Wstęp do topologii algebraicznej

Zadania 1

(1) *(Metric topology.)* Let (X, d) be a metric space. A set $A \subseteq X$ is open if for every $x \in A$, and some $r > 0$ we have $B(x, r) \subseteq A$. Show that this defines a topology on X. Show that a metric space (with the metric topology) is separable iff it is second countable.

(2) *(Zariski topology.)* Let k be a field. For an ideal $I \subseteq k[X_1, \ldots, X_n]$ we define the *algebraic set* $V(I) := \{(a_1, \ldots, a_n) \in k^n \mid f((a_1, \ldots, a_n)) = 0 \text{ for all } f \in I\}$. Show that complements of algebraic sets define a topology on k^n. When is this topology Hausdorff?

(3) *(Stone topology.)* Let $(B, \vee, \wedge, \neg, 0, 1)$ be a Boolean algebra. An *ultrafilter* is a maximal proper filter. The *Stone space* $S(B)$ is a set of all ultrafilters in B. Show that the sets of the form $U_p = \{f \in S(B) \mid p \in f\}$, for $p \in B$, provide a basis of open neighbourhoods for a topology on $S(B)$. Is it Hausdorff?

(4) *(Boundary of a tree.)* Let $T = (V_T, E_T)$ be an infinite tree. A *geodesic ray* is an infinite sequence $(v_0, v_1, v_2, \ldots)$ of vertices such that $v_{i+1} \neq v_i \neq v_{i+2}$ and $\{v_i, v_{i+1}\} \in E_T$, for all i. The *boundary of* T, denoted ∂T is the set of equivalence classes of geodesic rays, where $(v_0, v_1, v_2, \ldots) \sim (w_0, w_1, w_2, \ldots)$ if there are N, M such that $v_i = w_{i+M}$, for all $i \geqslant N$. For a geodesic ray $\gamma = (v_0.v_1, v_2, \ldots)$ and $k \geqslant 0$, we define $U_{\gamma,k} \subseteq \partial T$ as the set of classes of geodesic rays $(w_0, w_1, w_2, \ldots)$ such that $w_i = v_k$ and $w_{i+1} = v_{k+1}$ for some i. Show that the sets $U_{\gamma,k}$ provide a basis of open neighbourhoods for a topology on ∂T. Is this topology Hausdorff? Is it equivalent to metric topology?

(5) *(p–adic topology on $\mathbb{Z}$.)* Let p be a prime number. We define $U_{k,n} := k + p^n\mathbb{Z} = \{k + p^n a \mid a \in \mathbb{Z}\}$. Show that this gives a basis of open neighbourhoods of a topology on $\mathbb{Z}$. Is it equivalent to a metric topology?

(6) *(Order topology.)* Let $(X, <)$ be a totally ordered set. Consider the topology defined by subbase consisting of sets $\{x \mid a < x\}$. Show that it is normal.

(7) *(Profinite topology.)* Let G be a group. Define a topology on G by basis consisting of all left cosets of subgroups of finite index. When is this topology discrete? Hausdorff?

(8) Let X be a topological space, and let a sequence $(f_n) \subset C(X, [0, 1])$ converge uniformly to $f \colon X \to [0, 1]$. Show that f is continuous.