Domination and Products in Unstable Theories

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Groups and NIP
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Overview

Motivation and Main Result

T complete, κ large enough, $\mathfrak{U}$ a κ-monster. *Small* $=$ *of size* $< \kappa$.

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Theorem (Haskell, Hrushovski, Macpherson)

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In this talk:

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- What does such a theory look like.
- Properties of $\sim_{\mathrm{D}}$.
- Open questions, mostly in the NIP unstable case.

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Theorem (Haskell, Hrushovski, Macpherson)

In ACVF, $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \widetilde{\mathrm{Inv}}(k) \times \widetilde{\mathrm{Inv}}(\Gamma)$. $k :=$ residue field, $\Gamma :=$ value group (to be precise, they use $\overline{\mathrm{Inv}}(\mathfrak{U})$)

Theorem (M.)

There is a theory where $(\widetilde{\mathrm{Inv}}(\mathfrak{U}),\otimes)$ is not well-defined. Nor is $\overline{\mathrm{Inv}}(\mathfrak{U})$.

In this talk:

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Invariant Types

Canonical extension and product

Definition (A small)

p is A-*invariant* iff whether $p(x) \vdash \varphi(x; d)$ or not depends only on $\operatorname{tp}(d/A)$.

E.g. if p is A-definable or finitely satisfiable in A. Say $p \in S(\mathfrak{U})$ is *invariant* iff it is A-invariant for some *small* $A \subset \mathfrak{U}$.

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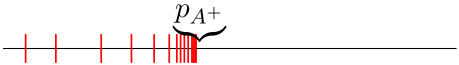
$p_{A^+}(x) \coloneqq \{x < d \mid d > A\} \cup \{x > d \mid d \not> A\}$



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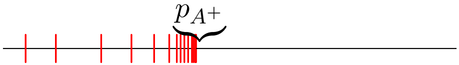

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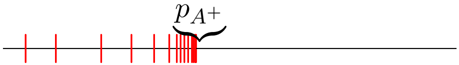

$$\varphi(x; d) \in (p \mid \mathfrak{U}B) \overset{\text{def}}{\iff} \text{for } \tilde{d} \in \mathfrak{U} \text{ such that } d \equiv_A \tilde{d}, \text{ we have } \varphi(x; \tilde{d}) \in p.$$

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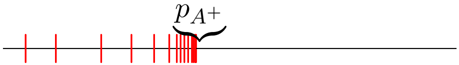

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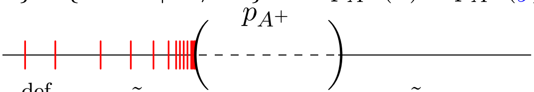

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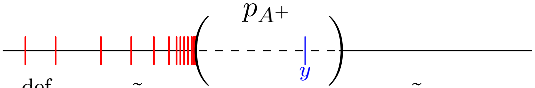

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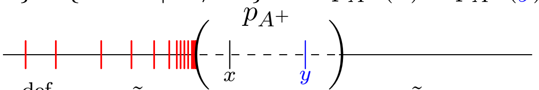

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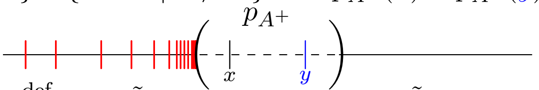

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Fact

$\otimes$ is associative. $\otimes$ commutative $\iff$ T stable (it's the usual $(a, b) \vDash p \otimes q \iff a \mathop{\downarrow}_{\mathfrak{U}} b$).

Domination

Definition (Domination preorder on $S^{\mathrm{inv}}_{<\omega}(\mathfrak{U})$)

$p_x \geq_{\mathrm{D}} q_y$ iff there are a small $A \subset \mathfrak{U}$ and $r \in S_{xy}(A)$ such that:

$$p,\ q \text{ are } A\text{-invariant},\ r \supseteq (p \restriction A) \cup (q \restriction A), \text{ and } p(x) \cup r(x,y) \vdash q(y)$$

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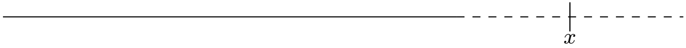
Example (DLO, all types below are $\emptyset$-invariant)

