INTRODUCTION TO STABLE GROUPS - LECTURE NOTES

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Contents

1. Model-theoretic background: stability, forking, Morley rank

We work in a monster model $\mathfrak{C}$ of a complete theory T in a language L. We will assume T has **(strong) elimination of imaginaries**, that is, for any $\emptyset$-definable in T equivalence relation E on a definable set X, there is a $\emptyset$-definable set Y and a $\emptyset$-definable function $f : X \to Y$ such that $\models \forall x, y(E(x, y) \leftrightarrow f(x) = f(y))$.

Let $\mathop{\smile\!\!\!\!\!\!|}\,$ denote the relation of **forking independence**, that is, $A \mathop{\smile\!\!\!\!\!\!|}_C B$ if $\mathrm{tp}(A/BC)$ does not fork over C. If $A \subseteq B$, $p \in S(A)$ and $q \in S(B)$, then we say that q is a **non-forking extension** of p (and write $p \subseteq_{nf} q$) when $p \subseteq q$ and q does not fork over A.

A **global type** is a complete type over the monster model $\mathfrak{C}$.

A type $p \in S(A)$ is called **stationary** if it has only one global non-forking extension, which we denote by $\tilde{p}$.

Fact 1.1. *If T is stable, then $\mathop{\smile\!\!\!\!\!\!|}\,$ has the following properties:*

(1) *(Invariance)* $A \mathop{\smile\!\!\!\!\!\!|}_C B \iff f(A) \mathop{\smile\!\!\!\!\!\!|}_{f(C)} f(B)$ *for any* $f \in \mathrm{Aut}(\mathfrak{C})$.
(2) *(Symmetry)* $A \mathop{\smile\!\!\!\!\!\!|}_C B \iff B \mathop{\smile\!\!\!\!\!\!|}_C A$
(3) *(Monotonicity) If* $A' \subseteq A$ *and* $B' \subseteq B$ *then* $A \mathop{\smile\!\!\!\!\!\!|}_C B \Rightarrow A' \mathop{\smile\!\!\!\!\!\!|}_C B'$.
(4) *(Finite character)* $A \mathop{\smile\!\!\!\!\!\!|}_C B$ *iff* $A_0 \mathop{\smile\!\!\!\!\!\!|}_C B_0$ *for all finite* $A_0 \subseteq A$ *and* $B_0 \subseteq B$.
(5) *(Transitivity) If* $B_1 \subseteq B_2 \subseteq B_3$ *then* $A \mathop{\smile\!\!\!\!\!\!|}_{B_1} B_3$ *iff* $A \mathop{\smile\!\!\!\!\!\!|}_{B_1} B_2$ *and* $A \mathop{\smile\!\!\!\!\!\!|}_{B_2} B_3$.
(6) *(Normality)* $A \mathop{\smile\!\!\!\!\!\!|}_C B \iff A \mathop{\smile\!\!\!\!\!\!|}_C BC$.
(7) *(Stationarity) If A is algebraically closed, then any* $p \in S(A)$ *is stationary (remember we are assuming elimination of imaginaries). In particular, if M is a model then any* $p \in S(M)$ *is stationary.*
(8) *(Extension) For any* A, B, C *there is* $A' \equiv_C A$ *with* $A' \mathop{\smile\!\!\!\!\!\!|}_C BC$.
(9) *(Local character) There exists a cardinal λ such that for any a and B there is* $C \subseteq B$ *with* $|C| \leq \lambda$ *such that* $a \mathop{\smile\!\!\!\!\!\!|}_C B$. *In fact, one can take* $\lambda = |T|$.

Moreover, in any theory T, if there is a ternary relation $\mathop{\smile\hskip-0.9em\raise0.4ex\hbox{$\vert$}}^*$ satisfying the above properties, then T is stable and $\mathop{\smile\hskip-0.9em\raise0.4ex\hbox{$\vert$}}^* = \mathop{\smile\hskip-0.9em\raise0.4ex\hbox{$\vert$}}$.

Definition 1.2. The **Morley rank** of a formula ϕ defining a set S, denoted $\mathrm{RM}(\phi)$ or $\mathrm{RM}(S)$, is an ordinal or -1 or ∞, defined by first recursively defining what it means for a formula to have Morley rank at least α for some ordinal α:

- $\mathrm{RM}(S) \geq 0$ iff $S \neq \emptyset$.
- $\mathrm{RM}(S) \geq \alpha + 1$ iff there are pairwise disjoint definable subsets $(X_i)_{i<\omega}$ of S such that $\mathrm{RM}(X_i) \geq \alpha$ for each $i < \omega$.
- If λ is a limit ordinal then $\mathrm{RM}(S) \geq \lambda$ iff $\mathrm{RM}(S) \geq \alpha$ for every $\alpha < \lambda$.

Finally, $\mathrm{RM}(S) = \alpha$ when $\mathrm{RM}(S) \geq \alpha$ and for no $\beta > \alpha$ one has $\mathrm{RM}(S) \geq \beta$. Also, we set $\mathrm{RM}(S) = \infty$ if $\mathrm{RM}(S) \geq \alpha$ for every $\alpha \in Ord$. If $\mathrm{RM}(S) \in Ord$, then the **Morley degree** of S, denoted by $\mathrm{DM}(S)$, is the maximal number of definable sets of Morley rank $\mathrm{RM}(S)$ into which S can be partitioned.

If $\pi(x)$ is a (partial) type, we put $\mathrm{RM}(\pi(x)) := \min\{\mathrm{RM}(\phi(x)) : \pi(x) \vdash \phi(x)\}$ and $\mathrm{DM}(\pi(x)) := \min\{\mathrm{DM}(\phi(x)) : \pi(x) \vdash \phi(x), \mathrm{RM}(\phi(x)) = \mathrm{RM}(\phi(x))\}$.

A one-sorted structure M (or its theory $Th(M)$) is called **strongly minimal** if $\mathrm{RM}(x = x) = \mathrm{DM}(x = x) = 1$ where x is a single variable of the only sort of M. Equivalently, every definable subset of any model $\mathfrak{C} \models Th(M)$ is either finite of co-finite.

Exercise 1.3. *If $T = DLO_0$ is the theory of dense linear orders without endpoints, then for any $a < b$ we have $\mathrm{RM}(a < x < b) = \infty$.*

Fact 1.4. *Suppose X_1 and X_2 are definable. Then:*
(0) $\mathrm{RM}(X_1) = 0$ iff X_1 is finite and nonempty.
(1) If $X_1 \subseteq X_2$, then $\mathrm{RM}(X_1) \leq \mathrm{RM}(X_2)$.
(2) $\mathrm{RM}(X_1 \cup X_2) = \max(\mathrm{RM}(X_1), \mathrm{RM}(X_2))$.
(3) If there is a definable bijection between X_1 and X_2, then $\mathrm{RM}(X_1) = \mathrm{RM}(X_2)$.
(4) If $X_2 = f(X_1)$ for some $f \in \mathrm{Aut}(\mathfrak{C})$, then $\mathrm{RM}(X_1) = \mathrm{RM}(X_2)$.

