Strong exponential closure for \((\mathbb{C}, e^x)\)

Giuseppina Terzo

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*joint work with Paola D’Aquino and Antongiulio Fornasiero
**Definition:** An exponential ring, or $E$-ring, is a pair $(R, E)$ where $R$ is a ring (commutative with 1) and

$$E : (R, +) \rightarrow (\mathcal{U}(R), \cdot)$$

a morphism of the additive group of $R$ into the multiplicative group of units of $R$ satisfying

- $E(x + y) = E(x) \cdot E(y)$ for all $x, y \in R$
- $E(0) = 1$.

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The model theoretic analysis of the exponential function over a field started with a problem left open by Tarski in the 30’s, about the decidability of the reals with exponentiation. Only in the mid 90’s Macintyre and Wilkie gave a positive answer to this question assuming Schanuel’s Conjecture. Complex exponentiation involves much deeper issues, and it is much harder to approach, as it inherits the Godel incompleteness and undecidability phenomena via the definition of the set of periods. Despite this negative results there are still many interesting and natural model-theoretic aspects to analyze.
Comparing $\mathbb{R}$, $exp$ and $\mathbb{C}$, $exp$

$Th(\mathbb{R}, exp)$ decidable modulo (SC)
(Macintyre-Wilkie '96)

$Th(\mathbb{R}, exp)$ model-complete
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As regards definability the above ideas show that definability in the complex exponential field is as complicated as definability in the ring $\mathbb{Z}$. For this reason, work stopped early on the logic of complex exponentiation, and was only taken up again after a wonderful discovery of Zilber early this century.
Zilber’s programme: Looks for a canonical algebraically closed field of characteristic 0 with exponentiation.

$(K, E)$ is a Zilber field if:

- $K$ is an algebraically closed field of characteristic 0;
- $E : (K, +) \to (K^\times, \cdot)$ is a surjective homomorphism and there is $\omega \in K$ transcendental over $\mathbb{Q}$ such that $\ker E = \mathbb{Z}\omega$;
- **Schanuel’s Conjecture (SC)** Let $\lambda_1, \ldots, \lambda_n \in K$ be linearly independent over $\mathbb{Q}$. Then $\mathbb{Q}(\lambda_1, \ldots, \lambda_n, E(\lambda_1), \ldots, E(\lambda_n))$ has transcendence degree (t.d.) at least $n$ over $\mathbb{Q}$;
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Pseudo exponential field or Zilber field

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We say $V \subseteq G_n(K)$ is normal if $\dim[M]V \geq k$ for any $k \times n$ integer matrix of rank $k$.

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We say that $V \subseteq G_n(K)$ is free if there are no $m_1, \ldots, m_n \in \mathbb{Z}$ and $a, b \in K$ where $b \neq 0$ such that $V$ is contained in

$\{(\bar{x}, \bar{y}) : m_1x_1 + \ldots + m_nx_n = a\}$ or $\{(\bar{x}, \bar{y}) : y_1^{m_1} \cdot \ldots \cdot y_n^{m_n} = b\}$. 
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**Strong Exponential Closure**

For all finite $A \subseteq K$ if $V \subseteq G_n(K)$ is irreducible, free and normal with $\dim V = n$ there is $(\bar{z}, E(\bar{z})) \in V$ a generic point in $V$ over $A$; Equivalently, there are infinitely algebraically independent such points in $V$.

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For all finite $A \subseteq K$ if $V \subseteq G_n(K)$ is irreducible, free and normal with $\dim V = n$ and defined over the definable closure of $A$, $\{(\bar{z}, E(\bar{z})) \in V : \text{generic over } A\}$ is countable.

