Definable Valuation on dependent fields

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Main Question

Question

Let K be a dependent field.
Under which conditions does K admit a non-trivial valuation ring ($\mathcal{O} \neq K$) definable in $\mathcal{L}_{ring} = (0, 1; +, -, \cdot)$?

Example

For the p-adic valuation on $\mathbb{Q}_p$ we have

$$\mathcal{O}_{v_p} := \{x \in \mathbb{Q}_p \mid v_p(x) \geq 0\} = \{x \in \mathbb{Q}_p \mid \exists y\ y^2 - y = p \cdot x^2\}.$$

Definition

Let (K, v) be a valued field. We say v is a definable valuation if there exists an $\mathcal{L}_{\text{ring}} = \{0, 1; +, -, \cdot\}$ formula φ such that

$$\mathcal{O}_v := \{x \in K \mid v(x) \geq 0\} = \{x \in K \mid \varphi(x)\}$$

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Example

Let K be an algebraically closed field. Then the only definable valuation on K is the trivial valuation.

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Fact

Let K be a field.
There exists a non-trivial valuation on K if and only if there exists no finite field F such that K/F is an algebraic field extension.

From now on we assume that no fields are algebraic extensions of finite fields.

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Fact and Notation

Let (K, v) be a valued field.
Then $\mathcal{B}_v := \{\{x \in K \mid v(x-a) > \gamma\} \mid \gamma \in \Gamma, a \in K\}$ is an open basis of a topolology $\mathcal{T}_v$ on K.

Definition and Lemma

Let K a field and $\mathcal{N} \subseteq \mathcal{P}(K)$ such that

(V 1) $\bigcap \mathcal{N} := \bigcap_{U \in \mathcal{N}} U = \{0\}$ and $\{0\} \notin \mathcal{N}$

(V 2) $\forall U, V \in \mathcal{N}\ \exists W \in \mathcal{N}\ W \subseteq U \cap V$

(V 3) $\forall U \in \mathcal{N}\ \exists V \in \mathcal{N}\ V - V \subseteq U$

(V 4) $\forall U \in \mathcal{N}\ \forall x, y \in K\ \exists V \in \mathcal{N}\ (x + V) \cdot (y + V) \subseteq x \cdot y + U$

(V 5) $\forall U \in \mathcal{N}\ \forall x \in K^{\times}\ \exists V \in \mathcal{N}\ (x + V)^{-1} \subseteq x^{-1} + U$

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Then

$$\mathcal{T}_{\mathcal{N}} := \{U \subseteq K \mid \forall x \in U\ \exists V \in \mathcal{N}\ x + V \subseteq U\}$$

is a *V-topology* on K.
$\mathcal{N}$ is a basis of zero neighbourhoods of $\mathcal{T}_{\mathcal{N}}$.

Theorem

A topology is a V-topology if and only if it is induced by a non-trivial valuation or by a non-trivial absolute value.

Theorem (Koenigsmann)

Let (K, v) be a valued field. Let v be non-trivial and henselian. Then there exists a non-trivial definable valuation on K if and only if K is not real closed and not separably closed.

Conjecture

Let K be a dependent field with $\sqrt{-1} \in K$. Assume that for all finite field extensions L/K and all $q \in \mathbb{N}$ prime
$(L^\times : (L^\times)^q) = \#\{a \cdot (L^\times)^q \mid a \in L^\times\} < \infty$.
Then either K is algebraically closed or there exists a non-trivial definable valuation on K.

Fact

Let L/K be a finite field extension and v a non-trivial definable valuation on L. Then $v|_K$ is a non-trivial definable valuation on K.

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Let L/K be a finite field extension. If K is dependent, then L is dependent as well.

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Proposition

Let K be a field, $\sqrt{-1} \in K$ and $G = (K^\times)^q \neq K^\times$ for $q \neq char(K)$. Let $\left(K^\times : (K^\times)^q\right) < \infty$. Let $\mathcal{N}_G := \{\bigcap_{i=1}^n a_i \cdot (G+1) \mid n \in \mathbb{N}, a_i \in K^\times\}$.
If $\mathcal{N}_G$ is a basis of neighbourhoods of zero of a V-topology, then there exists a non-trivial definable valuation on K.

Lemma

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Let $\mathfrak{T}$ be a theory and let A be a set of parameters. Let $\mathcal{L}_x(A)$ denote the set of all A definable sets in the variable x.

Lemma

$\mathcal{L}_x(A)$ is an algebra.