Proof. (0),(2),(3),(4):Exercise.
(1): It is enough to prove that for every ordinal α, if $\mathrm{RM}(X_1) \geq \alpha$ then $\mathrm{RM}(X_2) \geq \alpha$. We shall prove this by induction on α. For $\alpha = 0$, if $\mathrm{RM}(X_1) \geq 0$ then $X_1 \neq \emptyset$, so $X_2 \neq \emptyset$ and hence $\mathrm{RM}(X_2) \geq 0$

For the inductive step, if $\mathrm{RM}(X_1) \geq \alpha + 1$ witnessed by pairwise disjoint definable subsets Y_i of X with $\mathrm{RM}(Y_i) \geq \alpha$, we get that the sets Y_i are also contained in X_2, hence they witness that $\mathrm{RM}(X_2) \geq \alpha + 1$.

If λ is a limit ordinal and $\mathrm{RM}(X_1) \geq \lambda$, then for any $\alpha < \lambda$ we have that $\mathrm{RM}(X_1) \geq \alpha$, hence by the inductive assumption $\mathrm{RM}(X_2) \geq \alpha$ as well. This shows that $\mathrm{RM}(X_2) \geq \lambda$. □

Definition 1.5. Let $X \neq \emptyset$ be a $\emptyset$-definable set. We say that a partial type $\pi(x)$ is **generic** in X, if $\pi(x) \vdash x \in X$ and $\mathrm{RM}(\pi(x)) = \mathrm{RM}(X)$.

Corollary 1.6. *Let $X \neq \emptyset$ be a set definable in T. Then any generic partial type $\pi(x) \vdash x \in X$ over A extends to a complete generic type in X over A.*

Proof. Let

$$p(x) := \pi(x) \cup \{\neg\phi(x) \in L(A) : \mathrm{RM}(\phi(x)) < \mathrm{RM}(G)\}.$$

If $p(x)$ is inconsistent, then there are $\phi_0(x), \dots, \phi_{n-1}(x)$ with $\mathrm{RM}(\phi_i(x)) < \mathrm{RM}(X)$ for each $i < n$ such that $\pi(x) \vdash \bigvee_{i<n} \phi_i(x)$. So there is some $\pi(x) \vdash \psi(x)$ such that

$\psi(x) \vdash \bigvee_{i<n} \phi_i(x)$. Then $\mathrm{RM}(\psi(x)) \geq \mathrm{RM}(\pi(x)) = \mathrm{RM}(X)$, but, by Fact 1.4(2), $\mathrm{RM}(\bigvee_{i<n} \phi_i(x)) = \max_{i<n} \mathrm{RM}(\phi_i(x)) < \mathrm{RM}(X)$. A contradiction to Fact 1.4(1).

Now any $p' \in S(A)$ extending $p(x)$ is a generic type in X extending $\pi(x)$. □

From now on let us assume that the language L is countable.

Definition 1.7. A theory T is called ω-stable if for any countable M we have $|S(M)| \leq \omega$.

Fact 1.8. *T is ω-stable iff $\mathrm{RM}(\phi(x)) < \infty$ for any formula $\phi(x)$.*

Fact 1.9. *Suppose T is ω-stable.*

- *If $p \in S(A)$ and $q \in S(B)$ with $p \subseteq q$, then $p \subseteq_{nf} q \iff \mathrm{RM}(p) = \mathrm{RM}(q)$.*
- *A type $p \in S(A)$ is stationary iff $\mathrm{DM}(p) = 1$.*

Example 1.10.
- *If $F \models ACF_p$ then $\mathrm{RM}(F) = 1 = \mathrm{DM}(F)$ (i.e. F is strongly minimal)*
- *If $K \models DCF_0$, then $\mathrm{RM}(K) = \omega$, and the field of constants of K is strongly minimal.*

2. Stable groups

Usually when we say a 'stable group' we tacitly fix some stable theory T with strong elimination of imaginaries in which G is definable, and work in the monster model $\mathfrak{C}$ of T.

Recall a theory T is stable iff it does not have **order property (OP)**, that is, there do not exist a formula $\phi(x; y)$ and parameters $(a_i, b_i)_{i<\omega}$ such that $\models \phi(a_i, b_j) \iff i < j$. Equivalently, there do not exist $\phi(x; y)$ and $(a_i, b_i)_{i<\omega}$ such that $\models \phi(a_i, b_j) \iff i \leq j$ ($\leq$ in place of $<$).

Proposition 2.1. *Let G be a stable semigroup with both left and right cancellation. Then G is a group.*

Proof. Let $a \in G$ and consider the formula $\phi(x, y) = \exists z xz = y$. As $G \models \phi(a^n, a^m)$ for any $n < m < \omega$ and G does not have OP, it follows that there are some $n \geq m$ with $\phi(a^n, a^m)$. Thus there is some $c \in G$ with $a^n c = a^m$. Put $e = a^{n-m}c$ (if $n = m$ we mean $e = c$). So $a^m e = a^m$.

Claim 1. *e is a neutral element in G.*

Proof. Take any $c \in G$. As $a^m e = a^m$, we also have $a^m ec = a^m c$. By cancellation, $ec = c$, so e is left-neutral. Similarly we can find right-neutral element e', but then $e = ee' = e'$, so e is both left- and right-neutral. □

By the claim $G \models \phi(a^n, a^m)$ also for $n = m$, hence $G \models \phi(a^n, a^m)$ for any $n \leq m < \omega$. Thus, again using that G does not have AP, we have $G \models \phi(a^n, a^m)$ for some $n > m$, so there is some $c \in G$ with $a^n c = a^m = a^m e$. By cancellation, $a^{n-m}c = e$, so $a' := a^{n-m-1}c$ is right-inverse to a. Similarly, we can find a left-inverse to a, call it a''. Then $a'' = a''aa' = a'$, so a' is inverse to a. □

Recall T is **NIP** if there is no formula $\phi(x; y)$ and parameters $a_{i<\omega}$, $(b_W)_{W\subseteq\omega}$ such that $\models \phi(a_i, b_W) \iff i \in W$.

