we consider the Riesz transform R_λ) bounded in $L^p(dm_\lambda)$ for some $1 < p < \infty$, find necessary and sufficient conditions for a weight v in order to have the existence of a nontrivial weight u such that T becomes bounded from $L^p(vdm_\lambda)$ into $L^p(udm_\lambda)$. In the case of the Riesz transform considered in this paper we prove (see Theorem 9.3) that the condition is that

$$\int_0^\pi v(x)^{-\frac{1}{p-1}} dm_\lambda(x) < \infty.$$

To obtain this result, we consider a vector-valued extension of R_λ to the operator taking values in the Banach space l^p. Then we apply the analogous vector-valued version of the technique of splitting the operator into its local and global parts, to get the boundedness in $L^p_{\ell^p}$ of the extension (see section 3 for the definitions).

Guide to the paper. The structure of the paper is as follows. In section 2 we state precisely the notation and all the necessary definitions in order to introduce the objects of our interest. In particular, the Riesz transform R_λ is defined. In section 3 we introduce a "general machinery" in order to handle the "local" and "global" parts of the operators. In section 4 we study the behavior of the Riesz kernel according to this partition, and in section 5 we show that the Riesz transform R_λ is given as a principal value for polynomial functions and its boundedness properties. In section 6 we prove a multiplier theorem.

Section 7 is devoted to study the maximal function $R^*_\lambda f(\theta)$ and prove that the maximal operator $f \mapsto R^*_\lambda f$ is bounded on every $L^p(dm_\lambda)$, for $1 < p < \infty$ and of weak type $(1,1)$. The next section, section 8 deals with the study of the oscillation and ρ-variation operators. Finally, in section 9 we consider weighted inequalities and answer the question mentioned above (Theorem 9.3).

2. The Riesz transform

2.1. **Poisson kernel.** We are going to describe here some fundamental properties of our Poisson kernel P. By using (1.4) and (1.6), observe that

$$Pf(e^{-t},\theta) = e^{-t\lambda} f(e^{-t},\theta). \tag{2.1}$$

The results in [10] for $f(r,\theta)$ lead easily to prove that the Poisson integral Pf is given by a kernel, which can be computed explicitly. Namely, for any $\theta \in [0,\pi]$ we have

$$Pf(r,\theta) = r^\lambda \int_0^\pi P(r,\theta,\phi) f(\phi) dm_\lambda(\phi) \tag{2.2}$$

where $r = e^{-t}$ and

$$P(r,\theta,\phi) = \frac{\lambda}{\pi}(1-r^2)\int_0^\pi \frac{\sin^{2\lambda-1} t}{\big(1 - 2r(\cos\theta\cos\phi + \sin\theta\sin\phi\cos t) + r^2\big)^{\lambda+1}} dt. \tag{2.3}$$

Another easy consequence of (2.1) and Theorem 2, section 6, p.31 in [10] is the following theorem.

Theorem 2.4. *[Theorem 2, [10]] Let $f \in L^p(dm_\lambda)$, $1 \le p \le \infty$. Then*

(a) $\|Pf(r,\cdot)\|_{L^p(dm_\lambda)} \le r^\lambda \|f\|_{L^p(dm_\lambda)}$,
(b) $\|Pf(r,\cdot) - f\|_{L^p(dm_\lambda)} \to 0$ *as* $r \to 1$,
(c) $\lim_{r\to 1} Pf(r,\theta) = f(\theta)$ *almost everywhere,*
(d) $\|\sup_{r<1} |Pf(r,\cdot)|\|_{L^p(dm_\lambda)} \le A_p \|f\|_{L^p(dm_\lambda)}$.

2.2. **The Riesz Transform.** In this section we compute the kernel of the Riesz transform. By using (1.1), one shows that

$$\|R_\lambda f\|_{L^2(dm_\lambda)} \le C\,\|f\|_{L^2(dm_\lambda)}. \tag{2.5}$$

With the same proof, we get the boundedness of the operator for any $f \in L^2(dm_\lambda)$ and so the definition (1.7) makes sense for every $f \in L^2(dm_\lambda)$. Observe that a similar argument gives that the operator $(L_\lambda)^{-1/2}$ is also well defined in $L^2(dm_\lambda)$ with the parallel formula to the one above. In the following lemma we show that $(L_\lambda)^{-1/2}$ is defined by a kernel almost everywhere, and that the Riesz transform has an associated kernel in the sense of Calderón-Zygmund. This last statement means, for T a linear operator, that there exists a function $k(\theta,\phi)$ such that for every $f \in L^2([0,\pi]; dm_\lambda)$ and θ outside the support of f, $Tf(\theta) = \int k(\theta,\phi) f(\phi)\, dm_\lambda(\phi)$.

Lemma 2.6. *Given $f \in L^1(dm_\lambda)$, for almost every $\theta \in [0,\pi]$, we have that*

$$(L_\lambda)^{-\frac{1}{2}} f(\theta) = \int_0^\pi W_\lambda(\theta,\phi) f(\phi)\, dm_\lambda(\phi), \text{ where } W_\lambda(\theta,\phi) = \int_0^1 r^{\lambda-1} P(r,\theta,\phi)\, dr. \tag{2.7}$$

Given $f \in L^1(dm_\lambda)$ and θ outside the support of f, we have that

$$R_\lambda f(\theta) = \int_0^\pi R_\lambda(\theta,\phi) f(\phi)\, dm_\lambda(\phi), \quad \text{where} \quad R_\lambda(\theta,\phi) = \int_0^1 r^{\lambda-1} P_\theta(r,\theta,\phi)\, dr, \tag{2.8}$$

where P_θ stands for the partial derivative of P with respect to θ.

Proof. To prove the first formula, we use the integral formula

$$s^{-a} = \frac{1}{\Gamma(a)} \int_0^\infty e^{-ts} t^a \frac{dt}{t}$$

to define the negative square root of L_λ :

$$\begin{aligned} L_\lambda^{-1/2} f(\theta) &= \frac{1}{\Gamma(1)} \int_0^\infty e^{-t\sqrt{L_\lambda}} f(\theta) t \frac{dt}{t} = \int_0^\infty e^{-t\lambda} f(e^{-t},\theta) dt \\ &= \int_0^1 \int_0^\pi r^{\lambda-1} P(r,\theta,\phi) f(\phi)\, dm_\lambda(\phi)\, dr, \end{aligned} \tag{2.9}$$

after applying the change of variables $r = e^{-t}$. We get here (2.7) if we can change the order of integration in the former integral. Before we proceed further let us introduce some notation:

$$\begin{aligned} a &= \cos\theta\cos\phi + \sin\theta\sin\phi\cos t = \cos(\theta-\phi) - \sin\theta\sin\phi(1-\cos t), \\ D_r &= 1 - 2ra + r^2. \end{aligned} \tag{2.10}$$

Observe that $a \le \cos(\theta-\phi)$, and that for negative a, $D_r \ge 1$, therefore we always have that $D_r \ge 1 - \cos(\theta-\phi)^2 \ge 0$. In particular, this implies that $r^{\lambda-1}P(r,\theta,\phi)$ is positive and bounded above by a constant times $r^{\lambda-1}/(1-\cos(\theta-\phi)^2)^{\lambda+1}$. Let us observe that the expression $f(r,\theta)=\int_0^\pi P(r,\theta,\phi)f(\phi)\,dm_\lambda(\phi)$ is well defined for almost every $\theta\in[0,\pi]$ since by the results in [10], namely Lemmas 5.2 and 5.3, pg. 28, we have that $|f(r,\theta)|\le(|f|)^*(\theta)$, where

$$f^*(\theta)=\sup_{h\neq 0;\ 0\le\theta+h\le\pi}\frac{\int_\theta^{\theta+h}|f(\phi)|\,dm_\lambda(\phi)}{\int_\theta^{\theta+h}dm_\lambda(\phi)}$$

and this function is bounded in $L^p(dm_\lambda)$ for every $p\in(1,\infty]$ and of weak type $(1,1)$. In particular, for $f\in L^1(dm_\lambda)$, it is finite for almost every θ . Thus, we can see that the integrand in (2.9) belongs to $L^1(dm_\lambda\times dr)$ for $f\in L^1([0,\pi];dm_\lambda)$.

Once we change the order of integration in (2.9), to get the expression of $R_\lambda(\theta,\phi)$ we just have to be able to put the derivative ∂_θ inside the integral. We prove it by showing that $r^{\lambda-1}P_\theta(r,\theta,\phi)f(\phi)\in L^1(dm_\lambda\times dr)$. Observe that $|\partial_\theta a|\le 2$ and that then $|r^{\lambda-1}P_\theta(r,\theta,\phi)|$ can be bounded above by a constant for θ outside the support of f, what gives the result. □

Remark 2.11. *One can also use the Cauchy–Riemann equations* (1.8) *to prove the above lemma. Namely, in view of equations $\partial_\theta Pf(e^{-t},\theta)=-\partial_t R_\lambda f(e^{-t},\theta)$ and $R_\lambda f(0,\theta)=0$, we obtain*

$$R_\lambda f(e^{-t},\theta)=\int_t^\infty \partial_\theta P_t f(\theta)dt=\int_t^\infty\int_0^\pi e^{-\lambda t}P_\theta(e^{-t},\theta,\phi)f(\phi)d\phi dt.$$

Further, if θ does not belong to the support of f, we can make $r=e^{-t}$ and then argue as before to justify making r going to 1 *to obtain* (2.8).

Remark 2.12. *Observe that for any bounded function f, $f^*(\theta)\le\|f\|_{L^\infty}$ is a well defined function, finite for every $\theta\in[0,\pi]$. Thus, by repeating the proof of* (2.7), *we have that it holds for every $\theta\in[0,\pi]$.*

Next, let us state here the main results concerning the Riesz transform, namely that it is a principal value (Theorem 2.13) and its boundedness properties in L^p, $1<p<\infty$, and in weak L^1 (Theorem 2.14). We will prove them in section 5.

