Strong exponential closure for $(\mathbb{C}, e^x)$

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Exponential rings

Definition: An exponential ring, or E-ring, is a pair (R, E) where R is a ring (commutative with 1) and

$$E : (R, +) \to (\mathcal{U}(R), \cdot)$$

a morphism of the additive group of R into the multiplicative group of units of R satisfying
- $E(x + y) = E(x) \cdot E(y)$ for all $x, y \in R$
- $E(0) = 1$.

1. (K, E) where K is any ring and $E(x) = 1$ for all $x \in K$.
2. $(\mathbb{R}, exp)$; $(\mathbb{C}, exp)$;

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Motivations

The model theoretic analysis of the exponential function over a field started with a problem left open by Tarski in the 30's, about the decidability of the reals with exponentiation. Only in the mid 90's Macintyre and Wilkie gave a positive answer to this question assuming Schanuel's Conjecture. Complex exponentiation involves much deeper issues, and it is much harder to approach, as it inherits the Godel incompleteness and undecidability phenomena via the definition of the set of periods. Despite this negative results there are still many interesting and natural model-theoretic aspects to analyze.

Comparing $(\mathbb{R}, exp)$ and $(\mathbb{C}, exp)$

$Th(\mathbb{R}, exp)$ decidable
modulo (SC)
(Macintyre-Wilkie '96)

$Th(\mathbb{C}, exp)$ undecidable
$\mathbb{Z} = \{x : \forall y(E(y) = 1 \rightarrow E(xy) = 1)\}$

$Th(\mathbb{R}, exp)$
model-complete
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$Th(\mathbb{C}, exp)$
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$Th(\mathbb{R}, exp)$ o-minimal
good description of
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- Is $Th(\mathbb{C}, exp)$ quasi-minimal?
- What are the automorphisms of $(\mathbb{C}, exp)$?
- Is $\mathbb{R}$ definable in $(\mathbb{C}, exp)$?

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The complex exponential field

Macintyre 1996

$x \in \mathbb{Q}$ iff $\exists t, u, v((v-u)t = 1 \wedge e^v = e^u = 1 \wedge (vx = u))$

Laczkovich 2002

For any $x \in \mathbb{Q}$

$x \in \mathbb{Z}$ iff $\exists z(e^z = 2 \wedge e^{zx} \in \mathbb{Q})$

$Th_\exists(\mathbb{C}, e^x)$ is undecidable.

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Zilber's programme

As regards definability the above ideas show that definability in the complex exponential field is as complicated as definability in the ring $\mathbb{Z}$. For this reason, work stopped early on the logic of complex exponentiation, and was only taken up again after a wonderful discovery of Zilber early this century.

Pseudo exponential field or Zilber field

Zilber's programme: Looks for a canonical algebraically closed field of characteristic 0 with exponentiation.

(K, E) is a Zilber field if:

- K is an algebraically closed field of characteristic 0;
- $E : (K, +) \longrightarrow (K^{\times}, \cdot)$ is a surjective homomorphism and there is $\omega \in K$ transcendental over $\mathbb{Q}$ such that $\ker E = \mathbb{Z}\omega$;
- **Schanuel's Conjecture (SC)** Let $\lambda_1, \ldots, \lambda_n \in K$ be linearly independent over $\mathbb{Q}$. Then $\mathbb{Q}(\lambda_1, \ldots, \lambda_n, E(\lambda_1), \ldots, E(\lambda_n))$ has transcendence degree (t.d.) at least n over $\mathbb{Q}$;
- Axioms giving criteria for solvability of systems of exponential equations.

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Normal and free

Definition

We say $V \subseteq G_n(K)$ is normal if $dim[M]V \geq k$ for any $k \times n$ integer matrix of rank k.

Definition

We say that $V \subseteq G_n(K)$ is free if there are no $m_1, \ldots, m_n \in \mathbb{Z}$ and $a, b \in K$ where $b \neq 0$ such that V is contained in $\{(\overline{x}, \overline{y}) : m_1 x_1 + \ldots + m_n x_n = a\}$ or $\{(\overline{x}, \overline{y}) : y_1^{m_1} \cdot \ldots \cdot y_n^{m_n} = b\}$.

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axioms

- **(Strong Exponential Closure)** For all finite $A \subseteq K$ if $V \subseteq G_n(K)$ is irreducible, free and normal with dim $V = n$ there is $(\bar{z}, E(\bar{z})) \in V$ a generic point in V over A; Equivalentely, there are infinitely algebraically independent such points in V.
- **(Countable Closure)** For all finite $A \subseteq K$ if $V \subseteq G_n(K)$ is irreducible, free and normal with dim $V = n$ and defined over the definable closure of A, $\{(\bar{z}, E(\bar{z})) \in V : \text{ generic over } A\}$ is countable.