$\mathrm{tp}(x > \mathfrak{U})$



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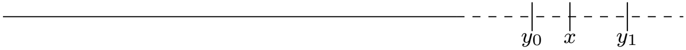

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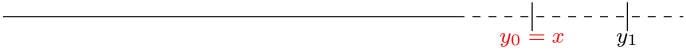

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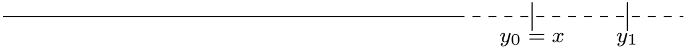

Example (Random Graph)

$p \geq_{\mathrm{D}} q \iff p \supseteq q$ after renaming/duplicating variables and ignoring realised ones.

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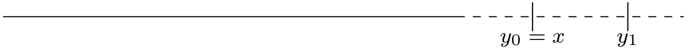

Example (Random Graph, or a set with no structure (*degenerate domination*))

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Is $\otimes$ a Congruence with respect to $\sim_{\mathrm{D}}$?

Or: is $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ well-defined?

Lemma

If p_0 is invariant and $p_0 \cup r \vdash p_1$, then (p_1 is invariant and) $(p_0 \mid B) \cup r \vdash (p_1 \mid B)$. In particular, if $p_0 \geq_{\mathrm{D}} p_1$, then $p_0 \otimes q \geq_{\mathrm{D}} p_1 \otimes q$.

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Does $q_0 \geq_{\mathrm{D}} q_1$ imply $p \otimes q_0 \geq_{\mathrm{D}} p \otimes q_1$?

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The answer is affirmative if T is for instance stable, binary, or if $\geq_{\mathrm{D}}$ can always be witnessed by definable functions. These are all instances of a (fairly ugly) general condition called *stationary domination* (more on this here).

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Does $q_0 \geq_{\mathrm{D}} q_1$ imply $p \otimes q_0 \geq_{\mathrm{D}} p \otimes q_1$?

The answer is affirmative if T is for instance stable, binary, or if $\geq_{\mathrm{D}}$ can always be witnessed by definable functions. These are all instances of a (fairly ugly) general condition called *stationary domination* (more on this here).

Fact

If for T the answer to the question above is "yes", then $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes, \leq_{\mathrm{D}})$ is an ordered monoid. The neutral element (and minimum) is the (unique) class of realised types, and nothing else is invertible ($p \otimes q$ realised $\Longrightarrow p, q$ both realised!).

Examples

(In all of these stationary domination holds and $\widetilde{\mathrm{Inv}}(\mathfrak{U}) = \overline{\mathrm{Inv}}(\mathfrak{U})$)

T strongly minimal (see here)

$(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes, \leq_{\mathrm{D}}) \cong (\mathbb{N}, +, \leq)$.

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Random Graph (see here)

$\sim_{\mathrm{D}}$ is degenerate, $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ resembles $(S^{\mathrm{inv}}_{<\omega}(\mathfrak{U}), \otimes)$, e.g. it is noncommutative.

A Counterexample

(with SOP and IP_2)

Idea: DLO

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Idea: 2-coloured DLO

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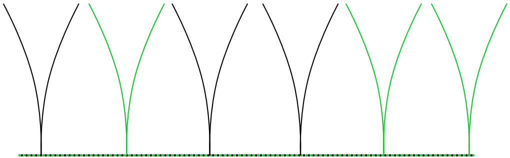

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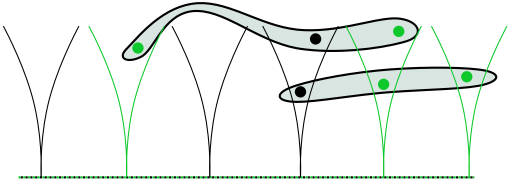

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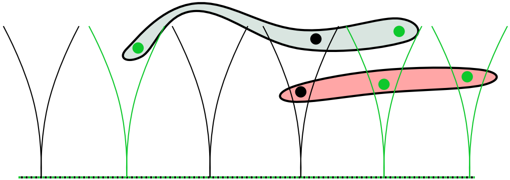