**Remark:**

Zilber finds an axiomatization of the class of pseudo exponential fields $L_{\omega_1\omega}(Q)$

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The class of pseudo exponential fields is quasiminimal (Bays-Kirby);
The class of pseudo exponential fields has automorphism different from identity and conjugation.
The class of pseudo exponential fields has a unique model in every uncountable cardinality. (Bays-Kirby)

Zilber’s Conjecture: The unique model of cardinality $2^{\aleph_0}$ is $(\mathbb{C}, e^x)$. 
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A positive answer would imply

- Is $\mathbb{R}$ definable in $(\mathbb{C},\exp)$ NO
- Is $(\mathbb{C},\exp)$ quasi-minimal? YES
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**Theorem (Bays and Kirby)**

If $(\mathbb{C},\exp)$ is exponentially algebraically closed then it is quasiminimal

**Theorem (Boxall ’18)**

Let $X$ a subset of $\mathbb{C}$ defined by $\exists y (P(x,y) = 0)$, where $P$ is a term formed from language $\{+ \times, \exp\}$ together with parameters from $\mathbb{C}$. Then either $X$ or $\mathbb{C} \setminus X$ is countable.
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- \( \lambda = 1 \) transcendence of \( e \); [Hermite (1873)]
- \( \lambda = 2\pi i \) transcendence \( \pi \); [Lindemann (1882)]
- \( \lambda_1 = \pi, \lambda_2 = \pi i \), algebraically independent \( \pi, e^\pi \) [Nesterenko (1996)]
- If \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are algebraic numbers linearly independent over \( \mathbb{Q} \), then \( e^{\lambda_1}, \ldots, e^{\lambda_n} \) are algebraically independent over \( \mathbb{Q} \) [Lindemann-Weierstrass (1885)]
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Schanuel’s conjecture

Schanuel’s Conjecture is currently considered out of reach, except for some very special cases.

- $\lambda = 1$ transcendence of $e$; [Hermite (1873)]
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Strong exponential closure

**Remark**
Assuming Schanuel’s Conjecture the axiom of strong exponential closure for \((\mathbb{C}, e^x)\) is the only impediment to prove Zilber’s Conjecture.

**Simplest case**
Given \(p(x, y) \in \mathbb{C}[X, Y]\) irreducible where both \(x\) and \(y\) appear, is there a generic solution \(p(z, e^z) = 0\)?
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If \( p(x, y) \in \mathbb{C}[x, y] \) is irreducible and depends on \( x \) and \( y \) then \( f(z) = p(z, e^z) \) has infinitely many zeros.

Proof

Follows from Hadamard Factorization theorem with Henson and Rubel’s result (proved independently by Van den Dries) which said that the map from exponential terms to entire function is injective.
**Theorem (Marker)**

If \( p(x, y) \in \mathbb{C}[x, y] \) is irreducible and depends on \( x \) and \( y \) then \( f(z) = p(z, e^z) \) has infinitely many zeros.

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Infinite solutions

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Theorem (Marker)

(SC). If \((z, e^z)\) and \((w, e^w)\) are solutions of \(p(x, y) \in \mathbb{Q}_{alg}[x, y]\) then \(z, w\) are algebraically independent over \(\mathbb{Q}\).

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Some ideas due also to Gunayadin and Martin-Pizarro.
Let $f$ be the analytic function $f(z) = p(z, e^z, e^{e^z}, \ldots, e^{e^{e^{\cdots e^z}}})$ over $\mathbb{C}$.

**Definition**

A solution $a$ of $f$ is generic over $L$ (for $L$ a finitely generated extension of $\mathbb{Q}$ containing the coefficients of $p$) if

$$t.d.\left(a, e^a, e^{e^a}, \ldots, e^{e^{e^{\cdots e^a}}}\right) = n,$$

where $n$ is the number of iterations of exponentiation which appear in the polynomial $p$. 
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Conjecture 1. Assuming Schanuel’s Conjecture. Let \( p(x, y_1, \ldots, y_n) \in \mathbb{Q}_{\text{alg}}[x, y_1, \ldots, y_n] \) a nonzero irreducible polynomial depending on \( x \) and the last variable \( y_n \). Then
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has a generic solution.

**Remark**

Strong Exponential Closure in \( \mathbb{C} \) implies a positive answer.
Past results: three iterations

**Theorem (DFT)**

(Sc) Let \( p(x, y_1, y_2, y_3) \in \mathbb{Q}^{alg}[x, y_1, y_2, y_3] \) be a nonzero irreducible polynomial depending on \( x \) and \( y_3 \). Then, there exists a generic solution of

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We consider when \( f(z) = p(z, e^z, e^{e^z}, e^{e^{e^z}}) \). The corresponding system in six variables \((z_1, z_2, z_3, w_1, w_2, w_3)\) is:

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V = \begin{cases} 
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\end{cases}
\]

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thought of as an algebraic set \( V \) in \( G_2(\mathbb{C}) \).