Remark:

Zilber finds an axiomatization of the class of pseudo exponential fields $L_{\omega_1\omega}(Q)$

- Q is the "quantifier exist uncountably many";
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Quasiminimality

Theorem (Zilber)

- The class of pseudo exponential fields is quasiminimal (Bays-Kirby);
- The class of pseudo exponential fields has automorphism different from identity and conjugation.

Categoricity result

Theorem (Zilber)

The class of pseudo exponential fields has a unique model in every uncountable cardinality. (Bays-Kirby)

Zilber's Conjecture: The unique model of cardinality $2^{\aleph_0}$ is $(\mathbb{C}, e^x)$.

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Answer

A positive answer would imply

- Is $\mathbb{R}$ definable in $(\mathbb{C}, exp)$ NO
- Is $(\mathbb{C}, exp)$ quasi-minimal? YES
- Are there automorphisms of $(\mathbb{C}, exp)$ different from identity and conjugation? YES

Theorem (Bays and Kirby)

If $(\mathbb{C}, exp)$ is exponentially algebraically closed then it is quasiminimal

Theorem (Boxall '18)

Let X a subset of $\mathbb{C}$ defined by $\exists \overline{y}(P(x, \overline{y}) = 0)$, where P is a term formed from language $\{+, \times, exp\}$ together with parameters from $\mathbb{C}$. Then either X or $\mathbb{C} \setminus X$ is countable.

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Do Zilber's axioms hold in $\mathbb{C}$?

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Schanuel's conjecture

Schanuel's Conjecture is currently considered out of reach, except for some very special cases.

- $\lambda = 1$ transcendence of e; **[Hermite (1873)]**
- $\lambda = 2\pi i$ transcendence π; **[Lindemann (1882)]**
- $\lambda_1 = \pi, \lambda_2 = \pi i$, algebraically independent π, e^π **[Nesterenko (1996)]**
- If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are algebraic numbers linearly independent over $\mathbb{Q}$, then $e^{\lambda_1}, \ldots, e^{\lambda_n}$ are algebraically independent over $\mathbb{Q}$ **[Lindemann-Weierstrass (1885)]**

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- $\lambda_1 = \pi, \lambda_2 = \pi i$, algebraically independent π, e^{π} **[Nesterenko (1996)]**
- If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are algebraic numbers linearly independent over $\mathbb{Q}$, then $e^{\lambda_1}, \ldots, e^{\lambda_n}$ are algebraically independent over $\mathbb{Q}$ **[Lindemann-Weierstrass (1885)]**

Strong exponential closure

Remark

Assuming Schanuel's Conjecture the axiom of strong exponential closure for $(\mathbb{C}, e^x)$ is the only impediment to prove Zilber's Conjecture.

Simplest case

Given $p(x, y) \in \mathbb{C}[X, Y]$ irreducible where both x and y appear, is there a generic solution $p(z, e^z) = 0$?

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Infinite solutions

THEOREM (Marker)

If $p(x, y) \in \mathbb{C}[x, y]$ is irreducible and depends on x and y then $f(z) = p(z, e^z)$ has infinitely many zeros.

Proof

Follows from Hadamard Factorization theorem with Henson and Rubel's result (proved independently by Van den Dries) which said that the map from exponential terms to entire function is injective.

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(SC). If (z, e^z) and (w, e^w) are solutions of $p(x, y) \in \mathbb{Q}^{alg}[x, y]$ then z, w are algebraically independent over $\mathbb{Q}$.

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Generic solutions

Let f be the analytic function $f(z) = p(z, e^z, e^{e^z}, \ldots, e^{e^{e^{\cdots^{e^z}}}})$ over $\mathbb{C}$.

Definition

A solution a of f is generic over L (for L a finitely generated extension of $\mathbb{Q}$ containing the coefficients of p) if

$$t.d._L(a, e^a, e^{e^a}, \ldots, e^{e^{e^{\cdots^{e^a}}}})) = n,$$

where n is the number of iterations of exponentiation which appear in the polynomial p.

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Conjecture 17'

Conjecture 1. *Assuming Schanuel's Conjecture. Let* $p(x, y_1, \ldots, y_n) \in \mathbb{Q}^{alg}[x, y_1, \ldots, y_n]$ *a nonzero irreducible polynomial depending on* x *and the last variable* y_n*. Then*

$$p(z, e^z, e^{e^z}, \ldots, e^{e^{e^{\cdots^{e^z}}}})) = 0$$

has a generic solution.