Lemma 2.2. *(Baldwin-Saxl condition) Let G be a group with NIP. Then for any formula $\phi(x;y)$ there is $n_\phi < \omega$ such that for any groups $H_1, \ldots, H_n \leq G$ definable by instances of $\phi(x;y)$ the intersection $H_1 \cap \cdots \cap H_n$ is equal to the intersection at most n_ϕ-many groups among $H_1, \ldots, H_n$.*

Proof. Suppose not, so for any $n < \omega$ we have $H_1, \ldots, H_n$ whose intersection is not equal to the intersection of any $n-1$ of them. Thus for any $i \in \{1, 2, \ldots, n\}$ there is some $g_i \in \bigcap_{j \neq i} H_j \setminus \bigcap_{j \in \{1,\ldots,n\}} H_j = \bigcap_{j \neq i} H_j \setminus H_i$. For $W \subseteq \{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\}$ with $i_1 < \cdots < i_m$ put $g_W := g_{i_1} \ldots g_{i_m}$. Then $g_W \in H_i \iff i \notin W$. As n was arbitrary, this together with compactness gives that $\phi(x;y)$ has IP. □

Corollary 2.3. *(Chain condition on intersections) Let G be a stable group and $\phi(x,y)$ a formula. Then there is no strictly descending chain of subgroups of G*

$$H_0 \geq H_1 \geq H_2 \ldots$$

each of which is an intersection of finitely many groups defined by an instance of $\phi(x,y)$. Moreover, there is a uniform bound on the length of such finite chain.

Proof. By Lemma 2.2 all H_i's are definable by instances of the formula $\psi(x; y_1, \ldots, y_{n_\phi}) := \bigwedge_{i \in \{1,\ldots,n_\phi\}} \phi(x; y_i)$. As G is stable, ψ does not have strict order property, i.e. its instances cannot form an infinite chain under inclusion, and by compactness there is a bound on the length of such a chain. □

3. ω-stable groups

We assume G is an ω-stable group.

Proposition 3.1. *(Descending chain condition) There is no strictly descending chain $G \geq H_0 \geq H_1 \geq \ldots$ of definable subgroups of G.*

Proof. Suppose there is such a chain. Then for any i we have that either $\mathrm{RM}(H_{i+1}) < \mathrm{RM}(H_i)$ or $\mathrm{RM}(H_{i+1}) = \mathrm{RM}(H_i)$ and $\mathrm{DM}(H_{i+1}) < \mathrm{DM}(H_i)$). This means that $(\mathrm{RM}(H_i), \mathrm{DM}(H_i))_{i<\omega}$ is a strictly descending sequence of elements the well-ordered class $(\mathrm{Ord} \times \omega, <_{\mathrm{lex}})$, where $<_{\mathrm{lex}}$ is the lexicographic order. This is a contradiction. (more explicitly, the non-increasing sequence $(\mathrm{RM}(H_i))_{i<\omega}$ or ordinals must stabilise from some point on, so from that point on $D(H_i)$ is a decreasing sequence of natural numbers, a contradiction). □

Corollary 3.2. *G has a smallest definable subgroup of finite index, called the* **connected component** *of G and denoted G^0.*

Proof. If not, then we can inductively find a decreasing chain of definable subgroups of G of finite index, contradicting Proposition 3.1. □

Exercise 3.3. *G^0 is normal in G and invariant under automorphisms of G.*

Lemma 3.4. *G has at most $\mathrm{DM}(G)$-many global generic types.*

Proof. If $p_1, \ldots, p_n$ are global generic types, then there are pairwise inconsistent formulas $\phi_1 \in p_1, \ldots, \phi_n \in p_n$. Then $\phi_i(x) \wedge x \in G \in p_i$, so we get by genericity of p_i that $\mathrm{RM}(\phi_i(x) \wedge x \in G) \geq \mathrm{RM}(G)$ for every i. This shows that $\mathrm{DM}(G) \geq n$. □

Note that G acts naturally on the set $S(\mathfrak{C}) \cap [x \in G]$ of global types in G by: $g \cdot \mathrm{tp}(a/G) = \mathrm{tp}(ga/G)$ (here a comes from a bigger monster model $\mathfrak{C}' \succ \mathfrak{C}$). More generally, G acts on the set of stationary types: for any stationary $p \in S(A) \cap [x \in G]$, define $g \cdot p := \mathrm{tp}(g \cdot a/A)$ for $a \models p$ such that $a \mathop{\smile\!\!\!\!\!\!\mid}_A g$.

Exercise 3.5. *Prove that this is indeed a well-defined action.*

Definition 3.6. Let $p \in S(A) \cap [x \in G]$ be stationary. Then we define the stabiliser of p in G as follows: $\mathrm{Stab}_G(p) := \{g \in G : g \cdot p = p\}$.

Exercise 3.7. *If p is stationary type with the unique global non-forking extension $\tilde{p}$, then* $\mathrm{Stab}_G(p) = \mathrm{Stab}_G(\tilde{p})$.

Exercise 3.8. $\mathrm{Stab}_G(p)$ *is a definable subgroup of G for any stationary type $p \in S(A) \cap [x \in G]$.*

Exercise 3.9. $\mathrm{RM}(a/A, g) = \mathrm{RM}(g \cdot a/A, g)$ *for any $a, g \in G$ and any parameter set A.*

By Exercise 3.9, if p is a stationary generic type then so is $g \cdot p$. Thus G acts on the set of generic stationary types in G.

Proposition 3.10. *If $p \in S(A) \cap [x \in G]$ is a generic stationary type, then* $\mathrm{Stab}_G(p)$ *has finite index in G.*

Proof. By Lemma 3.4, G has only finitely many generics over A (as each of them extends to a different global generic). Thus $G \cdot p$ is finite, so $[G : \mathrm{Stab}_G(p)]$ is finite by the Orbit-Stabiliser Theorem. □

Proposition 3.11. *If G is connected, then it has only one global generic type.*

Proof. Let $\mathrm{tp}(a/\mathfrak{C})$ and $\mathrm{tp}(b/\mathfrak{C})$ be global generic types in G (here a, b live in some bigger monster model $\mathfrak{C}' \succ \mathfrak{C}$). We may assume $a \perp_{\mathfrak{C}} b$. Thus $a \cdot \mathrm{tp}(b/\mathfrak{C}) = \mathrm{tp}(a \cdot b/\mathfrak{C})$. As G is connected, we get by Proposition 3.10 that $\mathrm{Stab}_G(p) = G$, so $a \in \mathrm{Stab}_G(p)$ and hence $\mathrm{tp}(a \cdot b/\mathfrak{C}) = \mathrm{tp}(b/\mathfrak{C})$. Similarly, $b^{-1} \in \mathrm{Stab}_G(\mathrm{tp}(a^{-1}/\mathfrak{C}))$, so $\mathrm{tp}(a^{-1}/\mathfrak{C}) = \mathrm{tp}(b^{-1}a^{-1}/\mathfrak{C})$, so applying $^{-1}$ we get that $\mathrm{tp}(a/\mathfrak{C}) = \mathrm{tp}(a \cdot b/\mathfrak{C}) = \mathrm{tp}(b/\mathfrak{C})$. □

Note that, as the $[G : G^0] < \omega$, we can find representatives $g_1, \ldots, g_d$ of all cosets of G_0 in $G(\mathfrak{C})$, hence every global type contains the formula $x \in g_i \cdot G^0$ for some i.