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We prove that there exists a solution of $p$. In fact there are infinitely many.

The solution is a generic solution.
Proof

1. We prove that there exists a solution of $p$. In fact there are infinitely many.

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We prove that there exists a solution of $p$. In fact there are infinitely many.

The solution is a generic solution.
Theorem (Katzberg)

A non constant polynomial $F(z) \in \mathbb{C}[z]^E$ has always infinitely many zeros unless it is of the form

$$F(z) = (z - \alpha_1)^{n_1} \cdots (z - \alpha_n)^{n_n} e^{g(z)},$$

where $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$, $n_1, \ldots, n_n \in \mathbb{N}$, and $g(z) \in \mathbb{C}[z]^E$. 

Solutions of exponential polynomials over $\mathbb{C}$
Solutions of exponential polynomials over \( \mathbb{C} \)

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Theorem (D’Aquino, Fornasiero, T.)

Let \( f(z) = p(z, e^z, e^{e^z}, \ldots, e^{e^{\cdots e^z}}) \), where \( p(x, y_1 \ldots, y_k) \) is an irreducible polynomial over \( \mathbb{C}[x, y_1 \ldots, y_k] \). Then the function \( f \) has infinitely many solutions in \( \mathbb{C} \) unless \( p(x, y_1 \ldots, y_k) = g(x) \cdot y_1^{n_{i_1}} \cdots y_k^{n_{i_k}} \), where \( g(x) \in \mathbb{C}[x] \).

Proof

It is an immediate consequence of Katzberg’s result.

No restrictions on the coefficients of \( p(x, y_1 \ldots, y_k) \) and it is unconditionally (no Schanuel’s Conjecture).
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Solutions of exponential functions

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It is an immediate consequence of Katzberg’s result.

No restrictions on the coefficients of \( p(x, y_1, \ldots, y_k) \) and it is unconditionally (no Schanuel’s Conjecture).
Masser’s result

**Theorem (Masser)**

Let $P_1(x), \ldots, P_n(x) \in \mathbb{C}[x]$, where $x = x_1, \ldots, x_n$. Then there exist $z_1, \ldots, z_n \in \mathbb{C}$ such that

\[
\begin{align*}
  e^{z_1} &= P_1(z_1, \ldots, z_n) \\
  e^{z_2} &= P_2(z_1, \ldots, z_n) \\
  &\vdots \\
  e^{z_n} &= P_n(z_1, \ldots, z_n)
\end{align*}
\]  

We have to show that the function $F : \mathbb{C}^n \to \mathbb{C}^n$ defined as

\[F(x_1, \ldots, x_n) = (e^{x_1} - P_1(x_1, \ldots, x_n), \ldots, e^{x_n} - P_n(x_1, \ldots, x_n))\]

has a zero in $\mathbb{C}^n$. 
**Theorem (Masser)**

Let $P_1(\bar{x}), \ldots, P_n(\bar{x}) \in \mathbb{C}[\bar{x}]$, where $\bar{x} = x_1, \ldots, x_n$. Then there exist $z_1, \ldots, z_n \in \mathbb{C}$ such that

\[
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(2)

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$$F(x_1, \ldots, x_n) = (e^{x_1} - P_1(x_1, \ldots, x_n), \ldots, e^{x_n} - P_n(x_1, \ldots, x_n))$$

does not have a zero in $\mathbb{C}^n$. 