Remark

Strong Exponential Closure in $\mathbb{C}$ implies a positive answer.

Past results: three iterations

Theorem (DFT)

(SC) Let $p(x, y_1, y_2, y_3) \in \mathbb{Q}^{alg}[x, y_1, y_2, y_3]$ be a nonzero irreducible polynomial depending on x and y_3. Then, there exists a generic solution of

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Three iterations

We consider when $f(z) = p(z, e^z, e^{e^z}, e^{e^{e^z}})$. The corresponding system in six variables $(z_1, z_2, z_3, w_1, w_2, w_3)$ is:

$$V = \begin{cases} p(z_1, z_2, z_3, w_3) = 0 \\ w_1 = z_2 \\ w_2 = z_3. \end{cases} \tag{1}$$

thought of as an algebraic set V in $G_2(\mathbb{C})$.

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(SC) If $p(x, y, z, w) \in \mathbb{Q}^{alg}[x, y, z, w]$ then the variety defined by V has a generic point.

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Proof

1. We prove that there exists a solution of p. In fact there are infinitely many.
2. The solution is a generic solution.

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Solutions of exponential polynomials over $\mathbb{C}$

Theorem (Katzberg)

A non constant polynomial $F(z) \in \mathbb{C}[z]^E$ has always infinitely many zeros unless it is of the form

$$F(z) = (z - \alpha_1)^{n_1} \cdot \ldots \cdot (z - \alpha_n)^{n_n} e^{g(z)},$$

where $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$, $n_1, \ldots, n_n \in \mathbb{N}$, and $g(z) \in \mathbb{C}[z]^E$.

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Theorem (D'Aquino, Fornasiero, T.)

Let $f(z) = p(z, e^z, e^{e^z}, \ldots, e^{e^{e^{\cdots^{e^z}}}})$, where $p(x, y_1 \ldots, y_k)$ is an irreducible polynomial over $\mathbb{C}[x, y_1 \ldots, y_k]$. Then the function f has infinitely many solutions in $\mathbb{C}$ unless $p(x, y_1 \ldots, y_k) = g(x) \cdot y_1^{n_{i_1}} \cdot \ldots \cdot y_k^{n_{i_k}}$, where $g(x) \in \mathbb{C}[x]$.

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It is an immediate consequence of Katzberg's result

No restrictions on the coefficients of $p(x, y_1 \ldots, y_k)$ and it is unconditionally (no Schanuel's Conjecture).

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Masser's result

Theorem (Masser)

Let $P_1(\overline{x}), \ldots, P_n(\overline{x}) \in \mathbb{C}[\overline{x}]$, where $\overline{x} = x_1, \ldots, x_n$. Then there exist $z_1, \ldots, z_n \in \mathbb{C}$ such that

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We have to show that the function $F : \mathbb{C}^n \to \mathbb{C}^n$ defined as $F(x_1, \ldots, x_n) = (e^{x_1} - P_1(x_1, \ldots, x_n), \ldots, e^{x_n} - P_n(x_1, \ldots, x_n))$ has a zero in $\mathbb{C}^n$.

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Lemma (Kantorovich)

Let $F : \mathbb{C}^n \to \mathbb{C}^n$ with $F(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n))$ be an entire function, and p_0 be such that $J(p_0)$, the Jacobian of F at p_0 is non singular. Let $\eta = |J(p_0)^{-1}F(p_0)|$ and U the closed ball of center p_0 and radius 2η. Let $M > 0$ be such that $|H(F)|^2 \leq M^2$ (where $H(F)$ denotes the Hessian of F). If $2M\eta|J(p_0)^{-1}| < 1$ then there is a zero of F in U.

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Generalization of Masser's result

A cone is an open subset $U \subseteq \mathbb{C}^n$ s. t. for every $1 \leq t \in \mathbb{R}$, if $\overline{x} \in U$ then $t\overline{x} \in U$.

Definition

An algebraic function is an analytic function $f : U \to \mathbb{C}$ s.t. there exists a nonzero polynomial $p(\overline{x}, u) \in \mathbb{C}[\overline{x}, u]$ with $p(\overline{x}, f(\overline{x})) = 0$ on all $\overline{x} \in U$. If, moreover, the polynomial p is monic in u, we say that f is integral algebraic.

Generalizzation of Masser's result

Theorem (Masser-DFT)

Let $f_1, \ldots, f_n : U \to \mathbb{C}$ be nonzero algebraic functions, defined on some cone U. Assume that $U \cap (2\pi i \mathbb{Z}^*)^n$ is Zariski dense in $\mathbb{C}^n$. Then

$$\begin{cases} e^{z_1} = f_1(z_1, \ldots, z_n) \\ e^{z_2} = f_2(z_1, \ldots, z_n) \\ \vdots \\ e^{z_n} = f_n(z_1, \ldots, z_n) \end{cases} \tag{3}$$

has a solution $\overline{a} \in U$.