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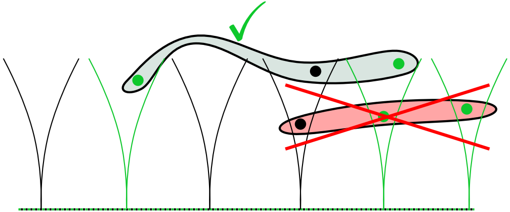

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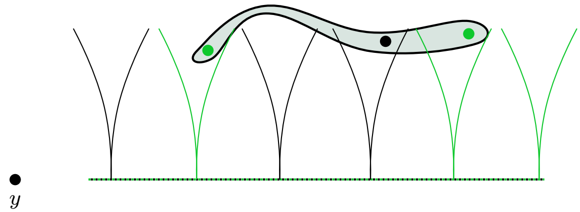

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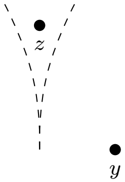

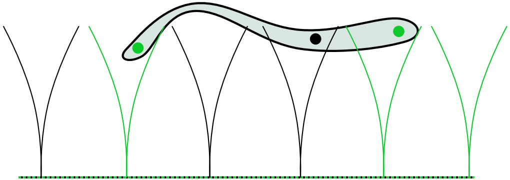

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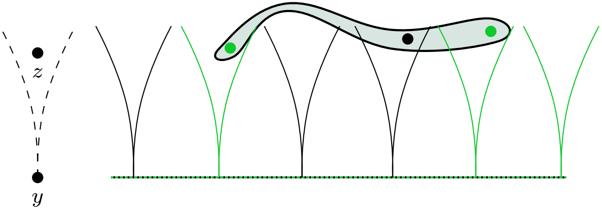

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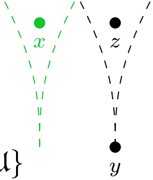

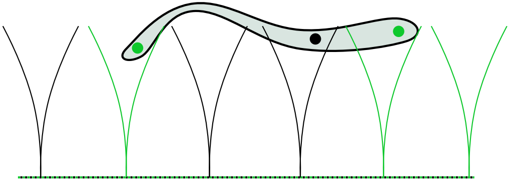

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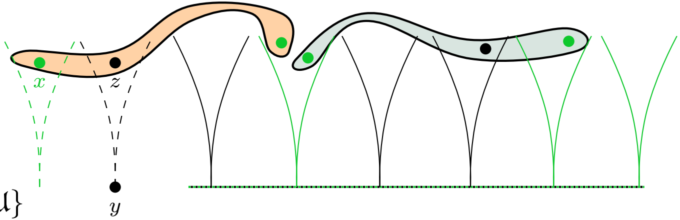

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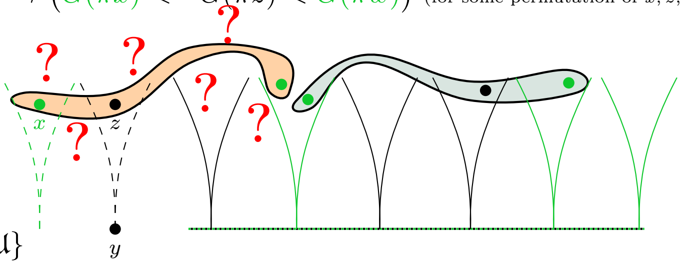

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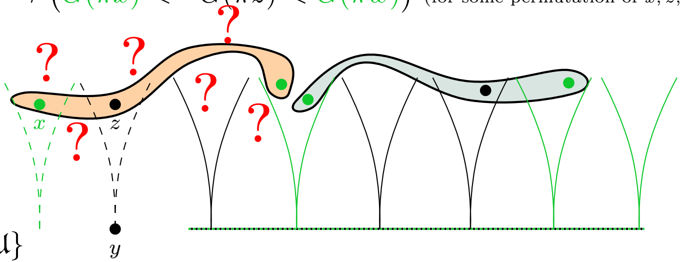

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There is a supersimple version here. Also works for a number of variations of $\sim_{\mathrm{D}}$.

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Domination equivalence is quite coarse; for instance it does not preserve Morley rank (generic equivalence relation), nor dp-rank (DLO).