Corollary 3.12. *In each coset of G^0 in G there is exactly one global generic type of G.*

Proof. As any coset of G_0 has the same Morley rank as G, it has at least one generic type of G. If p and q were distinct global generic types in a coset $g \cdot G^0$, then $g^{-1} \cdot p$ and $g^{-1} \cdot G$ would be distinct generic types in G^0, a contradiction to Proposition 3.11, □

Corollary 3.13. $\mathrm{DM}(G) = [G : G^0]$.

Recall that in a stable theory T, if $a \perp_M b$ where $M \models T$, then $\mathrm{tp}(a/MB)$ is finitely satisfiable in M (i.e. any formula in $\mathrm{tp}(a/Mb)$) has a realisation in M.

Proposition 3.14. *For any global type p in G we have* $\mathrm{RM}(p) \geq \mathrm{RM}(\mathrm{Stab}_G(p))$, *and if equality holds then* $\mathrm{Stab}_G(p)$ *is connected.*

Proof. Let $a \models p$ and $b \models q$ where q is the global generic type in $\mathrm{Stab}_G(p)^0$, with $a \perp_{\mathfrak{C}} b$. Then

$$\mathrm{RM}(\mathrm{Stab}_G(p)) = \mathrm{RM}(b/\mathfrak{C}) = \mathrm{RM}(b/\mathfrak{C}, a) = \mathrm{RM}(b \cdot a/\mathfrak{C}, a) \leq \mathrm{RM}(b \cdot a/\mathfrak{C}) = \mathrm{RM}(p)$$

where the last equality holds as $b \cdot p = p$.

Now if $\mathrm{RM}(b/\mathfrak{C}) = \mathrm{RM}(p)$, then we must have $\mathrm{RM}(b \cdot a/\mathfrak{C}, a) = \mathrm{RM}(b \cdot a/\mathfrak{C})$ so $b \cdot a \perp_{\mathfrak{C}} a$, so $\mathrm{tp}(a/\mathfrak{C}, b \cdot a)$ is finitely satisfiable in $\mathfrak{C}$. So, as the formula $(b \cdot a)^{-1} \cdot x^{-1} \in \mathrm{Stab}_G^0(p)$ belongs to $\mathrm{tp}(a/\mathfrak{C}, b \cdot a)$, it has a realisation $c \in \mathfrak{C}$. So $b \cdot a \in \mathrm{Stab}_G(p)^0 \cdot c$, hence, as $b \cdot a \models p$, we have $(x \in \mathrm{Stab}_G(p)^0 \cdot c) \in p$. Now if $\mathrm{Stab}_G(p)$ is not connected, then there is some $d \in \mathrm{Stab}_G(p) \setminus \mathrm{Stab}_G(p)^0$. But then, as $d \cdot \mathrm{Stab}_G(p)^0 \cdot c$ is disjoint from $\mathrm{Stab}_G(p)^0 \cdot c$, we get that $d \cdot p \neq p$, a contradiction. □

4. Zilber's Indecomposability Theorem

Definition 4.1. We say a definable set $A \subseteq G$ is **indecomposable**, if for any definable group $H \leq G$ we have $|A/H| > 1 \Rightarrow |A/H| \geq \omega$.

Proposition 4.2. *Any definable subset of G is a union of finitely many disjoint indecomposable sets.*

Proof. Follows essentially from the descending chain condition. □

Theorem 4.3. *(Zilber's Indecomposability Theorem) Suppose* $\mathrm{RM}(G) < \omega$ *and* $(A_i)_{i\in I}$ *is a family of indecomposable subsets of G with $e \in A_i$ for every $i \in I$.The then group $H := \langle A_i, i \in I\rangle$ is definable and connected. Moreover, $H = (A_{i_1} \cdot)^2$ for some $m \leq$* $\mathrm{RM}(G)$ *and $i_1, \dots, i_m \in I$.*

Proof. We can choose $i_1, \dots, i_m \in I$ with $m \leq \mathrm{RM}(G)$ such that for $B := A_{i_1} \dots A_{i_m}$ we have

$$\mathrm{RM}(A_i \cdot B) = \mathrm{RM}(B)$$

for every $i \in I$. Let $p \in S(\mathfrak{C}) \cap [x \in B]$ be such that $\mathrm{RM}(p) = \mathrm{RM}(B)$, and $H := \mathrm{Stab}_G(p)$.

Claim 2. *$A_i \subseteq H$ for every $i \in I$.*

Proof. If not, then, as $e \in A_i$, we have $|A_i/H| > 1$, hence $|A_i/H| \geq \omega$ by indecomposability of A_i, witnessed by some $(a_j)_{j<\omega}$ in A_i with $a_j^{-1}a_{j'} \notin H$ for $j \neq j'$. Hence $a_j^{-1}a_{j'} \cdot p \neq p$ so $a_{j'} \cdot p \neq a_j \cdot p$ for $j \neq j'$. But $a_j \cdot p \in S(\mathfrak{C}) \cap [x \in A_i \cdot B]$, so $\mathrm{RM}(A_i \cdot B) > \mathrm{RM}(p) = \mathrm{RM}(B)$, contradiction to the choice of B. □

By the claim we have $B \subseteq H$, hence $p \vdash (x \in H)$, so $\mathrm{RM}(p) \leq \mathrm{RM}(H)$, hence, by Proposition 3.14 we have $\mathrm{RM}(p) = \mathrm{RM}(H)$ and H is connected. Hence $\mathrm{RM}(H \setminus B) < \mathrm{RM}(B)$, which easily implies that $H = B^2$ (as any element of H is a product of two generics in H over the parameters of B). □

Remark 4.4.
- The assumption $e \in A_i$ cannot be omitted: otherwise we could obtain any group as $\langle A_i, i \in I\rangle$ taking A_i to be singletons.
- ZIT does not hold for ω-stable groups.
- We do not assume I to be small, in particular, it can be an infinite definable set.