Theorem 2.13. *The Riesz transform R_λ is given as a principal value on polynomial functions, i.e., for any polynomial function f and for almost every $\theta\in[0,\pi]$*

$$R_\lambda f(\theta)=\lim_{\varepsilon\to 0}\int_{|\theta-\phi|>\varepsilon}R_\lambda(\theta,\phi)f(\phi)dm_\lambda(\phi).$$

Theorem 2.14. *The Riesz transform R_λ is bounded on $L^p(dm_\lambda)$, for $1<p<\infty$ and from $L^1(dm_\lambda)$ into $L^{1,\infty}(dm_\lambda)$.*

Remark 2.15. *Given $p\in(1,\infty)$ and $f\in L^p(dm_\lambda)$ such that it has an associated Fourier series expansion $f\sim\sum a_nP_n^\lambda(\cos\theta)$, then*

$$R_\lambda f(\theta)\sim\sum -2\lambda\frac{a_n}{n+\lambda}\sin\theta P_{n-1}^{\lambda+1}(\cos\theta).$$

Indeed, consider the linear operators from $L^p(dm_\lambda)$ into $\mathbb{C}$ given by

$$\begin{aligned} U_nf &= \frac{-2\lambda a_n}{n+\lambda},\\ V_nf &= \|\sin\theta P_{n-1}^{\lambda+1}(\cos\theta)\|_{L^2(dm_\lambda)}^{-2}\int_0^\pi R_\lambda f(\phi)\sin\phi P_{n-1}^{\lambda+1}(\cos\phi)\,dm_\lambda(\phi).\end{aligned}$$

Let us note that U_n is trivially bounded by (1.3) *and Theorem 2.14, and that V_n is bounded also by Theorem 2.14 and the fact that $\sin\phi P_{n-1}^{\lambda+1}(\cos\phi)$ are in $L^{p'}(dm_\lambda)$. Now,* *observe that* $\{\sin\phi P_{n-1}^{\lambda+1}(\cos\phi)\}$ *form an orthogonal system in* $L^2(dm_\lambda)$*, and that therefore, for every polynomial function* f*,* $U_n f = V_n f$*. Since polynomial functions are dense in* $L^p(dm_\lambda)$ *for every* $p\in(1,\infty)$*, our thesis holds.*

3. General machinery

We start this section with an appropriate for our purposes covering lemma. First, we construct a family of balls on the interval $[0,\frac{\pi}{2}]$. Let us fix a very small $\delta>0$ throughout the paper, and define a sequence of points and balls as follows

$$\theta_0=\frac{\pi}{2},\quad \theta_{i+1}=\frac{\theta_i}{1+\delta},\quad B_i=\left\{\phi:\ \frac{1}{1+\delta}<\frac{\theta_i}{\phi}<1+\delta\right\},$$

Let us also consider balls $B(\theta_i,\delta')=\{\phi:\ (1+\delta')^{-1}<\theta_i/\phi<1+\delta'\}$. One can easily extend (by symmetry with respect to $\frac{\pi}{2}$) the above family to a set of balls covering the interval $(0,\pi)$, that will be denoted in the same way $\{B_i\}_{i=0}^{\infty}$.

Lemma 3.1. *For the collection of balls* $\{B_i\}_{i=0}^{\infty}$ *defined above we have*

(1) The collection $\{\overline{B}_i\}_{i=0}^{\infty}$ *of closed balls covers* $(0,\pi)$*.*
(2) There exists δ_0 *such that the balls* $B(\theta_i,\delta_0)$ *are pairwise disjoint.*
(3) For every $n\geq 1$*, if* $1+\delta_0=(1+\delta)^n$*, then the collection* $\{B(\theta_i,\delta_0)\}$ *has bounded overlap.*
(4) For every $n\geq 1$*, if* $1+\delta_0=(1+\delta)^n$*, then there exist a constant* C *depending only on* δ*,* n*, such that for every measurable set* $E\subset B(\theta_i,\delta_0)$*, we have that*

$$\frac{1}{C}\theta_i^{2\lambda}|E|\leq m_\lambda(E)\leq C\theta_i^{2\lambda}|E|.$$

Proof. By the symmetry of the construction it is enough to prove this properties for the balls with center in $[0,\pi/2]$. In order to prove (2) it is enough to take δ_0 satisfying $(1+\delta_0)^2<1+\delta$. In fact, we have

$$(1+\delta_0)\theta_{i+1}<\frac{(1+\delta)\theta_{i+1}}{1+\delta_0}=\frac{\theta_i}{1+\delta_0}.$$

We leave the details of the remaining proof to the reader. □

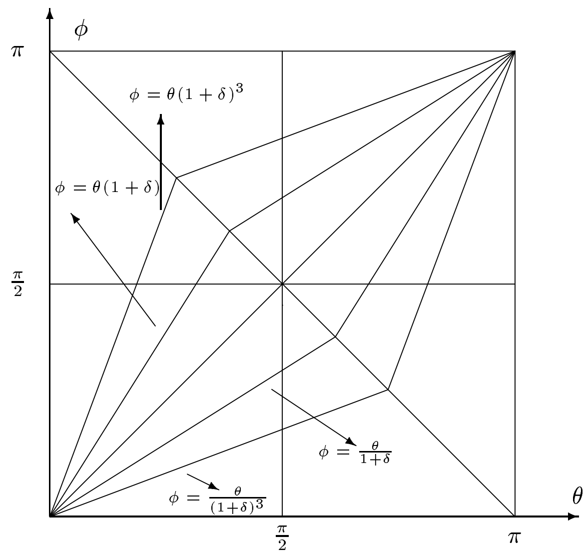
Consider a linear operator T mapping the space of polynomial functions into the space of measurable functions on $[0,\pi]$, satisfying the following assumptions:

(a) T either extends to a bounded operator on $L^q(d\phi)$ for some $1<q<\infty$, or of weak type (1,1) with respect to $d\phi$,
(b) there exists a measurable function K, defined in the complement of the diagonal in $[0,\pi]\times[0,\pi]$, such that for every function f and all θ outside the support of f,

$$Tf(\theta)=\int_0^{\pi}K(\theta,\phi)f(\phi)d\phi,$$

(c) for all (θ,ϕ), $\theta\neq\phi$, K verifies $|K(\theta,\phi)|\leq C|\theta-\phi|^{-1}$.

Let us denote by N_t the region constructed in the following way: consider the part of the region $M_t=\{(\theta,\phi):\ \frac{1}{1+t}<\frac{\theta}{\phi}<1+t\}$ under the diagonal of the square $[0,\pi]^2$ with negative slope, and define N_t to be this region together with its reflected with respect to the point $(\pi/2,\pi/2)$. It is not difficult to see that the regions N_{t_1} and N_{t_2}, with $t_1=\delta$ and $1+t_2=(1+\delta)^3$ have the shape shown in the next picture.



Observe that we get a region contained in M_t and symmetric with respect to the point $(\pi/2, \pi/2)$. This region will define the "local" and "global" parts of the operators.

Given an operator T satisfying (b), we define its global and local parts by

$$T_{\text{glob}} f(\theta) = \int_0^\pi K(\theta,\phi)(1 - 1_{N_{t_2}}(\theta,\phi)) f(\phi) d\phi, \tag{3.2}$$

$$T_{\text{loc}} f(\theta) = T f(\theta) - T_{\text{glob}} f(\theta). \tag{3.3}$$

We will need the following two lemmas, whose proofs are a reformulation of the proofs of Lemmas 3.2 and 3.3 in [5] using only properties of the covering from Lemma 3.1.

Lemma 3.4. *Let $d\nu$ be any positive measure in $[0,\pi]$, $\{f_j\}$ be a sequence of functions and define $f = \sum_j 1_{B_j} f_j$, where $\{B_j : j \in \mathbb{N}\}$ is the collection of balls in Lemma 3.1. Then, for all $\mu > 0$,*

$$\nu\{\theta : |f(\theta)| > \mu\} \le \sum_j \nu\{\theta \in B_j : |f_j(\theta)| > \mu/2\}, \tag{3.5}$$

and for any $1 \le q < \infty$

$$\|f\|_{L^q(d\nu)} \le 2\bigg(\sum_j \int_{B_j} |f_j(\theta)|^q \, d\nu(\theta)\bigg)^{1/q}, \tag{3.6}$$

Proof. Observe that any point $\theta \in [0,\pi]$ belongs to at most two balls of the covering $\{B_j\}$. Then, the level set $\{\theta : |f(\theta)| > \mu\}$ is a subset of $\bigcup_j \{\theta \in B_j : |f_j(\theta)| > \mu/2\}$, which gives (3.5). To prove (3.6), just observe that

$$\int_0^\pi |f(\theta)|^q \, d\nu(\theta) \le \int_0^\pi (2 \max_j |f_j(\theta)| 1_{B_j}(\theta))^q \, d\nu(\theta) \le 2^q \int_0^\pi \sum_j |f_j(\theta)|^q 1_{B_j}(\theta) \, d\nu(\theta).$$

□

Lemma 3.7. *Let T be a linear operator mapping polynomial functions into the space of measurable functions on $[0,\pi]$ verifying (a). Given the covering $\{B_j\}_{j=0}^\infty$, we define the* *operator*

$$T^1 f(\theta) = \sum_j 1_{B_j}(\theta)|T(1_{B_j'} f)(\theta)|, \quad \textit{where} \quad B_j' = B(\theta_j, \delta'),\ 1+\delta' = (1+\delta)^2.$$

Then, T^1 is also bounded from $L^1(d\phi)$ into $L^{1,\infty}(d\phi)$ or from $L^q(d\phi)$ into $L^q(d\phi)$, $1 < q < \infty$, as the case might be.

Proof. To prove that T^1 is of weak type $(1,1)$ with respect to $d\phi$ if T is, observe that

$$|\{\theta \in B_j :\ |T(1_{B_j'} f)(\theta)| > \mu/2\}| \le \frac{C}{\mu}\int_{B_j} |f(\phi)|\, d\phi.$$

We finish if we apply Lemma 3.4 and the bounded overlap property from Lemma 3.1. The proof of strong type (q,q) is analogous. □

The following theorem is one of the main tools in the proofs of the results in the paper.

Theorem 3.8. *Under the assumptions (a), (b) and (c) made on T above, the operator T_{loc} inherits from T either $L^q(d\phi)$-boundedness or the weak type (1,1) with respect to the Lebesgue measure, as the case might be.*

Proof. Assume that θ is in the ball B_j and define B_j' as in Lemma 3.7. Then

$$\begin{aligned} T_{\mathrm{loc}} f(\theta) =& T f(\theta) - T_{\mathrm{glob}} f(\theta) \\ =& T(f 1_{B_j'})(\theta) + T(f(1-1_{B_j'}))(\theta) - \int_0^\pi (1 - 1_{N_{t_2}}(\theta,\phi)) K(\theta,\phi) f(\phi) d\phi \\ =& T(f 1_{B_j'})(\theta) + \int_0^\pi (1_{N_{t_2}}(\theta,\phi) - 1_{B_j'}(\phi)) K(\theta,\phi) f(\phi) d\phi. \end{aligned}$$

By multiplying by 1_{B_j} and summing over j, we get that for any $\theta \in [0,\pi]$

$$|T_{\mathrm{loc}} f(\theta)| \le |T^1(f)(\theta)| + T^2(f)(\theta),$$

where T^1 is as in Lemma 3.7, and by that lemma it is bounded in L^q or of weak type $(1,1)$ as corresponds. Therefore, we only need to prove that T^2 is bounded, where

$$T^2(f)(\theta) = \int_0^\pi \sum_j 1_{B_j}(\theta) |1_{N_{t_2}}(\theta,\phi) - 1_{B_j'}(\phi)| |K(\theta,\phi)| |f(\phi)| d\phi.$$