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- Definability (over *some* small set, not necessarily the same as q)
- Finite satisfiability (in *some* small set, not necessarily the same as q)
- Generic stability (over *some* small set, not necessarily the same as q)
- Weak orthogonality to a fixed type

Generic stability is particularly interesting:

- It is possible to have $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \neq \widetilde{\mathrm{Inv}}(\mathfrak{U}^{\mathrm{eq}})$ (more g.s. types, e.g. DLO+dense eq. rel.).
- Using [Tan], strongly regular g.s. types are $\leq_{\mathrm{D}}$-minimal (among the nonrealised ones).
- $(\widetilde{\mathrm{Inv}}^{\mathrm{gs}}(\mathfrak{U}), \otimes, \leq_{\mathrm{D}})$ makes sense in any theory (can be trivial).

Questions/Work in Progress

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Thanks for listening!

https://arxiv.org/abs/1810.13279

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(this is not a proper bibliography, it's just a list of the sources mentioned in these slides)

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More examples: Branches

Example

Let T be the theory in the language $\{P_\sigma \mid \sigma \in 2^{<\omega}\}$ asserting that every point belongs to every $P_{\eta\restriction n}$ for exactly one $\eta \in 2^\omega$. Then $\widetilde{\operatorname{Inv}}(\mathfrak{U}) \cong \bigoplus_{2^{\aleph_0}} \mathbb{N}$.

Basically, $\widetilde{\operatorname{Inv}}(\mathfrak{U})$ here is counting how many new points are in a "branch".

More Examples: Generic Equivalence Relation

Equivalence relation E with infinitely many infinite classes (and no finite classes). A set of generators for $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ looks like this:

- a single $\sim_{\mathrm{D}}$-class $[\![0]\!]$ for realised types
- if $p_a(x) := \{E(x,a)\} \cup \{x \notin \mathfrak{U}\}$, then $[\![p_a]\!] = [\![p_b]\!]$ if and only if $\vDash E(a,b)$; corresponds to new points in an existing equivalence class
- a single $\sim_{\mathrm{D}}$-class $[\![p_g]\!]$, where $p_g := \{\neg E(x,a) \mid a \in \mathfrak{U}\}$; corresponds to new equivalence classes.

The product adds new points/new classes. So, if $\mathfrak{U}$ has κ equivalence classes,

$$\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \mathbb{N} \oplus \bigoplus_{\kappa} \mathbb{N}$$

More Examples: Cross-cutting Equivalence Relations

$T_n :=$ n generic equivalence relations E_i; intersection of classes of different E_i always infinite. Here $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ is generated by:

- a single $\sim_{\mathrm{D}}$-class $[\![0]\!]$ for realised types
- if $p_a(x) := \{E_i(x,a) \mid i < n\} \cup \{x \notin \mathfrak{U}\}$, then $[\![p_a]\!] = [\![p_b]\!]$ if and only if $\models \bigwedge_{i<n} E_i(a,b)$; corresponds to new points in E_i-relation with a for all i
- For each $i < n$, a class $[\![p_i]\!]$ saying x is in a new E_i class, but in existing E_j-classes for $j \neq i$ (does not matter which)

So

$$\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \prod_{i<n} \mathbb{N} \oplus \bigoplus_{\kappa} \mathbb{N}$$

Why $\prod$ instead of $\bigoplus$? If we allow, say, $\aleph_0$ equivalence relations, then

$$\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \prod_{i<\aleph_0}^{\mathrm{bdd}} \mathbb{N} \oplus \bigoplus_{\kappa} \mathbb{N}$$

◂ Back

Other Notions

One can define a finer equivalence relation:

Definition

$p \equiv_{\mathrm{D}} q$ is defined as $p \sim_{\mathrm{D}} q$, but by asking the same r to work in both directions: $p \cup r \vdash q$ and $q \cup r \vdash p$.

Another notion classically studied is (see e.g. [Poi]):

Definition

$p \geq_{\mathrm{RK}} q$ iff every model realising p realises q.