Corollary 4.5. *If $A \subseteq G$ is indecomposable, then $\langle A^{-1} \cdot A\rangle$ is definable and connected.*

Proof. Apply ZIT to $(a^{-1} \cdot A)_{a\in A}$ (note $a^{-1} \cdot A$ is indecomposable for every a). □

Exercise 4.6. *Let S be a definable group acting definably on an ω-stable group G. If $A \subseteq G$ is S-invariant and such that $|A/H| > 1 \Rightarrow |A/H| \geq \omega$ for every S-invariant definable group $H \leq G$, then A is indecomposable.*

Proof. Let $H \leq G$ be definable and such that $1 < |A/H| < \omega$. As H is S-invariant, for any $s \in S$ we have $|s \cdot A/H| = |s \cdot A/s \cdot H| = |A/H| < \omega$. By the descending chain condition $\bigcap_{s\in S} s \cdot H = \bigcap_{s\in S_0} s \cdot H$ for some finite $S_0 \subseteq S$. Hence $1 < |A/\bigcap_{s\in S} s \cdot H| < |A/H|^{|S_0|} < \omega$, a contradiction to the assumption. □

Recall a group G is called **simple** when it has no non-trivial proper normal subgroup.

Definition 4.7. We say a group G is **definably simple** if it has no non-trivial proper definable normal subgroup.

Corollary 4.8. *Let G be a group of finite Morley rank. If G is definably simple and non-abelian, then G is simple.*

Proof. We may assume G is infinite (if G is finite, every normal subgroup of G is definable). So, as G^0 is definable and normal in G, we must have $G^0 = G$ (as we cannot have $G^0 = \{e\}$ for infinite G). Let $a \in G \setminus \{e\}$. If a^G is finite, then $[G : C(a)] < \omega$, hence $C(a) = G$ by connectedness, so $Z(G) \neq \{e\}$ and hence $Z(G) = G$ as $Z(G)$ is a definable normal subgroup of G. A contradiction to the non-abelianity assumption.

If a^G is infinite, then $\{e\} \cup a^G$ is indecomposable by Exercise 4.6 applied to the conjugation action of G (so we only need to check the indecomposability of a^G with respect to $H = G$ and $H = \{e\}$, which clearly holds). Thus by ZIT $\langle a^G \rangle = \langle a^G \cup \{e\} \rangle$ is definable; as it is also normal, we must have $\langle a^G \rangle = G$. This clearly implies that G is normal. □

Corollary 4.9. *Let G be a simple group of finite Morley rank. Then G is almost strongly minimal, i.e. there exist strongly minimal sets $S_1, \ldots, S_n$ with $G \subseteq \mathrm{acl}(S_1 \cup \cdots \cup S_n)$.*

Proof. Let $A \subseteq G$ be a strongly minimal definable subset. By 4.2 we may assume A is indecomposable. Pick some $a \in A$ and let $B := a^{-1} \cdot A$. Then for any $g \in G$ we have that B^g $(:= g^{-1} \cdot B \cdot g)$ is indecomposable and $e \in B^g$, so by ZIT $H := \langle \bigcup_{g \in G} B^g \rangle = B^{g_1} \cdot \cdots \cdot B^{g_n}$ for some $g_1, \ldots, g_n$. As H is clearly normal in G, we must have that $B^{g_1} \cdot \cdots \cdot B^{g_n} = G$, so $G \subseteq \mathrm{acl}(B^{g_1}, \ldots, B^{g_n})$. As $B^{g_1}, \ldots, B^{g_n}$ are strongly minimal, we are done. □

Fact 4.10. *(Borovik) Let G be a group of finite Morley rank. Then* RM *in G satisfies for any definable $A \subseteq G^k$, $B \subseteq G^l$:*

- $\mathrm{RM}(X) \geq n + 1$ *iff there are pairwise disjoint definable $X_0, X_1, \cdots \subseteq X$ with* $\mathrm{RM}(X_i) \geq n$ *for every $i < \omega$ (this is just by definition)*
- *(definability) For any formula $\phi(x, c)$ and any $n < \omega$, the set $\{c \in \mathfrak{C} : \mathrm{RM}(\phi(x, c) = n)\}$ is definable.*
- *(additivity) If $f : X \to Y$ is a definable function and $\mathrm{RM}(f^{-1}(y)) = n$ for every $y \in Y$, then $\mathrm{RM}(X) = n + \mathrm{RM}(Y)$.*
- *(elimination of $\exists^\infty$) If $f : A \to B$ is definable, then there is m such that fibers of f of size $> m$ are infinite.*

Fact 4.11. *If in a group G there is a rank with the above properties, then $\mathrm{RM}(G) < \omega$ and the rank coincides with Morley rank.*

Corollary 4.12. *If $\mathrm{RM}(G) < \omega$ and $H \leq G$ is definable, then $\mathrm{RM}(G) = \mathrm{RM}(H) + \mathrm{RM}(G/H)$.*

5. Fields

5.1. **Macintyre's theorem.** Let K be an infinite ω-stable field (we allow additional structure on K, unless we call K a pure field). We aim to prove that K is algebraically closed.

Lemma 5.1. $\mathrm{DM}(K) = 1$, *hence $(K, +)$ and $(K^*, \cdot)$ are connected groups.*

Proof. Consider any $a \in K^*$. Then $x \mapsto a \cdot x$ is a definable automorphisms of $(K, +)$, hence $a \cdot (K, +)^0 = (K, +)^0$. This shows that $(K, +)^0$ is an ideal of K, hence equals K, so $(K, +)$ is connected. So $\mathrm{DM}(K) = 1$ by Proposition 3.13, hence also $\mathrm{DM}(K^*) = 1$ and hence $(K^*, \cdot)$ is connected as well. □

Lemma 5.2. *For any $n > 0$ and $a \in K$ the polynomial map $x^n - a$ is surjective on K, and if $char(K) = p > 0$, then $x^p - x$ is surjective on K as well.*

Hence K is perfect, Artin-Schreier closed and Kummer closed. Moreover, the same holds for any finite extension of K.

Proof. Let $a \in K^*$ be a generic. As a and a^n are interalgebraic, we get $\mathrm{RM}(a^n) = \mathrm{RM}(a)$, so a^n is generic in K^*. Hence, as $a^n \in (K^*)^n = \{x^n : x \in K^*\}$ and $(K^*)^n$ is a definable subgroup of $(K^*, \cdot)$, we must have that $(K^*)^n$ is a finite-index subgroup of K^*, thus, as K^* is connected, $(K^*)^n = K^*$, so $K^n = K$.

Similarly, if $char(K) = p > 0$, we have that a and $a^p - a$ are interalgebraic, and, as $f(x) = x^p - x$ is an additive homomorphism, we get that $f(K) \ni a^p - a$ is a subgroup of finite index in $(K, +)$, hence equals K.

The "moreover" clause follows as any finite extension of K is interpretable in K, and hence has finite Morley rank as well (by Fact 4.10(3)). □

Exercise 5.3. *Show that indeed any finite extension of K is interpretable in K.*

Now we will use Galois theory to conclude that K is algebraically closed. We will use the following fact.