Strong exponential closure

Theorem (Main Theorem DFT '18)

(SC) Let $V \subseteq \mathbb{C}^n \times (\mathbb{C}^*)^n$ an irreducible variety over $\mathbb{Q}^{alg}$ with $\dim V = n$. If $\pi_1(V)$ and $\pi_2(V)$ are dominant, then there exists a generic point of V of the form $(\overline{a}, e^{\overline{a}})$.

Remark

The result implies many cases of Zilber's Conjecture

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Positive answer to the conjecture

Remark

Let $p(x, y_1, \ldots, y_n) \in \mathbb{Q}^{alg}[x, y_1, \ldots, y_n]$ a nonzero irreducible polynomial depending on x and the last variable y_n. Let $p(z, e^z, e^{e^z}, \ldots, e^{e^{e^{\cdots^{e^z}}}}) = 0$, the corrisponding system in $2n$ variables is:

$$V = \begin{cases} p(x_1, \ldots, x_n, y_1, \ldots, y_n) = 0 \\ x_{i+1} = y_i \end{cases} \tag{4}$$

for all $i = 2, \ldots, n$.
V is a variety of dim $V = n$.

Positive answer to the conjecture

Main theorem implies Conjecture 1

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Let $V \subseteq \mathbb{C}^n \times (\mathbb{C}^*)^n$, over the algebraic closure of $\mathbb{Q}$.

Definition

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Existence of solutions

Theorem (Masser Brownawell-DFT)

Let $V \subseteq \mathbb{C}^n \times (\mathbb{C}^*)^n$ be an irreducible algebraic variety such that the projection onto the first coordinates $\pi_1(V)$ is Zariski dense in $\mathbb{C}^n$. Then the set $\{\overline{a} \in \mathbb{C}^n : (\overline{a}, e^{\overline{a}}) \in V\}$ is Zariski dense in $\mathbb{C}^n$.

Remark (1)

The result it is unconditionally (it doesn't use Schanuel's Conjecture)

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By Bays-Kirby result quasiminimality is true for $(\mathbb{C}, e^x)$ for some cases

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Proof of genericity

Ingredients:

1. Schanuel's conjecture
2. Masser's system

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Proof of genericity

Let be $(\overline{a}, e^{\overline{a}}) \in V$ and suppose that the point is not generic, i.e. $t.d._{\mathbb{Q}}(\overline{a}, e^{\overline{a}}) = m < n$.
Without loss of generality we can assume

$$|V_{\overline{a}}| < \infty \text{ and } |V^{e^{\overline{a}}}| < \infty$$

By Schanuel's Conjecture

$$l.d.(\overline{a}) \leq t.d._{\mathbb{Q}}(\overline{a}, e^{\overline{a}}) = m < n.$$

So, there exists $M \in \mathbb{Z}^n \times \mathbb{Z}^{n-m}$ such that $M \cdot \overline{a} = 0$.

Applying exponentiation we have the relation $e^{\overline{a}^M} = 1$. The above relations define:

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Let be $(\overline{a}, e^{\overline{a}}) \in V$ and suppose that the point is not generic, i.e. $t.d._{\mathbb{Q}}(\overline{a}, e^{\overline{a}}) = m < n$.
Without loss of generality we can assume

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In other words $\overline{a}$ is generic in L_M and $e^{\overline{a}}$ is generic in T_M.
We consider

$$W_N = \{(\overline{x}, \overline{y}) \in V : \overline{x} \in L_N \wedge \mid V_{\overline{x}} \mid < \infty \wedge \mid V^{\overline{y}} \mid < \infty\},$$

where $N \in \mathbb{C}^n \times \mathbb{C}^{n-m}$. If $N = M$ then $(\overline{a}, e^{\overline{a}}) \in W_M$.

$(W_N)_N$ is a definable family.

We observe that

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Let be W'_M the irreducible components of the Zariski closure of W_M containing the point $(\overline{a}, e^{\overline{a}})$.

Since $(\overline{a}, e^{\overline{a}}) \in W'_M$ is generic and $e^{\overline{a}} \in \pi_2(W'_M)$ then $\pi_2(W'_M) \subseteq T_M$, so its Zariski closure is contained in T_M.

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1. Weaken the hypothesis that $\pi_1(V)$ and $\pi_2(V)$ are dominant.
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