This operator is given by the kernel

$$H(\theta,\phi) = \sum_j 1_{B_j}(\theta) |1_{N_{t_2}}(\theta,\phi) - 1_{B_j'}(\phi)| |K(\theta,\phi)|,$$

which gives a bounded operator. We will study this kernel in each of the squares $[0,\pi/2]^2$, $[0,\pi/2]\times[\pi/2,\pi]$, $[\pi/2,\pi]\times[0,\pi/2]$ and $[\pi/2,\pi]^2$. In the first and fourth ones, $H(\theta,\phi)$ is supported in $N_{t_2}\setminus N_{t_1}$. Indeed, let us assume first $(\theta,\phi)\in[0,\pi/2]^2$. If $(\theta,\phi)\notin N_{t_2}$ then

$$\frac{1}{(1+\delta)^3} > \frac{\theta}{\phi} \quad \text{or} \quad \frac{\theta}{\phi} > (1+\delta)^3.$$

Hence, if $\theta \in B_j$, then ϕ cannot belong to B_j' and then $|1_{N_{t_2}}(\theta,\phi) - 1_{B_j'}(\phi)| = 0$. If $(\theta,\phi)\in N_{t_1}$ and $\theta\in B_j$ then

$$\frac{1}{1+\delta} < \frac{\theta}{\phi} < 1+\delta \qquad \text{and} \qquad \frac{1}{(1+\delta)^2} < \frac{\theta_j}{\phi} < (1+\delta)^2,$$

which implies that $\phi \in B'_j$ and $|1_{N_{t_2}}(\theta,\phi) - 1_{\delta' B_j}(\phi)| = 0$. For $(\theta,\phi) \in [\pi/2,\pi]^2$, we use that each term $1_{B_j}(\theta)|1_{N_{t_2}}(\theta,\phi) - 1_{B'_j}(\phi)|$ is symmetric with respect to $(\pi/2,\pi/2)$ and we get in this region the same result.

In the region where $(\theta,\phi) \in [0,\pi/2]\times[\pi/2,\pi]$, one can see by similar arguments as before that the support of $H(\theta,\phi)$ is contained in $N^c_{t_1}$. Since in that region there exists a constant ε such that $|\theta-\phi| > \varepsilon$, there $|K(\theta,\phi)| \le C_\varepsilon$ and by the finite overlapping property from Lemma 3.1, $H(\theta,\phi) \le C\, 1_{N^c_{t_1}}(\theta,\phi)$. By symmetry, we get the same bound in the region $[\pi/2,\pi]\times[0,\pi/2]$.

Since the part of the operator in the regions $[0,\pi/2]\times[\pi/2,\pi]$ and $[\pi/2,\pi]\times[0,\pi/2]$ is trivially bounded in $L^p(d\phi)$ for every p $(1 \le p \le \infty)$, to prove that T^2 also satisfies this property, it is enough to prove that

$$\sup_{\phi\in[0,\pi/2]} \int_0^{\pi/2} \frac{1_{N_{t_2}\setminus N_{t_1}}(\theta,\phi)}{|\theta-\phi|}\,d\theta < \infty, \qquad \sup_{\theta\in[0,\pi/2]} \int_0^{\pi/2} \frac{1_{N_{t_2}\setminus N_{t_1}}(\theta,\phi)}{|\theta-\phi|}d\phi < \infty.$$

Let us prove the first inequality, since the second one follows in an identical way. To this end, write:

$$\begin{aligned}\sup_\phi \int_0^{\pi} \frac{1_{N_{t_2}\setminus N_{t_1}}(\theta,\phi)}{|\theta-\phi|}d\theta &\le \sup_\phi \int_{\frac{\phi}{1+t_2}}^{\frac{\phi}{1+t_1}} \frac{1}{\phi-\theta}d\theta + \sup_\phi \int_{\phi(1+t_1)}^{\phi(1+t_2)} \frac{1}{\theta-\phi}d\theta \\ &= \log\frac{1-\frac{1}{1+t_1}}{1-\frac{1}{1+t_2}} + \log\frac{1+t_2}{1+t_1} < \infty.\end{aligned}$$

□

Similar arguments as the ones shown in the above proof lead to the following lemma.

Lemma 3.9. *Assume that an operator S defined on polynomial functions satisfies (b), with a kernel supported in N_{t_2}. Then, strong type (p,p) for Lebesgue measure and m_λ measure are equivalent. The same holds for weak type (p,p), $1 \le p \le \infty$.*

Proof. One can see that $|Sf(\theta)| \le \sum_j 1_{B_j}|S(f1_{\tilde B_j})(\theta)|$, where B_j are as in Lemma 3.1 and $\tilde B_j = B(\theta_j,\tilde\delta)$, for $1+\tilde\delta = (1+\delta)^4$, just proceeding as in the proof of Theorem 3.8. Then, by using Lemma 3.4, and parts (3) and (4) from Lemma 3.1, one can easily get the result. □

In sections 7 and 9 we will have to consider linear operators taking values in some Banach spaces. We denote by $L^p_B(\mu) = L^p_B(\mu,[0,\pi])$, $p<\infty$, the Bochner-Lebesgue space consisting of all B-valued (strongly) measurable functions f defined on $[0,\pi]$ such that $\|f\|^p_{L^p_B(\mu)} = \int_0^\pi \|f(\theta)\|^p_B d\mu(\theta) < \infty$. For $p=\infty$ we write L^∞_B for the space of all f such that $\|f\|_{L^\infty_B} = \operatorname{supess}\|f(\theta)\|_B < \infty$. Similarly, the space $L^{p,\infty}_B(\mu) = \text{weak-}L^p_B(\mu)$ is formed by all B-valued functions f such that $\|f\|_{L^{p,\infty}_B(\mu)} = \sup_{t>0} t\mu(\{\theta\in[0,\pi] : \|f(\theta)\|_B > t\})^{1/p} < \infty$.

We note here that the "heritage" Theorem 3.8 remains valid in the vector valued case with the same proof. More precisely, given B_1, B_2 Banach spaces, let T be a linear operator defined in the space of polynomial functions with B_1-valued coefficients and taking values in the space of B_2-valued and strongly measurable functions on $[0,\pi]$, satisfying conditions (a'), (b') and (c') which are, in fact, (a), (b) and (c) with appropriate changes, i.e., (a') T extends to a bounded operator either from $L^q_{B_1}(d\mu)$ into $L^q_{B_2}(d\mu)$ for some $1<q<\infty$, or from $L^1_{B_1}(d\mu)$ into $L^{1,\infty}_{B_2}(d\mu)$,

(b') there exists a $\mathcal{L}(B_1,B_2)$-valued measurable function K, defined in the complement of the diagonal in $[0,\pi]\times[0,\pi]$, such that for every function $f\in L^\infty_{B_1}$ and θ outside the support of f,

$$Tf(\theta)=\int_0^\pi K(\theta,\phi)f(\phi)d\phi.$$

(c') the function K satisfies $\|K(\theta,\phi)\|\le C|\theta-\phi|^{-1}$. for all $\theta\neq\phi$.

In this "vector valued language" we can obtain Lemmas 3.4 and 3.7, Theorem 3.8 and Lemma 3.9 in a similar way (just changing absolute values by norms). For further use, let us state the vector valued version of Theorem 3.8 as follows.

Theorem 3.10. *If T is an operator satisfying (a'), (b') and (c'). Define T_{loc} and T_{glob} as in 3.3 and 3.2. Then T_{loc} inherits from T either the L^q-boundedness or the weak type (1,1) as the case might be. Besides, the corresponding boundedness holds for both Lebesgue and m_λ measures.*

4. Estimates on the Riesz kernel

4.1. **The local part.** In this section we study the behavior of the kernel of the Riesz transform, $R_\lambda(\theta,\phi)$ related to the kernel of the classical conjugate function in the torus, $\mathcal{C}(\theta,\phi)=\frac{1}{2\pi}\frac{1}{\tan((\theta-\phi)/2)}$, when considered on the local part. We are going to prove that the difference

$$|R_\lambda(\theta,\phi)(\sin\phi)^{2\lambda}-C_0\mathcal{C}(\theta,\phi)|,$$

where C_0 is appropriately chosen real number, can be estimated in the local region by some "nice" function $M(\theta,\phi)$ such that the local operator $\int_0^\pi M(\theta,\phi)f(\phi)\,d\phi$ is bounded on every $L^p(d\phi)$, for $1\le p\le\infty$.

We will use the notation for a and D_r introduced in (2.10), as well as the following ones

$$\begin{aligned} b &= \partial_\theta a=-\sin\theta\cos\phi+\cos\theta\sin\phi\cos t=-\sin(\theta-\phi)-\cos\theta\sin\phi(1-\cos t),\\ \Delta_r &= 1-2r\cos(\theta-\phi)+r^2,\\ \sigma_r &= r\sin\theta\sin\phi. \end{aligned}$$

Also, we shall write D, Δ, σ instead of D_1, Δ_1, σ_1. As in [16] we are going to make use of the following simple identity, which is valid for every real $\alpha>-1$

$$\int_0^1 \frac{r^\alpha(1-r^2)}{D_r^{\alpha+2}}dr=\frac{1}{(\alpha+1)D^{\alpha+1}}. \tag{4.1}$$

By the arguments in the proof of Lemma 2.6, (4.1) and (2.3), we may simplify the kernel of the Riesz transform as follows

$$R_\lambda(\theta,\phi)=\frac{2\lambda}{\pi}\int_0^\pi \frac{b\sin^{2\lambda-1}t}{D^{\lambda+1}}dt. \tag{4.2}$$

Define

$$N(\theta,\phi)=1+\log^+\left(\frac{\sin\theta\sin\phi}{1-\cos(\theta-\phi)}\right). \tag{4.3}$$

Then, the estimate we get is the following

Lemma 4.4. *There exist constants C_0 and $C > 0$ such that in the local region we have the following estimate*

$$|R_\lambda(\theta,\phi)(\sin\phi)^{2\lambda} - C_0\mathcal{C}(\theta,\phi)| \le CM(\theta,\phi),$$

where $M(\theta,\phi) = N(\theta,\phi)(\sin\phi)^{-1}$.