This behaves best in totally transcendental theories (because of prime models). It corresponds to $p(x) \cup \{\varphi(x, y)\} \vdash q(y)$.
But even there, modulo $\sim_{\mathrm{RK}}$ it is *not* true that every type decomposes as a product of $\geq_{\mathrm{RK}}$-minimal types (but in non-multidimensional totally transcendental theories every type decomposes as a product of strongly regular types).
A classical example where $\geq_{\mathrm{D}}$ differs from $\geq_{\mathrm{RK}}$: generic equivalence relation with a bijection s such that $\forall x\, E(x, s(x))$. Back

Hrushovski's Counterexample

Example (Hrushovski)

In DLO plus a dense-codense predicate P, $\overline{\mathrm{Inv}}(\mathfrak{U})$ is not commutative.

Proof idea.

Let $p(x) := \{P(x)\} \cup \{x > \mathfrak{U}\}$ and $q(y) := \{\neg P(x)\} \cup \{y > \mathfrak{U}\}$. Then p, q do not commute, even modulo $\equiv_{\mathrm{D}}$ (but they do modulo $\sim_{\mathrm{D}}$).
The predicate P forbids to "glue" variables. One will be "left behind": e.g. if $r \vdash x_0 < y_0 < y_1 < x_1$, knowing that $y_1 > \mathfrak{U}$ does not imply $x_0 > \mathfrak{U}$. $\square$

In this case, for each cut C there are generators $[\![p_{C,P}]\!]$ and $[\![p_{C,\neg P}]\!]$, with relations

- $[\![p_{C,P}]\!] \otimes [\![p_{C,P}]\!] = [\![p_{C,\neg P}]\!] \otimes [\![p_{C,P}]\!] = [\![p_{C,P}]\!]$
- (same relations swapping P and $\neg P$)
- $[\![p_{C_0,-}]\!] \otimes [\![p_{C_1,-}]\!] = [\![p_{C_1,-}]\!] \otimes [\![p_{C_0,-}]\!]$ whenever $C_0 \neq C_1$.

Back

Stable Case

In a stable theory, $\leq_{\mathrm{D}}$, $\sim_{\mathrm{D}}$ and $\equiv_{\mathrm{D}}$ can be expressed in terms of forking:

Definition (See e.g. [Pil])

$a \triangleright_E b$ iff, for all c,

$$a \mathop{\downarrow}_E c \Longrightarrow b \mathop{\downarrow}_E c$$

$p \triangleright_E q$ (p *dominates* q over E) iff there are $a \models p$ and $b \models q$ such that $a \triangleright_E b$

$p \bowtie_E q$ (p and q are *domination equivalent*) iff $p \triangleright_E q \triangleright_E p$, i.e. there are

$\underbrace{a}_{\models p} \triangleright_E \underbrace{b}_{\models q} \triangleright_E \underbrace{c}_{\models p}$

$p \doteq_E q$ (p and q are *equidominant* over E) iff there are $a \models p$ and $b \models q$ such that $a \triangleright_E b \triangleright_E a$

These are well-behaved with non-forking extensions: we can drop $_E$.

Comparison

Proposition (T stable)

The previous definitions of $\leq_{\mathrm{D}} = \triangleleft$, $\sim_{\mathrm{D}} = \bowtie$ and $\equiv_{\mathrm{D}} = \doteq$.

Remark

The proof uses crucially stationarity of types over models.

In almost all examples we saw before, $\sim_{\mathrm{D}}$ coincides with $\equiv_{\mathrm{D}}$.

Exception: in DLO with a predicate, $(\overline{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ is not commutative, while $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ is (in fact, it is the same as in DLO).

Fact (See [Wag, Example 5.2.9])

Even in the stable case, $\sim_{\mathrm{D}}$ and $\equiv_{\mathrm{D}}$ are generally different.

Classical Results

In the thin case (generalises superstable), this is classical (e.g. [Pil]):

Theorem (T thin)

$\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is a direct sum of copies of $\mathbb{N}$.
If T is moreover superstable, $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ is generated by $\{[\![p]\!] \mid p \text{ regular}\}$.

Superstability (even just thinness) implies that $\equiv_{\mathrm{D}}$ and $\sim_{\mathrm{D}}$ coincide.

The behaviour of $\geq_{\mathrm{D}}$ in general seems related to the existence of some kind of prime models (in the stable case, "prime a-models" are the way to go). This seems to hint that, maybe, o-minimal theories are a good context to investigate.
Also, some suitable generalisation of the Omitting Types Theorem would help.

