INTRODUCTION TO STABLE GROUPS - PROBLEM LIST 1

(1) Suppose X_1 and X_2 are definable. Show that:
(a) $\mathrm{RM}(X_1) = 0$ iff X_1 is finite and nonempty.
(b) $\mathrm{RM}(X_1 \cup X_2) = \max(\mathrm{RM}(X_1), \mathrm{RM}(X_2))$ (you can use that $X_1 \subseteq X_2 \implies \mathrm{RM}(X_1) \leq \mathrm{RM}(X_2)$, which was proved in the lecture).
(c) If there is a definable bijection between X_1 and X_2, then $\mathrm{RM}(X_1) = \mathrm{RM}(X_2)$.
(d) If $X_2 = f(X_1)$ for some $f \in \mathrm{Aut}(\mathfrak{C})$, then $\mathrm{RM}(X_1) = \mathrm{RM}(X_2)$.

(2) Let $\mathfrak{C}$ be a monster model of the theory DLO_0 of dense linear order without endpoints. Let $a < b \in \mathfrak{C}$ and let $\phi(x) = (a < x < b)$. Prove that $\mathrm{RM}(X) = \infty$.

(3) Let G be a group definable in an ω-stable theory. Prove that the connected component G^0 of G (that is, the smallest definable subgroup of G of finite index) is normal in G and $\emptyset$-definable (hint: as we know that G^0 is definable, it is enough to show that G is invariant under automorphisms to conclude that it is $\emptyset$-definable).

(4) For any stationary $p \in S(A) \cap [x \in G]$, define $g \cdot p := \mathrm{tp}(g \cdot a/A)$ for $a \models p$ such that $a \mathop{\smile\!\!\!\!\!\!\mid}_A g$. Prove that $\mathrm{Stab}_G(p) := \{g \in G : g \cdot p = p\}$ is definable.
Hints: (i) WLOG $p \in S(\mathfrak{C})$, as $\mathrm{Stab}_G(p) = \mathrm{Stab}_G(\tilde{p})$ (recall $\tilde{p}$ is the unique global nonforking extension of p). (ii) As p is stationary, $\mathrm{DM}(p) = 1$, so there is a formula $\phi \in p$ with $\mathrm{RM}(p) = \mathrm{RM}(\phi)$ and $\mathrm{DM}(\phi) = 1$. (iii) Use definability of types in stable theories.