Proof. Observe that $R_\lambda(\pi-\theta,\pi-\phi) = -R_\lambda(\theta,\phi)$, $\mathcal{C}(\pi-\theta,\pi-\phi) = -\mathcal{C}(\theta,\phi)$ and $M(\pi-\theta,\pi-\phi) = M(\theta,\phi)$. Therefore, if we get the comparison in the region $(\theta,\phi) \in [0,\pi/2]\times[0,\pi]$ we get the same inequality in the region $[\pi/2,\pi]\times[0,\pi]$. Thus, we will restrict ourselves to prove the inequality for $(\theta,\phi)\in[0,\pi/2]\times[0,\pi]$. In the local region for this range of (θ,ϕ), we get the following estimates, that will be used throughout the proof: there exists a constant C such that

$$\frac{1}{C}\sin\theta \le \sin\phi \le C\sin\theta, \qquad |\sin(\theta-\phi)| \le C\sin\phi.$$

The proof of these inequalities is trivial, although the argument differs when $\phi\in[0,\pi/2]$ than when $\phi\in[\pi/2,\pi]$. We will also handle integrals of the form

$$I^\alpha_\beta = \int_0^{\frac{\pi}{2}} \frac{t^\alpha}{(\Delta + \frac{\sigma t^2}{2})^\beta}dt.$$

By using the change of variables $t = \Delta^{\frac12}\sigma^{-\frac12}u$, and the former inequalities, for any $\lambda>0$, if $\alpha = 2\lambda+1$ and $\beta=\lambda+1$, the above integral can be estimated as follows

$$I^{2\lambda+1}_{\lambda+1} \le C(\sin\theta\sin\phi)^{-(\lambda+1)}N(\theta,\phi).$$

First we get rid of a part of the kernel (4.2), by observing that

$$\left|(\sin\phi)^{2\lambda}\frac{2\lambda}{\pi}\int_{\pi/2}^{\pi}\frac{b(\sin t)^{2\lambda-1}}{D^{\lambda+1}}\,dt\right| \le C(\sin\phi)^{2\lambda+1}\int_{\pi/2}^{\pi}\frac{(\sin t)^{2\lambda-1}}{\sigma^{\lambda+1}t^{2\lambda+2}}\,dt \le \frac{C}{\sin\phi}$$

Next, let us proceed with the comparison of the kernels in several steps.

Step 1. Define

$$R^1_\lambda(\theta,\phi) = \frac{2\lambda}{\pi}\int_0^{\pi/2}\frac{bt^{2\lambda-1}}{D^{\lambda+1}}dt.$$

Then using that $|\sin^{2\lambda-1}t - t^{2\lambda-1}| = O(t^{2\lambda+1})$, we obtain

$$|R_\lambda(\theta,\phi) - R^1_\lambda(\theta,\phi)|(\sin\phi)^{2\lambda} \le C(\sin\phi)^{2\lambda+1}I^{2\lambda+1}_{\lambda+1} \le CM(\theta,\phi).$$

Step 2. Define

$$R^2_\lambda(\theta,\phi) = \frac{2\lambda}{\pi}\int_0^{\pi/2}\frac{bt^{2\lambda-1}}{\overline{D}^{\lambda+1}}dt,$$

where

$$\overline{D} = 2(1-\cos(\theta-\phi) + \sin\theta\sin\phi\frac{t^2}{2}).$$

Then by the Taylor expansion, for $t\in[0,\pi]$.

$$\Big|\frac{1}{D^{\lambda+1}} - \frac{1}{\overline{D}^{\lambda+1}}\Big| \le C\frac{\sin\theta\sin\phi t^4}{\overline{D}^{\lambda+2}}.$$

Thus, we have

$$|R^1_\lambda(\theta,\phi) - R^2_\lambda(\theta,\phi)|(\sin\phi)^{2\lambda} \le c(\sin\phi)^{2\lambda+3}I^{2\lambda+3}_{\lambda+2} \le CM(\theta,\phi).$$

Step 3. Define

$$R^3_\lambda(\theta,\phi) = \frac{2\lambda}{\pi}\int_0^{\pi/2}\frac{-\sin(\theta-\phi)t^{2\lambda-1}}{\overline{D}^{\lambda+1}}dt.$$

Then

$$|R^2_\lambda(\theta,\phi) - R^3_\lambda(\theta,\phi)|(\sin\phi)^{2\lambda} \le C(\sin\phi)^{2\lambda+1} I^{2\lambda+1}_{\lambda+1} \le CM(\theta,\phi).$$

STEP 4. Compute:

$$\begin{aligned} R^3_\lambda(\theta,\phi)(\sin\phi)^{2\lambda} &= (\sin\phi)^{2\lambda}\frac{2\lambda}{\pi}\int_0^{\frac{\pi}{2}} \frac{-\sin(\theta-\phi)t^{2\lambda-1}}{\overline{D}^{\lambda+1}}dt \\ &= -\frac{2\lambda}{\pi}\sin(\theta-\phi)(\sin\phi)^{2\lambda}\Delta^{-1}\sigma^{-\lambda}\Big(C - \int_{\frac{\pi}{2}\sigma^{\frac12}\Delta^{-\frac12}}^\infty \frac{u^{2\lambda-1}}{(1+u^2)^{\lambda+1}}du\Big) \\ &= C_0\Big(\frac{\sin\phi}{\sin\theta}\Big)^\lambda \mathcal{C}(\theta,\phi) + I, \end{aligned}$$

where

$$|I| \le C\frac{\sin(\theta-\phi)}{\Delta}\int_{\frac{\pi}{2}\sigma^{\frac12}\Delta^{-\frac12}}^\infty \frac{1}{u^3}du \le C|\sin(\theta-\phi)|\sigma^{-1} \le \frac{C}{\sin\phi},$$

and finally notice

$$\Big|\Big(\frac{\sin\phi}{\sin\theta}\Big)^\lambda \mathcal{C}(\theta,\phi) - \mathcal{C}(\theta,\phi)\Big| \le \frac{C}{\sin\phi}.$$

□

Lemma 4.5. *The local operator defined for all $f \in L^1(d\phi)$ and $\theta \in [0,\pi]$ as*

$$Mf(\theta) = \int_0^\pi M(\theta,\phi)f(\phi)\,d\phi,$$

where $M(\theta,\phi)$ is defined in Lemma 4.4 is bounded on $L^p(d\phi)$ for $1 \le p \le \infty$.

Proof. To prove the lemma it is enough to check the following conditions, for $c = (1+\delta)^3$

$$\sup_{\theta\in[0,\pi]}\int_{\frac{\theta}{c}}^{c\theta} M(\theta,\phi)d\phi < \infty, \qquad \sup_{\phi\in[0,\pi]}\int_{\frac{\phi}{c}}^{c\phi} M(\theta,\phi)d\theta < \infty.$$

By the change of variables $x\theta = \phi$, we have that

$$\begin{aligned} \sup_{\theta\in[0,\pi]}\int_{\frac{\theta}{c}}^{c\theta}\frac{N(\theta,\phi)}{\sin\phi}d\phi &\le C\Big(1 + \sup_{\theta\in[0,\pi]}\int_{\frac{\theta}{c}}^{c\theta}\log^+\frac{\theta\phi}{(\theta-\phi)^2}\frac{d\phi}{\phi}\Big) \\ &\le C + C\int_{\frac{1}{c}}^{c}\log^+\frac{x}{(1-x)^2}\frac{dx}{x} < \infty. \end{aligned}$$

□

4.2. **The global part.** Using the symmetry of the Riesz kernel $R_\lambda(\theta,\phi) = -R_\lambda(\pi-\theta,\pi-\phi)$, we may restrict ourselves to consider the global part of the Riesz transform only for $\theta \in [0,\pi/2]$. Observe (see the picture in page 8) that for $\theta \in [0,\pi/2]$, we can give a bound above for $R_{\lambda,\mathrm{glob}}$ with three integrals as follows, where $c = (1+\delta)^3$:

$$\begin{aligned} |R_{\lambda,\mathrm{glob}}f(\theta)| &\le \int_0^\pi |R_\lambda(\theta,\phi)|(1 - 1_{N_{t_2}}(\theta,\phi))|f(\phi)|\,dm_\lambda(\phi) \\ &= \int_{\{\theta>c\phi\}} + \int_{\{\frac{\pi}{2}>\phi>c\theta\}} + \int_{[\pi/2,\pi]} = I + II + III. \end{aligned} \tag{4.6}$$

For the last integral, observe that there exists a $\varepsilon > 0$ such that in the intersection $[0,\pi/2]\times[\pi/2,\pi]$ with the global region, $|\theta-\phi|>\varepsilon$. Therefore, $|R_\lambda(\theta,\phi)|\le C_\varepsilon$ as can be easily seen from (4.2), and we get

$$III \le C_\varepsilon \int_{[\pi/2,\pi]} |f(\phi)|\, dm_\lambda(\phi) \le C_\delta \|f\|_{L^1(dm_\lambda)},$$

that is, this part is bounded in all $L^p(dm_\lambda)$, $1\le p\le\infty$. Let us now handle the integral I. If $\theta > c\phi$ and $\theta,\, \phi\in[0,\pi/2]$, then $\sin(\theta-\phi)\sim\sin\theta$ and $|b|\le\sin(\theta-\phi)$. We will use as well that for θ and ϕ in this region, $D=1-a=1-\cos(\theta-\phi)\ge C(\theta-\phi)^2\ge C\sin(\theta-\phi)^2$. Hence, in this part we obtain that

$$|R_\lambda(\theta,\phi)| \le \frac{C\sin(\theta-\phi)}{(\sin(\phi-\theta))^{2\lambda+2}} = \frac{C}{(\sin\theta)^{2\lambda+1}}.$$

We will state the boundedness of this operator in the following lemma.

Lemma 4.7. *The operator*

$$M^1 f(\theta) = \int_0^\pi M^1(\theta,\phi) f(\phi)\, dm_\lambda(\phi), \text{ where } M^1(\theta,\phi) = \frac{1}{(\sin\theta)^{2\lambda+1}} 1_{\{\theta>c\phi\}\cap[0,\pi/2]^2}(\theta,\phi)$$

for $c>1$, is of weak type (1,1) and strong type (p,p) for $1<p<\infty$ with respect to the measure dm_λ.

Proof. First we prove the weak type. Since the operator is linear, we can restrict ourselves to prove the weak type $(1,1)$ for positive functions with $\|f\|_{L^1(dm_\lambda)}=1$. In this case, $|M^1 f(\theta)| \le \frac{1}{(\sin\theta)^{2\lambda+1}}$. Take any $\mu\le 2$. Then,

$$m_\lambda(\{\theta :\, M^1 f(\theta) > \mu\}) \le m_\lambda([0,\pi]) \le \frac{C}{\mu}.$$

For $\mu>2$, observe that $\frac{1}{(\sin\theta)^{2\lambda+1}} > \mu$ if and only if $\theta < \arcsin\left((1/\mu)^{1/(2\lambda+1)}\right)$, since in that region $\sin\theta$ has a well defined increasing inverse. Also, in this region $(\theta\le\pi/2-\varepsilon)$, there exists C such that $0<C<\cos\theta$. Therefore,

$$\begin{aligned} m_\lambda(\{\theta :\, M^1 f(\theta) > \mu\}) &= m_\lambda(\{\theta :\, \theta < \arcsin\left((1/\mu)^{1/(2\lambda+1)}\right)\}) \\ &\le \frac{1}{C}\int_0^{\arcsin\left((1/\mu)^{1/(2\lambda+1)}\right)} \cos\theta(\sin\theta)^{2\lambda}\, d\theta = \frac{1}{C\mu} \end{aligned}$$

Now we have to prove that M^1 is bounded on $L^\infty(dm_\lambda)$, and then the result will follow from Marcinkiewicz interpolation theorem. This boundedness holds, since

$$\sup_{0\le\theta\le\pi/2} \int_0^{\frac{\pi}{2}} \frac{(\sin\phi)^{2\lambda}}{(\sin\theta)^{2\lambda+1}} 1_{\{\theta>c\phi\}}(\theta,\phi)\, d\phi < \infty.$$

□

To give a bound for II, observe that in the region where $\phi > c\theta$ and $\theta,\, \phi\in[0,\pi/2]$, we have that $b\le C\sin(\phi-\theta)$ and also $|\sin(\theta-\phi)|\sim\sin\phi$. Since here it also holds that $D\ge C\sin(\theta-\phi)^2$ we obtain

$$|R_\lambda(\theta,\phi)| \le \frac{C}{(\sin\phi)^{2\lambda+1}}.$$

Lemma 4.8. *The operator*

$$M^2 f(\theta) = \int_0^\pi M^2(\theta,\phi) f(\phi)\, dm_\lambda(\phi), \text{ where } M^2(\theta,\phi) = \frac{1}{(\sin\phi)^{2\lambda+1}} 1_{\{\phi > c\theta\}\cap[0,\pi/2]^2}(\theta,\phi)$$

for $c>1$ is of weak type $(1,1)$ and strong type (p,p) for $1<p<\infty$ with respect to the measure dm_λ.

