Wstęp do topologii algebraicznej

Ćwiczenia 3

(1) Show that $[0,1], [0,1), (0,1), \mathbb{R}$ are connected and might be disconnected by removing a point from the interior.
(2) Show that $\mathbb{R}^k \setminus \{(0,\ldots,0)\}$ is connected, for $k \geqslant 2$.
(3) Let (f_n) be a sequence of continuous functions $f_n \colon X \to Y$ between metric spaces. Show that if (f_n) converges uniformly to a function $f \colon X \to Y$ then $f \in C(X,Y)$. If f_n are uniformly bounded then f is bounded.
(4) Show that a closed subset of a complete metric space is itself a complete metric space.
(5) Show that a closed, totally bounded subset of a complete metric space is compact. Is completeness essential?
(6) Let $X = \{a,b\}$ be a two-point space with the discrete topology. Show that the only subalgebras of $C(X,\mathbb{R})$ are: $C(X,\mathbb{R})$, $\{(0,0)\}$, and linear spans of $(0,1), (1,0)$, and $(1,1)$.
(7) Let $c_n = \left(\frac{-1}{2}\right)\left(\frac{1}{2}\right)\cdots\left(\frac{2n-3}{2}\right)\frac{1}{n!}$, for $n = 1,2,3,\ldots$.
    (a) Show that the series $1-\sum_{n=1}^{\infty} c_n t^n$ converges absolutely and uniformly on compact subsets of $(-1,1)$.
    (b) Show that the term-wise differentiated series $-\sum_{n=1}^{\infty} n c_n t^{n-1}$ converges absolutely and uniformly on compact subsets of $(-1,1)$.
    (c) Conclude that if $f(t) = 1-\sum_{n=1}^{\infty} c_n t^n$ then $f'(t) = -\sum_{n=1}^{\infty} n c_n t^{n-1}$, for $-1 < t < 1$.
    (d) Show that $f(t) = -2(1-t)f'(t)$, and that $(1-t)^{-1/2} f(t)$ is constant.
    (e) Show that $f(t) = (1-t)^{1/2}$, for $-1 < t < 1$.
    (f) Show that $\sum_{n=1}^{\infty} c_n = 1$, and conclude that $1-\sum_{n=1}^{\infty} c_n t^n$ converges absolutely and uniformly on $[-1,1]$ to $(1-t)^{1/2}$.
(8) Let X be a compact Hausdorff space, and let $\mathcal{A}$ be a closed subalgebra of $C(X)$. Show that for any $f,g \in \mathcal{A}$, we have $|f| \in \mathcal{A}$, $\min\{f,g\} \in \mathcal{A}$, and $\max\{f,g\} \in \mathcal{A}$.
    Hint: Consider the function $h = \frac{f}{||f||}$ and approximate $|h|$ by $P \circ h$, where P is a polynomial. Use the fact that $\min\{f,g\} = \frac{1}{2}(f+g-|f-g|)$.