Fact 5.4.
- *If $char(F) = p > 0$ and L/F is a cyclic extension of degree p, then L/F is an Artin-Schreier extension.*
- *If L/F is a cyclic extension of degree n, p does not divide n and F contains all n-th roots of 1, then L/F is a Kummer extension*

First, we claim that K contains all roots of unity. Suppose not, and n is minimal such that K does not contain some primitive n-th root of unity a. Then $K(a)$ is a normal separable extension of K of degree strictly less than n. We can find $L \subseteq K(a)$ with L/K cyclic of some order m with m either equal or coprime to p. By minimality, K contains all m-th roots of unity. Hence by Fact 5.4, K is either an Artin-Schreier or Kummer extension of K, a contradiction.

Suppose K is not algebraically closed, so it has a normal extension L of a finite degree n, which is also separable as K is perfect. Let H be a subgroup of $Gal(L/K)$ of a prime order q, and let $F = L^H$ be its field of invariants, so L is an extension of F of degree q. If $q \neq char(F)$, then since L contains all roots of unity, L is a Kummer extension of F, a contradiction. If $q = char(F)$, then L is an Artin-Schreier extension of F, a contradiction.

Remark 5.5. Superstable fields are also algebraically closed. Stable fields are not necessarily algebraically closed, but are conjectured to be separably closed.

5.2. **Fields of finite Morley rank.** Recall we allow additional structure on fields.

Exercise 5.6. *If K is a field of finite Morley rank, then K has no infinite proper definable subring.*

Corollary 5.7. *Suppose* $\mathrm{RM}(K) < \omega$ *and $char(K) = 0$.*

(1) *$(K, +)$ has no nontrivial proper definable subgroups.*
(2) *All definable endomorhpisms of $(K, +)$ are of the form $x \mapsto a \cdot x$ for some $a \in K$.*

Proof. (1) Suppose $A \subseteq K$ is a nontrivial definable additive subgroup. Let $R := \{a \in K : a \cdot A \subseteq A\}$. Then $\mathbb{Z} \subseteq R$ and R is a definable subring of K, so $R = K$ by Exercise 5.6, so $A \trianglelefteq K$ so $A = K$.
(2) Let h be a definable endomorphism of $(K, +)$. Let $R := \{a \in A : (\forall x \in K) h(a \cdot x) = a \cdot h(x)\}$. Again $\mathbb{Z} \subseteq R$, so h is K-linear, hence of the form $x \mapsto a \cdot x$ for $a := h(1)$. □

Remark 5.8. There exist fields of Morley rank 2 with an infinite definable proper multiplicative subgroup.

Corollary 5.9. *If* $\mathrm{RM}(K) < \omega$ *and* $char(K) = 0$*, then* K *is definably rigid, i.e. it has no nontrivial definable automorphism.*

Proof. If f were a nontrivial definable automorphism, then $Fix(f) \supseteq \mathbb{Z}$ would be a nontrivial proper definable subgroup of $(K, +)$ (in fact, $Fix(f)$ is always a definable subfield of K). □

Note that if $char(K) = p > 0$, then the Frobenius map and its powers are definable automorphisms of K.

Proposition 5.10. *Let* K *be an infinite field of finite Morley rank. Then there is no nontrivial definable group of automorphisms of* K*.*

Proof. Suppose S is such a group. By Corollary 5.9, we have $char(K) = p > 0$. any $s \in S \setminus \{e\}$ we have that $Fix(s) := \{a \in K : s \cdot a = a\}$ is finite, as otherwise it would be proper infinite definable subfield of K, which does not exist by Exercise 5.6. Hence S embeds into $Aut(\mathbb{F}_p^{alg})$. As $Aut(\mathbb{F}_p^{alg})$ is torsion-free, so is S. Let $s \in S \setminus \{e\}$. Then $s^{2^n} \neq e$ for any $n < \omega$, and we have that $Fix(s^{2^n})$ is a proper subset of $Fix(s^{2^{n+1}})$. So we obtain uniformly definable finite sets $Fix(s^{2^n})$ of arbitrarily large cardinality, contradicting Fact 4.10(3). □

6. Groups of finite Morley rank

Definition 6.1. Let G be a definable group acting definably on an abelian group A. We say that A is **minimal** if for any definable subgroup $B \leq A$, if $G \cdot B \subseteq B$, then either B is finite or $B = A$.

Theorem 6.2. *Suppose an infinite group* M *acts definably and faithfully on an abelian group* A *in a structure of finite Morley rank. Suppose* A *is* M*-minimal. Then there is a definable field* K *such that* A *is definably isomorphic to* $(K, +)$ *and* M *embeds in* $(K^*, \cdot)$ *and acts on* $A = (K, +)$ *by scalar multiplication.*

Proof. Write M multiplicatively and A additively.

As $Stab_M(A) = \{e\}$, we have by DCC that $Stab_M(a_1, \dots, a_n) = \{e\}$ for some $a_1, \dots, a_n \in A$. Hence the action of an $m \in M$ is determined by $(ma_1, \dots, ma_n)$. As M is infinite, it follows that there is $i \leq n$ with $M \cdot a_i$ infinite. Put $a := a_i$. By Exercise 4.6 and minimality $M \cdot a \cup \{0\}$ is indecomposable. Hence by ZIT $\langle M \cdot a\rangle = \{m_1 \cdot a + \cdots + m_k \cdot a : m_i \in M\}$ for some $k < \omega$. $\langle M \cdot a\rangle$ is M-invariant, so $\langle M \cdot a\rangle = A$. Let R be the endomorphisms ring of A generated by M. As M is commutative so is R. As A is generated by a as an R-module, it follows that an element of $r \in R$ is determined by $r(a)$ (if $r(a) = r'(a)$ then for every $r'' \in R$ we have $r(r''a) = r''(ra) = r''(r'a) = r'(r''a)$). Thus $R = \{m_1 + \cdots + m_n : m_i \in M\}$ is an interpretable ring (exercise).

We claim that R has no divisors of 0. Let $0 \neq r \in R$. Note that $\ker(R)$ is an M-invariant subgroup of M, hence it is finite (as otherwise $\ker(R) = A$ and $r = 0$). Thus $\mathrm{im}(r)$ (which is also an M-invariant subgroup of A) is infinite, and hence, by minimality of A, we have $\mathrm{im}(r) = A$. This implies R has no divisors of 0. Hence, by stability, $K := R$ is a field. As K acts on A, A is a linear space over K, and $\dim_K(A) = 1$ as $K \cdot a = A$ (even $M \cdot a = A$). Clearly $((M, \cdot) \leq (K^*, \cdot)$ acts on A by scalar multiplication. □

Theorem 6.3. *(Nesin) Let a connected group G act faithfully and definably on an abelian group A in a structure of finite Morley rank. Suppose $M \triangleleft G$ is infinite and definable, and $B \leq A$ is M-invariant and M-minimal, and A is generated by $\bigcup_{g\in G} g \cdot B$. Then there is a definable field K and a definable structure of a finite-dimensional K-vector space on A such that G acts linearly on A and M acts K-scalarly on A.*

Proof. We claim first that the action of M^0 on B is nontrivial: otherwise, for any $g \in G$ we have that $M^g = gM^0g^{-1}$ acts trivially on gB, so it acts trivially on A, so $M^0 = \{e\}$ by faithfulness, a contradiction as M is assumed to be infinite.