Proof. To prove weak type (1,1) we argue as in the previous lemma, since in this region $\sin\phi \geq C\sin\theta$. The second part holds, because M^2 is the adjoint of M^1 in $L^2(dm_\lambda)$. □

5. Proofs of Theorems 2.13 and 2.14

5.1. **Proof of Theorem 2.13.** Let f be a polynomial function. Then by the spectral definition of R_λ (1.7) and Remark 2.12, we can write

$$R_\lambda f(\theta) = (\partial_\theta (L_\lambda)^{1/2}) f(\theta) = (\partial_\theta[(L_\lambda)^{1/2} f])(\theta) = \partial_\theta \int_0^\pi W_\lambda(\theta,\phi) f(\phi)\, d\phi$$

for every $\theta \in [0,\pi]$, since the integral exists for every $\theta\in[0,\pi]$ (by Lemma 2.6 and Remark 2.12).

Next step is adding and subtracting the function $W(\theta,\phi) = \frac{1}{\pi}\log|\sin((\theta-\phi)/2)|$, the kernel of the operator $(\partial_\theta^2)^{-\frac12}$ in the torus. Let $\mathcal{C}(\theta,\phi)$ be the kernel of the classical conjugate function in the torus. Since $W(\theta,\phi)$ is an integrable function in ϕ, we have that $\int_0^\pi W(\theta,\phi) f(\phi)\, dm_\lambda(\phi)$ is well defined for every θ, and we can write

$$\begin{aligned} R_\lambda f(\theta) &= \partial_\theta \int_0^\pi (W_\lambda(\theta,\phi)\sin^{2\lambda}\phi - C_0 W(\theta,\phi)) f(\phi)\, d\phi \\ &\quad + C_0 \partial_\theta \int_0^\pi W(\theta,\phi) f(\phi)\, d\phi. \end{aligned} \tag{5.1}$$

The second term is a principal value, since it is the conjugate function in the torus. To handle the first term, observe that for $\theta\neq\phi$, $\partial_\theta W_\lambda(\theta,\phi) = R_\lambda(\theta,\phi)$ (since for $\theta \neq \phi$, $|r^{\lambda-1}P_\theta(r,\theta,\phi)| \leq Cr^\lambda/(1-\cos(\theta-\phi)^2)^{\lambda+1}$ and this function is dr-integrable). Thus, for almost every $\phi\in[0,\pi]$, we have that

$$\partial_\theta (W(\theta,\phi)\sin^{2\lambda}\phi - C_0 W(\theta,\phi)) = R_\lambda(\theta,\phi)\sin^{2\lambda}\phi - C_0\mathcal{C}(\theta,\phi).$$

Now, by using Lemma 4.4 and the arguments of subsection 4.2, we have that

$$\begin{aligned} &|R_\lambda(\theta,\phi)\sin^{2\lambda}\phi - C_0\mathcal{C}(\theta,\phi)| \\ &\leq\; M(\theta,\phi)1_{N_{t_2}}(\theta,\phi) + M^1(\theta,\phi)(\sin\phi)^{2\lambda} + M^2(\theta,\phi)(\sin\phi)^{2\lambda} + C1_{N_{t_2}^c}(\theta,\phi), \end{aligned}$$

where the constant C comes from the boundedness for III in (4.6) and the boundedness of $\mathcal{C}(\theta,\phi)$ in the global region. By the proofs of Lemmas 4.5, 4.7 and 4.8, the right-hand function is $d\phi$-integrable. Thus, by Lebesgue's dominated convergence theorem, the first term in (5.1) is also the limit of the truncated integrals, and therefore we get that for almost every $\theta\in[0,\pi]$,

$$\begin{aligned} &R_\lambda f(\theta) = \\ &= \lim_{\varepsilon\to 0}\int_{|\theta-\phi|>\varepsilon} (R_\lambda(\theta,\phi)\sin^{2\lambda}\phi - C_0\mathcal{C}(\theta,\phi)) f(\phi)\, d\phi + C_0 \lim_{\varepsilon\to 0}\int_{|\theta-\phi|>\varepsilon} \mathcal{C}(\theta,\phi) f(\phi)\, d\phi \\ &= \lim_{\varepsilon\to 0}\int_{|\theta-\phi|>\varepsilon} R_\lambda(\theta,\phi) f(\phi)\, dm_\lambda(\phi). \end{aligned}$$

5.2. **Proof of Theorem 2.14.** Let us point out that it is enough to prove the boundedness in L^p or in weak L^1 for a dense subclass of functions in those spaces. In this section, we consider polynomial functions $f \in L^p$ for any $1 \leq p < \infty$, which form a dense subset of L^p and for which Theorem 2.13 holds. Next, we define the global and local parts of the Riesz transform R_λ according to (3.2) and (3.3). The boundedness of the global part was obtained in subsection 4.2.

Consider the local part $R_{\lambda,\mathrm{loc}}$. We write

$$R_{\lambda,\mathrm{loc}} = (R_{\lambda,\mathrm{loc}} - C_0\mathcal{C}_{\mathrm{loc}}) + C_0\mathcal{C}_{\mathrm{loc}}, \tag{5.2}$$

where $\mathcal{C}$ denotes the classical conjugate function in the torus. Let us observe that if an operator T is given by a principal value, then its local part is also a principal value. Thus, by Lemmas 4.4 and 4.5 it follows that $R_{\lambda,\mathrm{loc}} - C_0\mathcal{C}_{\mathrm{loc}}$ is bounded on $L^p(d\phi)$ for $1 \leq p \leq \infty$. Let us recall that by classical results and Theorem 3.8, $\mathcal{C}_{\mathrm{loc}}$ is also bounded in $L^p(d\phi)$, $1 < p < \infty$ and of weak type $(1,1)$. Thus, $R_{\lambda,\mathrm{loc}}$ is bounded in $L^p(d\phi)$ $1 < p < \infty$ and of weak type $(1,1)$, and by Lemma 3.9 it follows that the same statement holds for $L^p(dm_\lambda(\phi))$ for $1 < p < \infty$ and for the weak type (1,1) with respect to dm_λ.

6. Comparison of the results for Muckenhoupt-Stein's conjugate function and the Riesz transform

The aim of this section is giving an argument to derive the boundedness in $L^p(dm_\lambda)$ for $p \in (1,\infty)$ obtained by Muckenhoupt and Stein in [10] for the conjugate function defined there, as a corollary of the boundedness obtained in Theorem 2.14 for the Riesz transform in the same range of p's. First step is observing that Muckenhoupt-Stein's conjugate function is defined, for a polynomial function $f(\theta) = \sum_{n=0}^{N} a_n P_n^\lambda(\cos\theta)$, as $\tilde{f}(\theta) = \lim_{r\to 1} \tilde{f}(r,\theta)$. By using (1.5) and the expression for the Riesz transform of f (1.7), we get that $\tilde{f}(\theta) = R_\lambda(T_m f)(\theta)$, where T_m is the multiplier operator defined for polynomial functions as

$$T_m f(\theta) = \sum_{n=0}^{N} a_n m_n P_n^\lambda(\cos\theta), \quad m_0 = 0, \quad m_n = -\frac{n+\lambda}{n+2\lambda} \quad \text{for} \quad n \geq 1.$$

Therefore, if we obtain that the multiplier T_m is bounded in $L^p(dm_\lambda)$ for every $p \in (1,\infty)$, we would get trivially the boundedness of the conjugate function in L^p (since polynomial functions are dense in all $L^p(dm_\lambda)$). This multiplier falls into the scope of the multiplier theorem proved by Muckenhoupt and Stein in [10], Theorem 10 in page 71. But this theorem gives the boundedness of T_m only in a restricted range of p's, so we need another multiplier theorem. We will restrict ourselves to state the multiplier theorem just for T_m, since it is not our interest at this point to prove a general theorem.

Lemma 6.1. *The multiplier operator T_m is bounded in $L^p(dm_\lambda)$ for every $p \in (1,\infty)$.*

Proof. Define, for $p \in (1,\infty)$ the space

$$X_p = (I - \pi_0)L^p(dm_\lambda),$$

where π_0 is the continuous projection (orthogonal in $L^2(dm_\lambda)$) onto the subspace generated by $P_0^\lambda(\cos\theta)$, which is a constant (see [17]). X_p is a closed subspace of L^p, the subspace of functions of null mean, and in particular a Banach space itself, where polynomial functions of the form $f(\theta) = \sum_{k=1}^{K} a_k P_k^\lambda(\cos\theta)$ are dense. Now, observe that for any polynomial function $f = \sum_{k=0}^{K} a_k P_k^\lambda(\cos\theta)$, $T_m f(\theta) = T_m(f - a_0 P_0^\lambda(\cos\theta))$, and therefore it is enough to get the boundedness of T_m for the polynomial functions in X_p.