Back

(Non-multi)Dimensionality

At least in the superstable case, independence of $\widetilde{\operatorname{Inv}}(\mathfrak{U})$ on $\mathfrak{U}$ already had a name:

Definition

T is *(non-multi)dimensional* iff no type is orthogonal to (every type that does not fork over) $\emptyset$.

If $\mathfrak{U}_0 \prec^+ \mathfrak{U}_1$ one has a map $\mathfrak{e}\colon \widetilde{\operatorname{Inv}}(\mathfrak{U}_0) \to \widetilde{\operatorname{Inv}}(\mathfrak{U}_1)$.

Proposition (T thin)

$\mathfrak{e}$ surjective $\iff$ T dimensional.

Question

Is this true under stability? It boils down to the image of $\mathfrak{e}$ being downward closed. I suspect this should follow from classical results. Back

Generically Stable Part

Proposition

$q \leq_{\mathrm{D}} p$ definable/finitely satisfiable/generically stable $\implies$ so is q.

As generically stable types commute with everything, in any theory the monoid generated by their classes is well-defined. (Warning: p generically stable $\not\Rightarrow$ $p \otimes p$ generically stable)

Hope

At least in special cases, get decompositions similar to $\widetilde{\operatorname{Inv}}(\mathfrak{U}) \cong \overbrace{\widetilde{\operatorname{Inv}}(k)}^{\text{g.s. part}} \times \widetilde{\operatorname{Inv}}(\Gamma)$.

Probably one should really work in T^{eq}:

Example

In $T = \mathsf{DLO}$+equivalence relation with (no finite classes and infinitely many) dense classes, $\widetilde{\operatorname{Inv}}(\mathfrak{U})$ grows when passing to T^{eq}, which has more generically stable types.

Question

How can the generically stable part look like?

Interaction with Weak Orthogonality

Definition

$p(x)$ is *weakly orthogonal* to $q(y)$ iff $p \cup q$ is complete.

Remark

Weakly orthogonal types commute.

Proposition

Weak orthogonality strongly negates domination: $q \perp^{\mathrm{w}} p_0 \geq_{\mathrm{D}} p_1 \implies q \perp^{\mathrm{w}} p_1$.
In particular if $q \perp^{\mathrm{w}} p \geq_{\mathrm{D}} q$ then q is realised.

Question

Under which conditions if $p \not\perp^{\mathrm{w}} q$ then they dominate a common nonzero class?

Known:

- Superstable (or *thin*) is enough. See here
- Fails in the Random Graph.

Action on Type Space

$f \in \mathrm{Aut}(\mathfrak{U})$ acts on $p \in S(\mathfrak{U})$ by changing parameters in formulas:

$$f \cdot p := \{\varphi(x, f(d)) \mid \varphi(x, d) \in p\}$$

Consider this action restricted to $\mathrm{Aut}(\mathfrak{U}/A)$.

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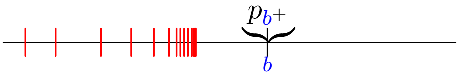
$T = \mathsf{DLO}$, consider $p_{b^+}(x) := \{x < d \mid d > b\} \cup \{x > d \mid d \leq b\}$



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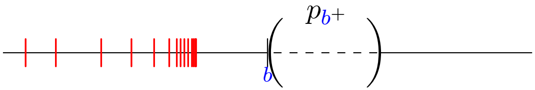

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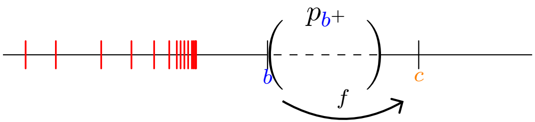

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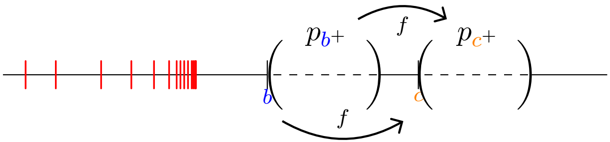