Proof

**Lemma (Kantorovich)**

Let $F : \mathbb{C}^n \to \mathbb{C}^n$ with

$$F(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n))$$

be an entire function, and $p_0$ be such that $J(p_0)$, the Jacobian of $F$ at $p_0$ is non singular. Let $\eta = |J(p_0)^{-1}F(p_0)|$ and $U$ the closed ball of center $p_0$ and radius $2\eta$. Let $M > 0$ be such that $|H(F)|^2 \leq M^2$ (where $H(F)$ denotes the Hessian of $F$). If $2M\eta|J(p_0)^{-1}| < 1$ then there is a zero of $F$ in $U$. 
**Lemma (Kantorovich)**

Let $F : \mathbb{C}^n \to \mathbb{C}^n$ with $F(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n))$ be an entire function, and $p_0$ be such that $J(p_0)$, the Jacobian of $F$ at $p_0$ is non singular. Let $\eta = |J(p_0)^{-1}F(p_0)|$ and $U$ the closed ball of center $p_0$ and radius $2\eta$. Let $M > 0$ be such that $|H(F)|^2 \leq M^2$ (where $H(F)$ denotes the Hessian of $F$). If $2M\eta|J(p_0)^{-1}| < 1$ then there is a zero of $F$ in $U$. 

Proof
A cone is an open subset $U \subseteq \mathbb{C}^n$ s. t. for every $1 \leq t \in \mathbb{R}$, if $\bar{x} \in U$ then $t\bar{x} \in U$.

**Definition**

An algebraic function is an analytic function $f : U \rightarrow \mathbb{C}$ s.t. there exists a nonzero polynomial $p(\bar{x}, u) \in \mathbb{C}[\bar{x}, u]$ with $p(\bar{x}, f(\bar{x})) = 0$ on all $\bar{x} \in U$. If, moreover, the polynomial $p$ is monic in $u$, we say that $f$ is integral algebraic.
Theorem (Masser-DFT)

Let $f_1, \ldots, f_n : U \to \mathbb{C}$ be nonzero algebraic functions, defined on some cone $U$. Assume that $U \cap (2\pi i \mathbb{Z}^*)^n$ is Zariski dense in $\mathbb{C}^n$. Then

$$
\begin{align*}
    e^{z_1} &= f_1(z_1, \ldots, z_n) \\
    e^{z_2} &= f_2(z_1, \ldots, z_n) \\
    & \vdots \\
    e^{z_n} &= f_n(z_1, \ldots, z_n)
\end{align*}
$$

(3)

has a solution $\bar{a} \in U$. 
**Theorem (Main Theorem DFT '18)**

Let $V \subseteq \mathbb{C}^n \times (\mathbb{C}^*)^n$ an irreducible variety over $\mathbb{Q}^{alg}$ with $\dim V = n$. If $\pi_1(V)$ and $\pi_2(V)$ are dominant, then there exists a generic point of $V$ of the form $(\bar{a}, e^{\bar{a}})$.

**Remark**

The result implies many cases of Zilber’s Conjecture.
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Remark

The result implies many cases of Zilber’s Conjecture.
Strong exponential closure

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**Remark**

The result implies many cases of Zilber’s Conjecture.
Positive answer to the conjecture

**Remark**

Let \( p(x, y_1, \ldots, y_n) \in \mathbb{Q}^{\text{alg}}[x, y_1, \ldots, y_n] \) a nonzero irreducible polynomial depending on \( x \) and the last variable \( y_n \). Let \( p(z, e^z, e^{e^z}, \ldots, e^{e^{\ldots e^z}}) = 0 \), the corresponding system in \( 2n \) variables is:

\[
V = \left\{ p(x_1, \ldots, x_n, y_1, \ldots, y_n) = 0 \right. \\
\left. x_{i+1} = y_i \right\}
\]

for all \( i = 2, \ldots, n \).

\( V \) is a variety of \( \dim V = n \).
Main theorem implies Conjecture 1
Let \( V \subseteq \mathbb{C}^n \times (\mathbb{C}^*)^n \), over the algebraic closure of \( \mathbb{Q} \).

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A point \((a, e^a) \in V\) is generic in \( V\) if

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t.d.\mathbb{Q}(a, e^a) = \dim V.
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**Definition**

A point $(a, e^a) \in V$ is generic in $V$ if

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1. There exists a solution (unconditionally)
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Existence of solutions

**THEOREM (Masser Brownawell-DFT)**

Let $V \subseteq \mathbb{C}^n \times (\mathbb{C}^*)^n$ be an irreducible algebraic variety such that the projection onto the first coordinates $\pi_1(V)$ is Zariski dense in $\mathbb{C}^n$. Then the set $\{\bar{a} \in \mathbb{C}^n : (\bar{a}, e^{\bar{a}}) \in V\}$ is Zariski dense in $\mathbb{C}^n$.