Let us consider first the case $p \geq 2$. To this end, let us observe that by spectral techniques, we have that

$$T_m f(\theta) = \sum_{n=1}^{N} a_n \Big(-\frac{1}{1+\frac{\lambda}{n+\lambda}} \Big) P_n^\lambda(\cos\theta) = -\frac{1}{1+\lambda L_\lambda^{-1/2}} f(\theta) = F(L_\lambda^{-1/2}) f(\theta),$$

where $F(z) = -\frac{1}{1+\lambda z}$. Now, observe that the spectrum of $L_\lambda^{-1/2}$ in X_p for $p > 2$ is

$$\sigma(L_\lambda^{-1/2}) = \Big\{ \frac{1}{n+\lambda} : \ n \geq 1 \Big\} \subsetneq \{ z : \ |z| < \frac{1}{\lambda} \},$$

and that the function F is holomorphic and admits a power series expansion $F(z) = \sum_{k=0}^\infty a_k z^k$ in the right-hand set. By using well known results on operator theory, see for example Theorem VII.3.10 in [3], we get

$$T_m = F(L_\lambda^{-1/2}) = \sum_{k=0}^{\infty} a_k (L_\lambda^{-1/2})^k. \tag{6.2}$$

Next step is giving a good bound for the norm of $L_\lambda^{-1/2}$ in X_p. This is given by the following lemma

Lemma 6.3. *For every p, $2 \leq p < \infty$, there exists $\varepsilon_p \in (0,1)$ such that for every $f \in X_p$,*

$$\|L_\lambda^{-1/2} f\|_{L^p(dm_\lambda)} \leq \frac{C}{\lambda + \varepsilon_p} \|f\|_{L^p(dm_\lambda)}.$$

Let us postpone the proof of this lemma, in order to clarify the reading of the proof of Lemma 6.1. By (6.2), we have that for any $f \in X_p$, $2 \leq p < \infty$,

$$\begin{aligned} \|T_m f\|_{L^p(dm_\lambda)} &\leq \sum_{k=0}^{\infty} |a_k| \, \|(L_\lambda^{-1/2})^k f\|_{L^p(dm_\lambda)} \\ &\leq \sum_{k=0}^{\infty} |a_k| \Big(\frac{C}{\lambda + \varepsilon_p} \Big)^k \|f\|_{L^p(dm_\lambda)} \leq C \|f\|_{L^p(dm_\lambda)}, \end{aligned}$$

since the series converges absolutely for $|z| < 1/\lambda$.

To prove the boundedness of T_m in X_p for $p \in (1,2)$, we use the standard duality argument. □

Proof of Lemma 6.3. First, observe that for any $f \in X_2$ polynomial function,

$$\begin{aligned} \|e^{-t\sqrt{L_\lambda}} f\|^2_{L^2(dm_\lambda)} &= \Big\| \sum_{k=1}^{K} a_k e^{-t(k+\lambda)} P_k^\lambda(\cos\theta) \Big\|^2_{L^2(dm_\lambda)} \\ &\leq e^{-2t(\lambda+1)} \sum_{k=1}^{K} |a_k|^2 = e^{-2t(\lambda+1)} \|f\|^2_{L^2(dm_\lambda)}. \end{aligned}$$

Next, for any $q > 2$ and any function in X_q, by (a) in Theorem 2.4, we have that $\|e^{-t\sqrt{L_\lambda}} f\|_{L^q(dm_\lambda)} \leq e^{-t\lambda} \|f\|_{L^q(dm_\lambda)}$. Take any p such that $2 < p < q$, $1/p = \varepsilon_p/2 + (1-\varepsilon_p)/q$, then by the interpolation theorem, we have that

$$\big\|e^{-t\sqrt{L_\lambda}}\big\|_{X_p \to L^p(dm_\lambda)} \leq C \big\|e^{-t\sqrt{L_\lambda}}\big\|^{\varepsilon_p}_{X_2 \to L^2(dm_\lambda)} \big\|e^{-t\sqrt{L_\lambda}}\big\|^{(1-\varepsilon_p)}_{X_q \to L^q(dm_\lambda)} \leq C e^{-t(\lambda+\varepsilon_p)}.$$

Finally, we get that

$$\|L_\lambda^{-1/2}\|_{X_p\to L^p(dm_\lambda)} \le \int_0^\infty \left\|e^{-t\sqrt{L_\lambda}}\right\|_{X_p\to L^p(dm_\lambda)} dt \le C\int_0^\infty e^{-t(\lambda+\varepsilon_p)}\,dt = \frac{C}{\lambda+\varepsilon_p}.$$

□

7. The maximal operator R_λ^*.

Define the following maximal operators

$$\begin{aligned} R_\lambda^* f(\theta) &= \sup_\varepsilon |R_{\lambda,\varepsilon} f(\theta)|, \quad \text{where} \quad R_{\lambda,\varepsilon} f(\theta) = \int_{|\theta-\phi|>\varepsilon} R_\lambda(\theta,\phi) f(\phi)\,d\phi,\\ \mathcal{C}^* f(\theta) &= \sup_\varepsilon |\mathcal{C}_\varepsilon f(\theta)|, \quad \text{where} \quad \mathcal{C}_\varepsilon f(\theta) = \int_{|\theta-\phi|>\varepsilon} \mathcal{C}(\theta,\phi) f(\phi)\,d\phi. \end{aligned}$$

The main result of this section is the boundedness of the first one. The boundedness of the second one is a well known fact, see [18].

Theorem 7.1. *R_λ^* is bounded on $L^p(dm_\lambda)$ for $1<p<\infty$, and of weak type (1,1) with respect to dm_λ.*

Proof. The first step is splitting R_λ^* as follows,

$$\begin{aligned} R_\lambda^* f(\theta) = \sup_\varepsilon |R_{\lambda,\varepsilon} f(\theta)| &\le \sup_\varepsilon \int_{|\theta-\phi|>\varepsilon} |R_{\lambda,\mathrm{loc}}(\theta,\phi)(\sin\phi)^{2\lambda} - C_0\mathcal{C}_{\mathrm{loc}}(\theta,\phi)||f(\phi)|d\phi\\ &+C_0 \sup_\varepsilon \Big|\int_{|\theta-\phi|>\varepsilon} \mathcal{C}_{\mathrm{loc}}(\theta,\phi) f(\phi) d\phi\Big| + \sup_\varepsilon \int_{|\theta-\phi|>\varepsilon} |R_{\lambda,\mathrm{glob}}(\theta,\phi)||f(\phi)|dm_\lambda(\phi). \end{aligned}$$

The first term is bounded above by the operator $Mf(\theta)$, as follows from Lemma 4.4. Then, Lemma 4.5 gives the boundedness of this term. The third factor can be estimated by the same arguments as in the proof of Theorem 2.14, see subsection 4.2, namely, getting a parallel inequality to (4.6). Then, Lemmas 4.7 and 4.8 give the boundedness of this term. The local part part of the classical maximal operator $\mathcal{C}^*_{\mathrm{loc}}$ is bounded as a consequence of Theorem 3.10 and the boundedness of $\mathcal{C}_{\mathrm{loc}}$. □

As a corollary of the former result, by classical arguments we can extend the definition of the Riesz transform as a principal value to all functions in $L^1(dm_\lambda)$.

Theorem 7.2. *For any function $f\in L^1(dm_\lambda)$ and almost every $\theta\in[0,\pi]$*

$$R_\lambda f(\theta) = \lim_{\varepsilon\to 0}\int_{|\theta-\phi|>\varepsilon} R_\lambda(\theta,\phi) f(\phi) dm_\lambda(\phi).$$

8. Oscillations

By the results in the previous sections, we know that our Riesz transform is the limit of the truncated operators $R_{\lambda,\varepsilon}$ almost everywhere. In general, for every family $\mathcal{T}=\{T_\varepsilon\}$ such that there exists $Tf=\lim_{\varepsilon\to0} T_\varepsilon f$, it is classical to measure the speed of this convergence by means of expressions involving differences of the type $|T_\varepsilon f - T_{\varepsilon'} f|$. Some of the operators used for this purpose are the so called oscillation operator, defined as

$$\mathcal{O}(\mathcal{T})f(x) = \left(\sum_{i=1}^\infty \sup_{t_{i+1}\le\varepsilon_{i+1}<\varepsilon_i<t_i} |T_{\varepsilon_{i+1}} f(x) - T_{\varepsilon_i} f(x)|^2\right)^{1/2},$$

for a fixed sequence $\{t_i\}$ decreasing to zero. For ρ in the range $2<\rho<\infty$, we have the so called ρ-variation operator,

$$\mathcal{V}_\rho(\mathcal{T})f(x)=\sup_{\{\varepsilon_i\}}\left(\sum_{i=1}^{\infty}|T_{\varepsilon_{i+1}}f(x)-T_{\varepsilon_i}f(x)|^\rho\right)^{1/\rho},$$

where the sup is taken over all sequences $\{\varepsilon_i\}$ decreasing to zero. Classical results are that when these operators are defined for the family of the truncated operators of the Hilbert transform or for the family of the Poisson integrals in $\mathbb{R}$, they are bounded in $L^p(\mathbb{R})$, $1<p<\infty$ and of weak type $(1,1)$ with respect to the Lebesgue measure (see [2] and [7] and the references therein). Our aim is proving similar results for the oscillation and ρ-variation of the family of the truncated operators of the Riesz transform. Let us call $\mathcal{R}_\lambda=\{R_{\lambda,\varepsilon}\}$ and $\overline{\mathcal{C}}=\{\mathcal{C}_\varepsilon\}$ to the families of the truncated operators associated to the Riesz transform and the classical conjugate function, respectively. First of all, let us point out that with the vector valued techniques shown in [7], one can get easily the boundedness of the oscillation and ρ-variation associated to the classical conjugate function in the torus from the known results for the Hilbert transform in the line. Let us denote by $\mathcal{H}=\{H_t\}$ the family of the truncated operators for the Hilbert transform in $\mathbb{R}$, that is,

$$H_tf(x)=\int_{|x-y|>t}\frac{f(y)}{x-y}\,dy.$$

Before proceeding further, let us observe that for any family $\mathcal{T}=\{T_\varepsilon\}$, $\mathcal{O}(\mathcal{T})f(x)\sim\mathcal{O}'(\mathcal{T})f(x)$ for almost every x, where

$$\mathcal{O}'(\mathcal{T})f(x)=\left(\sum_{i=1}^{\infty}\sup_{t_{i+1}<s\le t_i}|T_{t_{i+1}}f(x)-T_sf(x)|^2\right)^{1/2}.$$

Thus, it is enough to prove the desired boundedness properties for this operator instead of $\mathcal{O}$. Denote by $B=\ell^2_{L^\infty(0,\infty)}$ the space of sequences $u=\{u_i\}$ such that for every i, $u_i=u_i(s)$ is a function in $L^\infty(0,\infty)$, with $\|u\|_B=\|\{\|u_i\|_{L^\infty}\}\|_{\ell^2}$. Then, if we put $\mathcal{U}(\mathcal{T})f(x)=\{(T_{t_{i+1}}f(x)-T_sf(x))1_{(t_{i+1},t_i]}(s)\}$, we have that

$$\mathcal{O}'(\mathcal{T})f(x)=\|\{(T_{t_{i+1}}f(x)-T_sf(x))1_{(t_{i+1},t_i]}(s)\}\|_B=\|\mathcal{U}(\mathcal{T})f\|_B.$$

Also, let us call $\Theta=\big\{\varepsilon=\{\varepsilon_i\}:\ \varepsilon_i\downarrow 0\big\}$ the set of sequences decreasing to zero, and $B_\rho=B_{\ell^\rho}(\Theta)$, with ρ in the range $2<\rho<\infty$, the space of bounded functions $v=v(\varepsilon)$ defined on Θ, whose values are sequences in ℓ^ρ, $v(\varepsilon)=\{v_i(\varepsilon)\}$, and such that $\|v\|_{B_\rho}=\sup_{\varepsilon\in\Theta}\|\{v_i(\varepsilon)\}\|_{\ell^\rho}$. Then, if we put $\mathcal{V}(\mathcal{T})f(x)=\{T_{\varepsilon_{i+1}}f(x)-T_{\varepsilon_i}f(x)\}$, we have that

$$\mathcal{V}_\rho(\mathcal{T})f(x)=\|\{T_{\varepsilon_{i+1}}f(x)-T_{\varepsilon_i}f(x)\}\|_{B_\rho}=\|\mathcal{V}(\mathcal{T})f\|_{B_\rho}.$$