Thus $M/Fix_M(B)$ is an infinite abelian group acting faithfully on B, so by the previous theorem, if we let R be the ring of endomorphisms of B generated by M, then R acts K-scalarly on B for some definable field K. Then $K = R/\mathrm{ann}_R(B)$, so $\mathrm{ann}_R(B)$ is a maximal ideal of R, and so is $gIg^{-1} = \mathrm{ann}_R(gB)$ for any $g \in G$.

For any $g_1, \dots, g_n \in G$, we have that $I_1 = g_1Ig_1^{-1}, \dots, I_n = g_nIg_n^{-1}$ are maximal ideals. Hence, if they are pairwise distinct, then they are pairwise coprime, so $g_1B + \cdots + g_nB$ is a direct sum: if $x_1 + \cdots + x_n = 0$, then by the Chinese Remainder Theorem we can find $r \in \bigcap_{j\neq i} \mathrm{ann}_R(I_j)$ with $r \in 1 + \mathrm{ann}_R(I_i)$, so $x_i = r(x_1 + \cdots + x_n) = r \cdot 0 = 0$ for each i.

Hence by finiteness of Morley rank we have that $\{g \cdot I : g \in G\}$ is finite. As G acts definably and transitively on this finite set, we must actually have $gI = I$ for all $g \in G$. Thus I annihilates $\langle \bigcup_{g\in G} g \cdot B\rangle$, so $I = 0$. So $R = K$ is definable and acts definably on A (as $\mathrm{ann}(B) = \mathrm{ann}(A)$ so the action of an element of K on A is determined by its action on B, hence by its action on a single element of B). Also $dim_K(A) < \omega$ by finiteness of Morley rank. Finally, the action of G on M by conjugation induces a definable group of automorphisms of K, which must by trivial by Proposition 5.10. Hence the action of G on A is K-linear: for $g \in G$, $r \in K$ and $a \in A$ we have $g \cdot r \cdot a = g \cdot r \cdot g^{-1} \cdot g \cdot a = r \cdot g \cdot a$. □

Exercise 6.4. *Prove that if $Z(G)$ is finite and G is connected, then $Z(G/Z(G)) = \{e\}$.*

Definition 6.5. A definable group is **minimal** if it has no proper infinite definable subgroup.

Theorem 6.6. *(Reinecke) Let G be a connected, minimal ω-stable group. Then G is abelian.*

Proof. If $Z(G) = G$ we are done, so by minimality we may assume $Z(G)$ is finite. Then $Z(G/Z(G))$ is trivial by Exercise 6.4, so replacing G by $G/Z(G)$ we may assume that $Z(G) = \{e\}$.

Claim 3. *G has only one nontrivial conjugacy class.*

Proof. Take any $a \in G \setminus \{e\}$. Then $C_G(a)$ is finite, as otherwise by minimality of G it would be equal to G. Let $b \in G$ be a generic over a. Then b belongs to the finite set $b \cdot C_G(a) = \{x \in G : a^x = a^b\}$, so $b \in \mathrm{acl}(a, a^b)$, hence $\mathrm{RM}(b/a) \leq \mathrm{RM}(a^b/a)$, so a^b is a generic in G over a. Hence, as a^G is a definable over a and contains a^b, we must have that a^G is a generic subset of G. Similarly, a'^G is generic in G. Thus, as G is connected (and so $\mathrm{DM}(G) = 1$), we must have $a^G \cap a'^G \neq \emptyset$, so $a^G = a'^G$. □

By the claim, either $\forall x \in G x^2 = e$ or $\forall x \in G(x^2 = e \implies x = e)$. As the former implies that G is abelian, we may assume the latter. Let $a \in G\setminus\{e\}$. By the claim there is $c \in G$ with $a^c = a^{-1}$. Then $a \notin C_G(c)$, but $a^{c^2} = a$ so $a \in C_G(c^2)$ and so $C_G(c) \supsetneq C_G(c^2)$.

Now, as c is conjugate to c^2 in G, applying an inner automorphism sending c to c^2 we get that $C_G(c^2) \supsetneq C_G(c^4)$, and continuing applying this inner automorphisms we get

$$C_G(c) \supsetneq C_G(c^2) \supsetneq C_G(c^4) \supsetneq C_G(c^8) \supsetneq \dots,$$

contradicting NSOP. □

Corollary 6.7. (1) *If* RM$(G) = 1$, *then G^0 is abelian, so G is virtually abelian (i.e. it has an abelian subgroup of finite index).*
(2) *G is ω stable, then G has an infinite abelian subgroup.*

Theorem 6.8. *Suppose* RM$(G) < \omega$ *and $H \leq G$ is definable and connected. Then for any $A \subseteq G$, the group $[H, A] := \langle [h, a] : h \in H, a \in A\rangle$ is definable and connected, and equals $[H, a_1] \cdots [H, a_n]$ for some $a_1, \dots, a_n \in A$.*

Proof. Let $a \in A$.

Claim 4. *a^H is indecomposable in G.*

Proof. Let $K \leq G$ be definable and such that $K^h = K$ for every $h \in H$. Let

$$H_a := \{x \in H : a^h K = aK\}$$

Then F is a definable subgroup of H, and for any $x, y \in H$ we have that $a^x K = a^y K \iff xy^{-1} \in H_a$. Thus, if $|a^H/K| < \omega$ then $[H : H_a] < \omega$, so $H_a = H$ by connectedness of H, and hence $|a^H/K| = |\{aK\}| = 1$. □

By the claim and ZIT applied to the indecomposable sets $a^H a^{-1} = [H, a] \ni e$, we get that $[H, A] = \langle \bigcup_{a\in A}[H, a]\rangle = [H, a_1] \cdots [H, a_n]$ for some $a_1, \dots, a_n \in A$, and $[H, A]$ is a definable connected group. □

Applying Theorem 6.8 iteratively, we get:

Corollary 6.9. *Let G be a connected group with* RM$(G) < \omega$. *Then $G' = [G, G]$, $G'' = [G', G']$,... are connected, and $\Gamma_n(G)$ is connected for every n as well, where we define $\Gamma_{n+1}(G) := [\Gamma_n(G), G]$.*

Corollary 6.10. *If* RM$(G) < \omega$ *and G is connected, then the sequence $(\Gamma_n(G))_n$ stabilises after finitely many steps.*

Fact 6.11. *(Cherlin) If* RM$(G) \leq 2$ *and G is connected, then G is solvable (of step ≤ 3, that is $G''' = \{e\}$).*

Recall the algebraicity conjecture:

Conjecture 6.12. *(Cherlin-Zilber) Every simple group of finite Morley rank is an algebraic group over an algebraically closed field.*

A natural related question is: when does a finite Morley rank interpret an infinite field? This can be split into subcases with respect to how algebraically complicated G is.

