Grupy i kompleksy

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Ćwiczenia 1

(1) Which of the following formulas define an action of a group G on a set X?
   (a) For $G = \mathbb{Z}$, $X = \mathbb{R}$ let $g \cdot x$ be defined as gx, the product of real numbers, for $g \in G$ and $x \in X$.
   (b) For $G = GL_2(\mathbb{R})$, $X = M_2(\mathbb{R})$ let $g \cdot x$ be defined as $(gx + xg)/2$, for $g \in G$ and $x \in X$. Here gx and xg are matrix products.
   (c) For $G = GL_2(\mathbb{R})$, $X = M_2(\mathbb{R})$ let $g \cdot x$ be defined as gx, for $g \in G$ and $x \in X$.
   (d) For $G = GL_2(\mathbb{R})$, $X = M_2(\mathbb{R})$ let $g \cdot x$ be defined as xg, for $g \in G$ and $x \in X$.
   (e) For $G = PSL_2(\mathbb{R})$, $X = \mathbb{C}$ let $g \cdot z := \frac{az+b}{cz+d}$, for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ and $z \in X$.

(2) Show that for the left action $\mu\colon G \times X \to X$ of G on X, we have $\mu(g, \mu(g^{-1}, x)) = x$.

(3) Show that a left action of G on X is a homomorhism $G \to S(X)$ into the group of permutations of X.

(4) Let D_∞ be the *infinite dihedral group*. It is defined as the subgroup of the group of all self-bijections of $\mathbb{R}$ generated by translations $t(x) = x + 1$ and reflections $s(x) = -x$. Describe all elements of D_∞. The group D_∞ acts in a natural way on $\mathbb{R}$. Describe orbits and stbilizers of the following elements: $1, \frac{1}{2}, \frac{1}{3}$. Show that D_∞ is isomorhic to a semidirect product $\mathbb{Z} \rtimes \mathbb{Z}_2$.

(5) The multiplicative group $\mathbb{R}^*$ acts on $\mathbb{R}^2$ by the formula $t \cdot (x, y) = (tx, t^{-1}y)$, $t \in \mathbb{R}^*$, $x, y \in \mathbb{R}^2$. Describe orbits, stabilizers and fix-points of the action.

(6) Let a finite group G act on a set X. Show Burnside's Lemma:

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

(7) Let T be a finite (as a graph) tree. Show that $\mathrm{Aut}(T)$ acting on T always fixes a vertex or an edge. What about infinite trees?

(8) For a tree T, let $G < \mathrm{Aut}(T)$. Show that the subgraph spanned (induced) by vertices fixed by G is a tree.

(9) Let H be a finite index subgroup of G. Show that there exists a finite index normal subgroup of G contained in H.

   Hint: Consider the action of G on the cosets.

(10) What is a property of an analogue of the Cayley graph $\mathrm{Cay}(G, S)$ for a set S that does not generate G?

(11) Draw a Cayley graph Γ of $\mathbb{Z}$ and a Cayley graph Δ of $\mathbb{Z}^2$ having the following property. Combinatorial balls of large radius (e.g. of radius 10) in Γ and Δ are isomorphic.

(12) Does every Cayley graph have to be edge-transitive?

(13) Why the Petersen graph is not a Cayley graph?

(14) How to distinguish Cayley graphs of $\mathbb{Z}$ from the ones of $\mathbb{Z}^2$?

Hint: Look at the graphs "from far away", i.e. consider asymptotic (or coarse, or large-scale geometry) properties of the graphs.

(15) Show that every Cayley graph of an infinite group contains an isometric copy of $\mathbb{R}$.

(16) Draw a Cayley graph of the *Heisenberg group*:

$$H = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} | a, b, c \in \mathbb{Z} \right\}.$$

(17) Draw a Cayley graph of the Baumslag-Solitar group $BS(1, n)$, that is the subgroup of $GL(n, \mathbb{Q})$ generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}$.

(18) Prove the Sabidussi Theorem: A connected simplicial graph Γ is a Cayley graph of a group G iff there exists an action of G on Γ by graph automorphisms, which is simply transitive on the set of vertices of Γ.

(19) How to construct a Cayley graph of $G_1 \times G_2$ using Cayley graphs $\mathrm{Cay}(G_1, S_1)$ and $\mathrm{Cay}(G_2, S_2)$?