In the case that each T_t has an associated kernel K_t, then $\mathcal{U}(\mathcal{T})$ and $\mathcal{V}(T)$ have also an associated kernel:

$$\begin{aligned}\mathcal{U}(\mathcal{T})f(x) &= \{(T_{t_{i+1}}f(x)-T_sf(x))1_{(t_{i+1},t_i]}(s)\}\\ &= \int\big\{(K_{t_{i+1}}(x,y)-K_s(x,y))1_{(t_{i+1},t_i]}(s)\big\}f(y)\,d\nu(y),\end{aligned}$$

and

$$\mathcal{V}(\mathcal{T})f(x) = \int\big\{K_{\varepsilon_{i+1}}(x,y)-K_{\varepsilon_i}(x,y)\big\}f(y)\,d\nu(y).$$

With all these preliminaries, it is clear that the boundedness of the oscillation and ρ-variation operators associated to certain family, namely, $\overline{\mathcal{C}}$, is equivalent to the boundedness of $\mathcal{U}(\overline{\mathcal{C}})$ and $\mathcal{V}(\overline{\mathcal{C}})$. We can prove this boundedness by comparison with $\mathcal{U}(\mathcal{H})$ and $\mathcal{V}(\mathcal{H})$.

Proposition 8.1. *Let us consider $f : [0,\pi] \to \mathbb{R}$ and extend it as 0 to the whole $\mathbb{R}$. Then, there exists a positive constant C such that for every $\theta \in [0,\pi]$ and $p \in [1,\infty]$ we have that*

$$\|\mathcal{U}(\overline{\mathcal{C}})f(\theta) - \frac{1}{\pi}1_{[0,\pi]}(\theta)\mathcal{U}(\mathcal{H})f(\theta)\|_B \le C\|f\|_{L^p(d\phi)},$$

$$\|\mathcal{V}(\overline{\mathcal{C}})f(\theta) - \frac{1}{\pi}1_{[0,\pi]}(\theta)\mathcal{V}(\mathcal{H})f(\theta)\|_{B_\rho} \le C\|f\|_{L^p(d\phi)}.$$

Proof. We will prove the result only for $\mathcal{U}$, since the proof for $\mathcal{V}$ is completely analogous. In this case, we have that

$$\begin{aligned}
&\mathcal{U}(\overline{\mathcal{C}})f(\theta) - \frac{1}{\pi}1_{[0,\pi]}(\theta)\mathcal{U}(\mathcal{H})f(\theta)\\
&= \big\{(\overline{R}_{t_{i+1}}f(\theta) - \overline{R}_s f(\theta))1_{(t_{i+1},t_i]}(s)\big\} - \frac{1}{\pi}\big\{(H_{t_{i+1}}f(\theta) - H_s f(\theta))1_{(t_{i+1},t_i]}(s)\big\}\\
&= \int_0^\pi \frac{1}{2\pi}\bigg\{\bigg[\frac{1}{\tan((\theta-\phi)/2)} - \frac{1}{(\theta-\phi)/2}\bigg]\\
&\qquad\qquad \times 1_{\{t_{i+1}<|\theta-\phi|\le s\}}(|\theta-\phi|)1_{(t_{i+1},t_i]}(s)\bigg\} f(\phi)\,d\phi.
\end{aligned}$$

Notice that

$$\begin{aligned}
&\bigg\|\bigg\{\bigg[\frac{1}{\tan((\theta-\phi)/2)} - \frac{1}{(\theta-\phi)/2}\bigg]1_{\{t_{i+1}<|\theta-\phi|\le s\}}(|\theta-\phi|)1_{(t_{i+1},t_i]}(s)\bigg\}\bigg\|_B\\
&\qquad\qquad = \bigg|\frac{1}{\tan((\theta-\phi)/2)} - \frac{1}{(\theta-\phi)/2}\bigg|,
\end{aligned}$$

therefore by Minkowski's inequality, we have

$$\|\mathcal{U}(\overline{\mathcal{C}})f(\theta) - \frac{1}{\pi}1_{[0,\pi]}(\theta)\mathcal{U}(\mathcal{H})f(\theta)\|_B \le C\int_0^\pi \bigg|\frac{1}{\tan((\theta-\phi)/2)} - \frac{1}{(\theta-\phi)/2}\bigg||f(\phi)|\,d\phi.$$

And now, for $\theta,\phi \in [0,\pi]$, $\theta-\phi \in [-\pi,\pi]$ but both functions are odd. Therefore, we only have to give a bound for $|1/\tan(x/2) - 1/(x/2)|$ for $x \in [0,\pi]$, or equivalently, for $|1/\tan x - 1/x|$ for $x \in [0,\pi/2]$. In this range, clearly $|1/\tan x - 1/x| \le C$, and we get the result. □

The following theorem states the main results of this section.

Theorem 8.2. *For every $p \in (1,\infty)$ there exists a constant C such that*

$$\|\mathcal{O}(\mathcal{R}_\lambda)f\|_{L^p(dm_\lambda)} \le C\|f\|_{L^p(dm_\lambda)}, \qquad \|\mathcal{V}_\rho(\mathcal{R}_\lambda)f\|_{L^p(dm_\lambda)} \le C\|f\|_{L^p(dm_\lambda)}$$

and also

$$m_\lambda\big(\{\theta :\ \mathcal{O}(\mathcal{R}_\lambda)f(\theta) > \mu\}\big) \le \frac{C}{\mu}\|f\|_{L^1(dm_\lambda)}$$

$$m_\lambda\big(\{\theta :\ \mathcal{V}_\rho(\mathcal{R}_\lambda)f(\theta) > \mu\}\big) \le \frac{C}{\mu}\|f\|_{L^1(dm_\lambda)}$$

Proof. The proof of this theorem can be developed by using the same techniques applied in the proofs of Theorems 2.13 and 2.14, i.e., splitting the operators into their local and global parts. To this end, define for a family $\mathcal{T} = \{T_t\}$ as above, acting on functions over $[0, \pi]$, its local and global parts as $\mathcal{T}_{\text{loc}} = \{T_{t,\text{loc}}\}$ and $\mathcal{T}_{\text{glob}} = \{T_{t,\text{glob}}\}$.

Lemma 8.3. *Given a family of operators $\mathcal{T} = \{T_t\}$ on functions defined on $[0, \pi]$ such that each operator T_t is given by a kernel K_t, then we have*

$$\mathcal{U}(\mathcal{T})_{loc} = \mathcal{U}(\mathcal{T}_{loc}), \quad \mathcal{U}(\mathcal{T})_{glob} = \mathcal{U}(\mathcal{T}_{glob}) \quad \mathcal{V}(\mathcal{T})_{loc} = \mathcal{V}(\mathcal{T}_{loc}), \quad \mathcal{V}(\mathcal{T})_{glob} = \mathcal{V}(\mathcal{T}_{glob}).$$

Proof. Let us see the proof for the global part, since for the local part it follows from its definition and the linearity in $\mathcal{T}$ of the operators $\mathcal{U}$ and $\mathcal{V}$:

$$\mathcal{U}(\mathcal{T})_{\text{loc}} = \mathcal{U}(\mathcal{T}) - \mathcal{U}(\mathcal{T})_{\text{glob}} = \mathcal{U}(\mathcal{T}) - \mathcal{U}(\mathcal{T}_{\text{glob}}) = \mathcal{U}(\mathcal{T} - \mathcal{T}_{\text{glob}}) = \mathcal{U}(\mathcal{T}_{\text{loc}}).$$

By definition of the global part of an operator, we have

$$\begin{aligned}
\mathcal{U}(\mathcal{T})_{\text{glob}} f(\theta) &= \int_0^\pi \big\{(L_{t_{i+1}}(\theta,\phi) - L_s(\theta,\phi))1_{(t_{i+1},t_i]}(s)\big\}(1 - 1_{N_{t_2}}(\theta,\phi))f(\phi)\, d\phi, \\
&= \{(T_{t_{i+1},\text{glob}} f(\theta) - T_{s,\text{glob}} f(\theta))1_{(t_{i+1},t_i]}(s)\} = \mathcal{U}(\mathcal{T}_{\text{glob}})f(\theta).
\end{aligned}$$

A similar proof holds for $\mathcal{V}$. □

Now, we will prove that the local part of $\mathcal{U}(\mathcal{R}_\lambda)$ and $\mathcal{V}(\mathcal{R}_\lambda)$ differ from the local part of $\mathcal{U}(\overline{\mathcal{C}})$ and $\mathcal{V}(\overline{\mathcal{C}})$, respectively, in an operator which is bounded in $L^p(dm_\lambda)$ for $1 < p < \infty$ and of weak type $(1, 1)$.

Lemma 8.4. *There exists a positive constant C such that for every $\theta \in [0, \pi]$ we have that*

$$\|\mathcal{U}(\mathcal{R}_\lambda)_{loc} f(\theta) - C_0 \mathcal{U}(\overline{\mathcal{C}})_{loc} f(\theta)\|_B \le C \int_0^\pi M(\theta,\phi) 1_{N_{t_2}}(\theta,\phi)|f(\phi)|\, d\phi,$$

$$\|\mathcal{V}(\mathcal{R}_\lambda)_{loc} f(\theta) - C_0 \mathcal{V}(\overline{\mathcal{C}})_{loc} f(\theta)\|_{B_\rho} \le C \int_0^\pi M(\theta,\phi) 1_{N_{t_2}}(\theta,\phi)|f(\phi)|\, d\phi$$

where C_0 and $M(\theta, \phi)$ are as in Lemma 4.4.