- If G is virtually abelian, it never happens (roughly, by 1-basedness of G).
- If G is virtually nilpotent but not virtually abelian, there is an example by Baudisch where G does not interpret an infinite field; there are partial positive results here as well. (e.g. when $[G, G]$ is nontorsion).
- If G is virtually solvable but not virtually nilpotent, then yes.
- For G not virtually solvable, this is an open problem, there are partial positive results.

We will now focus on (3) and (4) above.

Proposition 6.13. *Suppose in a structure of finite Morley rank a definable solvable group G acts definably and faithfully on a definable connected abelian group A. Then for any definable $B \leq A$ which is either G-minimal or G'-minimal, the action of G' on B is trivial.*

Proof. We proceed by induction on the solvability class of G. If G is abelian then $G' = \{e\}$ acts trivially on B.

Case 1: B is G' minimal By induction, G'' acts trivially on B. Let $B_1 \leq B$ be G' minimal. Then B_1 is indecomposable in A (as it is enough to check it with respect to G'-invariant $H \leq A$, but then $H \cap B_1$ is finite or equal to A_1). Hence $\langle gB_1 : g \in G\rangle \leq A$ is definable. Suppose G' acts non-trivially on B. We apply Nesin Theorem to the action of the abelian group $G/G'' \trianglerighteq G'/G''$ on A_1, which is the same as the action of $G \trianglerighteq G'$ on A_1 (as G'' acts trivially on B hence on A_1). We obtain that G acts K-linearly on A_1 and G' acts K-scalarly on A_1 via $\Phi : G \to GL_n(K)$. But $\Phi(G') \subseteq SL_n(L) \cap \{\lambda \cdot Id : \lambda \imath K\}$ must be finite, so $G'/\ker(\Phi)$ is finite, hence trivial as G' is connected. Thus G' acts trivially on A_1, hence in particular on B_1.

Case 2: B is G-minimal Let $B_1 \leq B$ be G'-minimal. By Case 1 G' acts trivially on B_1, hence on $C := \langle gB_1 : g \in G\rangle$ (as $G' \trianglelefteq G$). As B_1 is indecomposable, C is definable (and G-invariant), hence by minimality of B, $C = B$, so G' acts trivially on B. □

Corollary 6.14. *If G is connected, solvable non-nilpotent group with $\mathrm{RM}(G) < \omega$, then $(G, \cdot)$ interprets an infinite field.*

Proof. Define $Z_1(G) = Z(G)$ and $Z_{n+1}(G) = \pi^{-1}Z(G/Z_n(G))$ where $\pi : G \to G/Z_n(G)$ is the quotient map. By Exercise 6.4, if $[Z_n : Z_{n+1}] < \omega$ then $Z(G/Z_n(G)) = \{e\}$. This must happen for some n as $\mathrm{RM}(G) < \omega$. Thus, as $Z_n(G)$ is nilpotent, we may assume $Z(G) = \{e\}$ (replacing G with $G/Z_n(G)$). As G is connected, so is $G^{(n)}$ for any n, so G^{n-1} is an infinite connected abelian normal subgroup of G where n is the solvability class of G. Let $A \leq G^{n-1}$ be G-minimal with respect to the action $G \curvearrowright G$ by conjugation. Then G' acts trivially on A by Proposition 6.13, so G/G' is an infinite connected abelian group acting on A definably. As A is not contained in $Z(G) = \{e\}$, the induced automorphism group is nontrivial (hence infinite, as it is connected). So we conclude by Theorem 6.1. □

Fact 6.15. *(Hrushovski) Let G be an infinite definable group of permutations of a strongly minimal set A definably in a stable thoery. Then $\mathrm{RM}(G) \in \{1, 2, 3\}$ and*
(1) If $\mathrm{RM}(G) = 1$ then G^0 and the action of G^0 on A is the action of G^0 on G^0 by translations.
(2) If $\mathrm{RM}(G) = 2$ then the action of G on A is the action $(K, +) \rtimes K^*$ on K by $x \mapsto ax + b$ for a definable field K.
(3) If $\mathrm{RM}(G) = 3$, then the action of G on A is the action of $PSL_2(K)$ on $P_1(K)$ by $x \mapsto \frac{ax+b}{cx+d}$.

Definition 6.16. A **bad** group is a connected non-solvable group of finite Morley rank whose every proper definable connected subgroup is nilpotent.

Corollary 6.17. *If G is a simple group with $\mathrm{RM}(G) = 3$ which is not a bad group, then $G = PSL_2(K)$ for a definable field K.*

Proof. As G is not bad, it has a proper connected non-nilpotent subgroup H, so $\mathrm{RM}(H) = 2$. Then G acts transitively and faithfully on the strongly minimal set G/H (faithfullness follow as $gaH = aH$ means $g \in aHa^{-1}$, but $\bigcap_{a\in G} aHa^{-1} \trianglelefteq G$ is trivial by simplicity), so we can apply Fact 6.15. □

Conjecture 6.18. *There does not exist any bad group.*

Fact 6.19. *(Frécon, 2016) There is no bad group of Morley rank 3, so Cherlin-Zilber Conjecture holds for groups of Morley rank ≤ 3.*

Conjecture 6.20. *(Borovik-Cherlin) Let G be a connected group acting in a structure of finite Morley rank on a set S generically $(n+2)$- transitively, with $n = \mathrm{RM}(S)$. Then (G, S) is isomorphic to the natural action of $PGL_{n+1}(F)$ on $P_n(F)$ for some algebraically closed field F.*

Note for $n = 1$ this follows from Fact 6.15 (after checking the actions in (1) and (2) there are not generically 3-transitive).

Freitag and Moosa prove Borovik-Cherlin conjecture for ACF_0, and then apply it to the connected component of the binding group $Aut(p(\mathfrak{C})/C(\mathfrak{C}))$, where C is the field of constants, in the proof of the following result (motivated by studying minimality of differential equations):

Fact 6.21. *(Freitag-Moosa) For every stationary finite rank type $p \in S(A)$ in DCF_0, $nmdeg(p) \leq U(p) + 1$, where $nmdeg(p)$ is the least k such that p has a nonalgebraic forking extension over $A \cup \{a_1, ..., a_k\}$, for some $a_1, ..., a_k$ realising p.*