◂ Back

Some Sufficient Conditions

Proposition

$q_0 \geq_{\mathrm{D}} q_1 \implies p \otimes q_0 \geq_{\mathrm{D}} p \otimes q_1$ is implied by any of the following:

- q_1 algebraic over q_0: every $c \vDash q_1$ is algebraic over some $b \vDash q_0$. E.g. $q_1 = f_* q_0$ for some definable function f. Reason: $\{c \mid (b, c) \vDash r\}$ does not grow with $\mathfrak{U}$.
- T is binary: $\bigcup \mathrm{tp}(a_i a_j) \vdash \mathrm{tp}(a_1, \ldots, a_n)$: few questions about $a \vDash p$ and $c \vDash q_1$.
- Or even *weakly binary*: $\mathrm{tp}(a/\mathfrak{U}) \cup \mathrm{tp}(b/\mathfrak{U}) \cup \mathrm{tp}(ab/M) \vDash \mathrm{tp}(ab/\mathfrak{U})$, e.g. theories that become binary after naming constants, like a circular order.
- T is stable.

A General Sufficient Condition

Any condition in the Proposition implies that if there is some $r \in S_{yz}(M)$ witnessing $q_0(y) \geq_{\mathrm{D}} q_1(z)$, then there is one such that, in addition, if

- $b, c \in \mathfrak{U}_1 \mathrel{^+\!\succ} \mathfrak{U}$ are such that $(b, c) \models q_0 \cup r$,
- $p \in S^{\mathrm{inv}}(\mathfrak{U}, M)$ and $a \models p(x) \mid \mathfrak{U}_1$,
- $r[p] := \mathrm{tp}_{xyz}(abc/M) \cup \{x = w\}$.

then $p \otimes q_0 \cup r[p] \vdash p \otimes q_1$. We call this *stationary domination.*

Proposition

$q_0 \geq_{\mathrm{D}} q_1 \implies p \otimes q_0 \geq_{\mathrm{D}} p \otimes q_1$ holds if

- q_1 is algebraic over q_0, or
- T is weakly binary, or
- T is stable.

Open Problems

- Understand if this holds under NIP.
- Understand if this is equivalent to good definition of $\widetilde{\mathrm{Inv}}(\mathfrak{U})$.

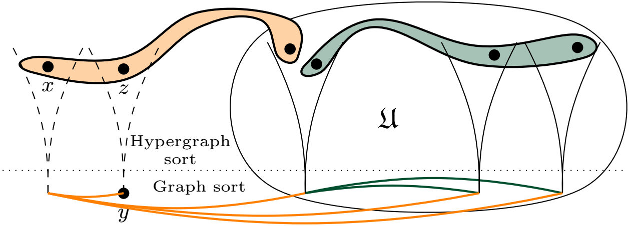
Another Counterexample

Ternary, supersimple, ω-categorical, can be tweaked to have degenerate algebraic closure

Replacing the densely coloured DLO with a random graph R_2 yields a supersimple counterexample of SU-rank 2; forking is $a \mathop{\smile\!\!\!\!\!\!\mid}_C b \iff (a \cap b \subseteq C) \wedge (\pi a \cap \pi b \subseteq \pi C)$.

$$R_3(x_0, x_1, x_2) \to \bigvee_{\sigma \in S_3} \underbrace{\left(R_2(\pi x_{\sigma 0}, \pi x_{\sigma 1}) \wedge R_2(\pi x_{\sigma 0}, \pi x_{\sigma 2}) \wedge \neg R_2(\pi x_{\sigma 1}, \pi x_{\sigma 2})\right)}_{\text{(exactly two edges between } \pi x_0, \pi x_1, \pi x_2)}$$

$$\begin{aligned} q_0(y) &:= \{\neg R_2(y, a) \mid a \in \mathfrak{U}\} \\ q_1(z) &:= \{\neg R_2(\pi z, a) \mid a \in \mathfrak{U}\} \\ r(y, z) &:= \{y = \pi z\} \cup \dots \\ p(x) &:= \{R_2(\pi x, a) \mid a \in \mathfrak{U}\} \\ &\quad \cup \{\neg R_3(x, a, b) \mid a, b \in \mathfrak{U}\} \end{aligned}$$



$q_0 \cup r \vdash q_1$: no hyperedges to decide. Same problem: $p \otimes q_0(x, y) \not\equiv_{\mathrm{D}} p \otimes q_1(t, z)$.