Proof. Since for every truncated operator we are away from the diagonal, observe that we can write

$$\begin{aligned}
&\mathcal{U}(\mathcal{R}_\lambda)_{\text{loc}} f(\theta) - C_0 \mathcal{U}(\overline{\mathcal{C}})_{\text{loc}} f(\theta) = \mathcal{U}(\mathcal{R}_{\lambda,\text{loc}})f(\theta) - C_0 \mathcal{U}(\overline{\mathcal{R}}_{\text{loc}})f(\theta) \\
&\quad = \big\{(R_{\lambda,t_{i+1},\text{loc}} f(\theta) - R_{\lambda,s,\text{loc}} f(\theta))1_{(t_{i+1},t_i]}(s)\big\} \\
&\qquad -C_0\big\{(\mathcal{C}_{t_{i+1},\text{loc}} f(\theta) - \mathcal{C}_{s,\text{loc}} f(\theta))1_{(t_{i+1},t_i]}(s)\big\} \\
&\quad = \int_0^\pi \big\{[R_\lambda(\theta,\phi)(\sin\phi)^{2\lambda} - C_0\mathcal{C}(\theta,\phi)] \\
&\qquad\qquad \times 1_{\{s<|\theta-\phi|\le t_{i+1}\}}(|\theta-\phi|)1_{(t_{i+1},t_i]}(s)\big\}1_{N_{t_2}}(\theta,\phi)f(\phi)\, d\phi.
\end{aligned}$$

Since

$$\begin{aligned}
&\big\|\big\{[R_\lambda(\theta,\phi)(\sin\phi)^{2\lambda} - C_0\mathcal{C}(\theta,\phi)]1_{\{s<|\theta-\phi|\le t_{i+1}\}}(|\theta-\phi|)1_{(t_{i+1},t_i]}(s)\big\}\big\|_B \\
&\qquad\qquad = |R_\lambda(\theta,\phi)(\sin\phi)^{2\lambda} - C_0\mathcal{C}(\theta,\phi)|,
\end{aligned}$$

by Minkowski's inequality and Lemma 4.4, we have that

$$\|\mathcal{U}(\mathcal{R}_\lambda)_{\text{loc}} f(\theta) - C_0 \mathcal{U}(\overline{\mathcal{C}})_{\text{loc}} f(\theta)\|_B \le \int_0^\pi M(\theta,\phi) 1_{N_{t_2}}(\theta,\phi)|f(\phi)|\, d\phi.$$

The proof for the ρ variation is completely analogous. □

By Proposition 8.1, we know the boundedness of $\mathcal{U}(\overline{\mathcal{C}})$ and $\mathcal{V}(\overline{\mathcal{C}})$ in $L^p(d\phi)$ for $1 < p < \infty$ and weak type $(1,1)$ with respect to $d\phi$. Now, one can see that $\mathcal{U}(\overline{\mathcal{C}})$ and $\mathcal{V}(\overline{\mathcal{C}})$ are vector valued operators as defined in section 3. Thus, by the vector valued version of the heritage theorem, Theorem 3.10, the local part of both of them inherits the boundedness properties mentioned above with respect to $d\phi$. By Lemmas 8.4 and 4.5, this gives the boundedness of the local part of $\mathcal{U}(\mathcal{R}_\lambda)$ and $\mathcal{V}(\mathcal{R}_\lambda)$ with respect to $d\phi$. Hence, by the vector valued version of Lemma 3.9, we obtain the boundedness of the local part of $\mathcal{U}(\mathcal{R}_\lambda)$ and $\mathcal{V}(\mathcal{R}_\lambda)$ also for dm_λ. In order to prove Theorem 8.2, it is only left to prove that the global parts are also bounded in $L^p(dm_\lambda)$. This can be obtained by using similar arguments as in Lemma 8.4 to handle the vector valued kernels, and then proceeding as in the proof of Theorem 2.14 for the global part, concretely obtaining the parallel inequality to 4.6. □

9. Weighted inequalities

Let us call *weight* to any strictly positive measurable function v. In this section we will answer the following question: find a necessary and sufficient condition on a given weight v, for the existence of a weight u such that the Riesz transform is bounded from $L^p(vdm_\lambda)$ into $L^p(udm_\lambda)$ for $1 < p < \infty$. We will use a well known technique based in ideas by Rubio de Francia. In particular, we shall need the following theorem due to this author (see [12]).

Theorem 9.1. *Let (X,μ) be a measure space, B a Banach space and T a sublinear operator from B into $L^s(X)$ such that the following inequality is satisfied for some $s < p$ and every sequence $\{f_j\} \subset B$*

$$\left\| \left(\sum_j |Tf_j|^p \right)^{1/p} \right\|_{L^s(X)} \le C_{p,s} \left(\sum_j \|f_j\|_B^p \right)^{1/p},$$

where $C_{p,s}$ is a constant depending on p and s. Then there exists a positive function u such that $u^{-1} \in L^{\frac{s}{p-s}}(X)$ and

$$\left(\int_X |Tf(\theta)|^p u(\theta) d\mu(\theta) \right)^{1/p} \le \|f\|_B.$$

Consider the extension of the Riesz transform R_λ to the operator defined for integrable functions $f = \{f_j\}_j$ taking values in ℓ^p, given by $R_\lambda f = \{R_\lambda f_j\}_j$. For this extended operator, we have the following lemma.

Lemma 9.2. *Let p be in the range $1 < p < \infty$, then*

$$R_\lambda : L^1_{l^p}(m_\lambda) \to L^{1,\infty}_{l^p}(m_\lambda)$$

Proof. It is known that the ℓ^p-valued extension of the classical conjugate function in the torus, $\mathcal{C}f = \{\mathcal{C}f_j\}_j$, maps $L^1_{l^p}(d\phi)$ into $L^{1,\infty}_{l^p}(d\phi)$ (see [1]). By the vector-valued version of the heritage theorem, Theorem 3.10, the local part of this extension of $\mathcal{C}$ is also of weak type $(1,1)$. Therefore by using identity (5.2), Lemma 4.4 and Lemma 4.5 we get that $R_{\lambda,loc}$ maps $L^1_{l^p}(d\phi)$ into $L^{1,\infty}_{l^p}(d\phi)$. We have used here that the operator in Lemma 4.5 is given by a positive kernel and therefore it can be extended to ℓ^p-valued functions for $1 \le p \le \infty$. An straightforward ℓ^p-valued extension of Lemma 3.9 gives the boundedness of $R_{\lambda,loc}$ from $L^1_{l^p}(m_\lambda)$ into $L^{1,\infty}_{l^p}(m_\lambda)$.

As for the global part we observe that since it is majorized by a positive operator (see section 4.2) then its vector-valued extension it is also bounded. □

Then, the condition on the weight v is given by the following result.

Theorem 9.3. *Let p be in the range $1 < p < \infty$. A weight v satisfies*

$$\int_0^\pi v(\theta)^{-\frac{1}{p-1}} dm_\lambda(\theta) < \infty$$

if and only if there exists a positive (measurable) function u such that

$$\int_0^\pi |R_\lambda f(\theta)|^p u(\theta) dm_\lambda(\theta) \le C \int_0^\pi |f(\theta)|^p v(\theta) dm_\lambda(\theta)$$

To get the part "only if" in Theorem 9.3, we follow the proof of Corollary 1.6 in [8]. By Lemma 9.2 and Kolmogorov's inequality we have that

$$\begin{aligned}\left\| \left(\sum_j |R_\lambda f_j|^p \right)^{1/p} \right\|_{L^s(dm_\lambda)} &\le C_s \sup_{t>0} t m_\lambda \left\{ \theta : \left(\sum_j |R_\lambda f_j(\theta)|^p \right)^{1/p} > t \right\} \\ &\le C_s \int_0^\pi \left(\sum_j |f_j(\theta)|^p \right)^{1/p} dm_\lambda(\theta) \le C_s \left(\sum_j \|f_j\|^p_{L^p(v dm_\lambda)} \right)^{1/p},\end{aligned}$$

where in the last inequality we have applied Holder inequality after multiplying inside the integral by $v^{1/p} v^{-1/p}$, and then used the condition on the weight. Therefore we are in the hypothesis of Theorem 9.1 with $B = L^p(v dm_\lambda)$ and the "only if" part follows.

The part "if" in Theorem 9.3 is a corollary of the following two lemmas, which conclude the proof. Define $\alpha = u^{1-p'}$, $\beta = v^{1-p'}$. Then, by duality, we obtain the following result.

Lemma 9.4. *If the Riesz transform $R_\lambda : L^p(v dm_\lambda) \to L^p(u dm_\lambda)$ then its adjoint operator $R'_\lambda : L^{p'}(\alpha dm_\lambda) \to L^{p'}(\beta dm_\lambda)$, where $\frac{1}{p} + \frac{1}{p'} = 1$.*

The final step in the proof of Theorem 9.3 is the following lemma.

Lemma 9.5. *Suppose $R'_\lambda : L^{p'}(\alpha dm_\lambda) \to L^{p'}(\beta dm_\lambda)$, then*

$$\int_0^\pi \beta(\theta) dm_\lambda(\theta) < \infty.$$

Proof. Observe that by using expression (4.2) and integration by parts, we have that

$$R'_\lambda\left(\frac{\pi}{2}, \frac{\pi}{4}\right) = R_\lambda\left(\frac{\pi}{4}, \frac{\pi}{2}\right) = \frac{\sqrt{2}\lambda}{\pi} \int_0^\pi \frac{\cos t (\sin t)^{2\lambda-1}}{(1 - \frac{\sqrt{2}}{2}\cos t)^{\lambda+1}} dt = C > 0.$$

Then, by continuity, there exists $\varepsilon > 0$ such that for $(\theta, \phi) \in [\pi/2 - \varepsilon, \pi/2 + \varepsilon] \times [\pi/4 - \varepsilon, \pi/4 + \varepsilon]$, $R'_\lambda(\theta, \phi) \ge C$. Now, choose a set A such that $\overline{A} \subset (\pi/4 - \varepsilon, \pi/4 + \varepsilon)$, $\int_A \alpha(\theta) dm_\lambda(\theta)$ is finite and $m_\lambda(A) > 0$. For $\theta \in [\pi/2 - \varepsilon, \pi/2 + \varepsilon]$, we then have

$$R'_\lambda 1_A(\theta) = \int_A R'_\lambda(\theta, \phi) dm_\lambda(\phi) = \int_A R_\lambda(\phi, \theta) dm_\lambda(\phi) \ge C \int_A dm_\lambda(\phi) = C.$$

Then,

$$\begin{aligned}\int_{\pi/2-\varepsilon}^{\pi/2+\varepsilon} \beta(\theta) dm_\lambda(\theta) &\le C \int_{\pi/2-\varepsilon}^{\pi/2+\varepsilon} (R'_\lambda 1_A(\theta))^{p'} \beta(\theta) dm_\lambda(\theta) \\ &\le C \int (1_A(\theta))^{p'} \alpha(\theta) dm_\lambda(\theta) = C \int_A \alpha(\theta) dm_\lambda(\theta) < \infty. \qquad (9.6)\end{aligned}$$

For $\theta \in [0, \pi/2 - \varepsilon]$, choose a set A such that $\overline{A} \subset (\pi/2 - \varepsilon/2, \pi/2)$, $\int_A \alpha(\theta) dm_\lambda(\theta)$ is finite and $m_\lambda(A) > 0$. For these range of θ and ϕ, we have that $R'_\lambda(\theta, \phi) \leq -C < 0$, and therefore $R'_\lambda(-1_A)(\theta) \geq C > 0$. Then, proceeding as in (9.6), we get that $\int_0^{\pi/2-\varepsilon} \beta(\theta)\, dm_\lambda(\theta) < \infty$.

Finally, repeating the above argument, we prove $\int_{\pi/2+\varepsilon}^{\pi} \beta(\theta)\, dm_\lambda(\theta) < \infty$. □

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