**REMARK (1)**

The result it is unconditionally (it doesn’t use Schanuel’s Conjecture)

**REMARK (2)**

By Bays-Kirby result quasiminimality is true for $(\mathbb{C}, e^x)$ for some cases.
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1. Schanuel’s conjecture
2. Masser’s system
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Let be \((\bar{a}, e^{\bar{a}}) \in V\) and suppose that the point is not generic, i.e.
\[ t.d._Q(\bar{a}, e^{\bar{a}}) = m < n. \]
Without loss of generality we can assume
\[ |V_{\bar{a}}| < \infty \text{ and } |V^{e^{\bar{a}}}| < \infty \]

By Schanuel’s Conjecture
\[ l.d.(\bar{a}) \leq t.d._Q(\bar{a}, e^{\bar{a}}) = m < n. \]

So, there exists \(M \in \mathbb{Z}^n \times \mathbb{Z}^{n-m}\) such that \(M \cdot \bar{a} = 0\).

Applying exponentiation we have the relation \(e^{\bar{a}^M} = 1\). The above relations define:
\[ L_M = \{\bar{x} : M \cdot \bar{x} = 0\} \text{ and } T_M = \{\bar{y} : \bar{y}^M = 1\}. \]
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We obtain that

\[ m = \dim L_M = \dim T_M, \]

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Moreover, \(|V_{\bar{a}}| < \infty\) and \(|V_{e\bar{a}}| < \infty\) imply \(e\bar{a}\) is algebraic over \(\mathbb{Q}(\bar{a})\), and \(\bar{a}\) is algebraic over \(\mathbb{Q}(e\bar{a})\), which means

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In other words $\bar{a}$ is generic in $L_M$ and $e^{\bar{a}}$ is generic in $T_M$. We consider

$$W_N = \{(\bar{x}, \bar{y}) \in V : \bar{x} \in L_N \land |V_x| < \infty \land |V_y| < \infty\},$$

where $N \in \mathbb{C}^n \times \mathbb{C}^{n-m}$. If $N = M$ then $(\bar{a}, e^{\bar{a}}) \in W_M$.

$(W_N)_N$ is a definable family.

We observe that

$$\dim W_M = \dim \pi_1(W_M) \leq \dim L_M.$$

Moreover $(\bar{a}, e^{\bar{a}}) \in W_M$ so, $\dim W_M \geq \dim L_M$. We have

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Let be $W'_M$ the irreducible components of the Zariski closure of $W_M$ containing the point $(\bar{a}, e^{\bar{a}})$.

Since $(\bar{a}, e^{\bar{a}}) \in W'_M$ is generic and $e^{\bar{a}} \in \pi_2(W'_M)$ then $\pi_2(W'_M) \subseteq T_M$, so its Zariski closure is contained in $T_M$.

Moreover, $e^{\bar{a}}$ is generic in $T_M$ and $e^{\bar{a}} \in \pi_2(W'_M)$, then we have

\[ T_M = \frac{\pi_2(W'_M)}{Zar}. \]
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We consider

\[ S_N = \{ \pi_2(W'_N)^{Zar} : W'_N \text{ irreducible component of } W_N \} \].

Let be \( U = \{ S_N : S_N \text{ tori } \} \).

Since \( U \) is a countable definable family in \((\mathbb{C}^*)^n\), and \( \mathbb{C} \) is \( \omega_1 \)-saturated then \( U \) is either finite or co-countable, and since it is countable then \( U \) is necessarily finite, i.e. \( U = \{ H_1, \ldots, H_l \} \). So, \( T_M = H_i \), for some \( i = 1, \ldots, l \).

We can avoid such tori adding finitely many inequalities in the Masser’s system which guarantees that the solution is a generic point.
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Next goals

1. Weaken the hypothesis that $\pi_1(V)$ and $\pi_2(V)$ are dominant.

2. Eliminate the hypothesis that $V$ is defined over $\mathbb{Q}^{alg}$ (work going on with D’Aquino, Fornasiero and Gunaydin)
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