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Strongly Minimal Theories

$(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ **well-defined by stability**

Example

If T is strongly minimal, $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes, \leq_{\mathrm{D}}) \cong (\mathbb{N}, +, \leq)$.

(for T stable, $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \mathbb{N} \Leftrightarrow T$ is *unidimensional*, e.g. countable and $\aleph_1$-categorical, or $\mathrm{Th}(\mathbb{Z}, +)$)

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In this case, $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is basically "counting the dimension". E.g.: in ACF_0 we have $p(x_1,\ldots,x_n)\sim_{\mathrm{D}} q(y_1,\ldots,y_m)\iff\operatorname{tr}\deg(x/\mathfrak{U})=\operatorname{tr}\deg(y/\mathfrak{U})$.
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Glue transcendence bases; recover the rest with one formula.

Taking products corresponds to adding dimensions: if $(a, b) \vDash p \otimes q$, then $\dim(a/\mathfrak{U}b) = \dim(a/\mathfrak{U})$, and in strongly minimal theories

$$\dim(ab/\mathfrak{U}) = \dim(b/\mathfrak{U}) + \dim(a/\mathfrak{U}b)$$

More generally, in superstable theories (or even *thin* theories), by classical results $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \bigoplus_{i<\lambda}(\mathbb{N}, +, \leq)$, for some λ.

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Dense Linear Orders

$(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ well-defined by binarity

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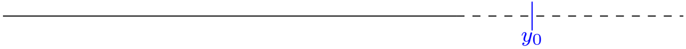
- Every element is idempotent: e.g. if $p(x) = \mathrm{tp}(x > \mathfrak{U})$, then $p(x) \sim_{\mathrm{D}} p(y_1) \otimes p(y_0)$ (seen before: glue x and y_0):



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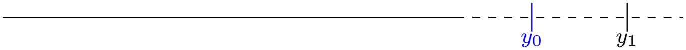

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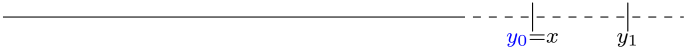

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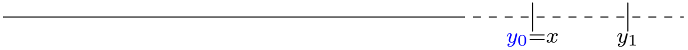

$\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is the free idempotent commutative monoid generated by the invariant cuts:

$$(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes, \leq_{\mathrm{D}}) \cong (\mathscr{P}_{\mathrm{fin}}(\{\text{invariant cuts}\}), \cup, \subseteq)$$

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Random Graph

$(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ well-defined by binarity

In the Random Graph, $\sim_{\mathrm{D}}$ is degenerate and $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ resembles closely $(S^{\mathrm{inv}}_{<\omega}(\mathfrak{U}), \otimes)$. For instance, it is not commutative:

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Example (All types $\emptyset$-invariant)

These types do not commute, even modulo $\sim_{\mathrm{D}}$:

$$q(y) := \{E(y, b) \mid b \in \mathfrak{U}\}$$
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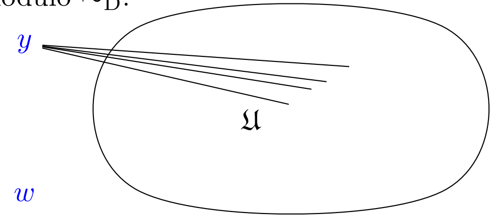

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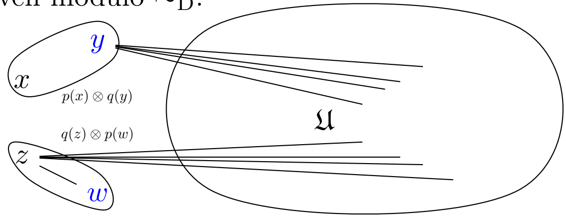

Proof Idea.

As $p_x \otimes q_y \vdash \neg E(x,y)$ and $q_z \otimes p_w \vdash E(z,w)$, gluing cannot work. But in the random graph domination is degenerate and there is not much more one can do. $\square$

More examples here ◂ Back