Definition

A *Keisler measure* μ over A in the variable x is a finitely additive probability measure on $\mathcal{L}_x(A)$.

Definition

Let K be a field and μ a Keisler measure on K. We say that μ is additively [multiplicatively] invariant if for all $x \in K$ [$x \in K^\times$] and all $X \in \mathcal{L}_x(A)$ we have $\mu(x + X) = \mu(X)$ [$\mu(x \cdot X) = \mu(X)$].

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Proposition

Let K be an dependent field. Then there exists an additively and multiplicatively invariant definable Keisler measure on K.

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Lemma

Let $a_0, \cdots, a_m \in K$ *and* $X \subseteq K$ *be definable.*
Then $\mu\left(\bigcap_{i=0}^{m}(a_i + X)\right) = \mu(X)$*.*

Proposition

Let $G \subsetneq K^\times$ a multiplicative group subgroup of K with $-1 \in G$. Assume $\mu(G) > 0$.
Then $\{0\} \notin \mathcal{N}_G$ i.e. $\{0\} \neq \bigcap_{i=1}^{n} a_i \cdot (G+1)$ for all $a_1, \ldots, a_n \in K^\times$.

Proof: (sketch) Let $a_1, \ldots, a_n \in K^\times$.

Let $i \in \{1, \ldots, n\}$ and $x \in G$.

We have $x \in \{t \in G \mid 1 \in (a_i \cdot G + t \cdot a_i)\}$ iff there exists $g \in G$ s.th. $1 = a_i \cdot g + x \cdot a_i$ iff $\frac{1}{a_i} - x \in G$ iff $x \in G + \frac{1}{a_i}$. Hence $\bigcap_{i=1}^{n} \{t \in G \mid 1 \in (a_i \cdot G + t \cdot a_i)\} = \bigcap_{i=1}^{n} \left(G + \frac{1}{a_i}\right) \cap G$.

$\mu\left(\bigcap_{i=1}^{m} \left(G + \frac{1}{a_i}\right) \cap (G + 0)\right) = \mu(G) > 0$ and therefore there exists $t_0 \in \bigcap_{i=1}^{m} \{t \in G \mid 1 \in (a_i \cdot G + t \cdot a_i)\}$.

As $t_0 \in G$ and G is a multiplicative group we have $0 \neq \frac{1}{t_0} \in \bigcap_{i=1}^{m} \left(a_i \cdot \frac{1}{t_0} \cdot G + a_i\right) = \bigcap_{i=1}^{m} (a_i \cdot G + a_i)$. $\square$

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$\mu\left(\bigcap_{i=1}^m \left(G + \frac{1}{a_i}\right) \cap (G + 0)\right) = \mu(G) > 0$ and therefore there exists $t_0 \in \bigcap_{i=1}^m \{t \in G \mid 1 \in (a_i \cdot G + t \cdot a_i)\}$.

As $t_0 \in G$ and G is a multiplicative group we have $0 \neq \frac{1}{t_0} \in \bigcap_{i=1}^m \left(a_i \cdot \frac{1}{t_0} \cdot G + a_i\right) = \bigcap_{i=1}^m (a_i \cdot G + a_i)$. $\square$

Lemma

Let $G = (K^\times)^q$ *for some prime* q *and* $(K^\times : (K^\times)^q) < \infty$.
Then $\mu(G) > 0$.

Corollary

Let $G = (K^\times)^q$ *for some prime* q *and* $(K^\times : (K^\times)^q) < \infty$.
Assume $-1 \in G$.
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Proposition (K.D.)

Let K be a dependent field, $\sqrt{-1} \in K$ and $G = (K^\times)^q \neq K^\times$ for $q \neq char(K)$. Let $\left(K^\times : (K^\times)^q\right) < \infty$.
Assume that

(V 3)' $\exists a_1, \ldots, a_n \in K^\times$
$\bigcap_{i=1}^n a_i \cdot (G+1) - \bigcap_{i=1}^n a_i \cdot (G+1) \subseteq G+1$

(V 4)' $\exists a_1, \ldots, a_n \in K^\times$
$\left(\bigcap_{i=1}^n a_i \cdot (G+1)\right) \cdot \left(\bigcap_{i=1}^n a_i \cdot (G+1)\right) \subseteq G+1$

(V 6)' $\exists a_1, \ldots, a_n \in K^\times \, \forall x, y \in K$
$x \cdot y \in \bigcap_{i=1}^n a_i \cdot (G+1)$
$\Rightarrow x \in G+1 \vee y \in G+1$

hold.
Then K admits a non-trivial definable valuation.