Grupy i kompleksy

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(1) Let S be an infinite set of generators of a finitely generated group G. Show that there exists a finite subset T of S generating G.

(2) Show that a quotient of a finitely generated group is finitely generated.

(3) Show that a finite index subgroup of a finitely generated group is finitely generated. What if we drop the finite index assumption?

(4) Show that for every n the free group F_n is a subgroup of F_2.

(5) Let $\{1\} \to K \to G \to Q \to \{1\}$ be an exact sequence of groups. Show that if both K and Q are finitely generated then so is G.

(6) Find a finite generating set for the wreath product $\mathbb{Z}_2 \wr \mathbb{Z}$.

(7) Show that the free group $F(S)$ over a set S satisfies the following *universal property*. For every map $\varphi\colon S \to H$ to a group H, there exists its unique extension $\overline{\varphi}\colon F(S) \to H$. Using this show that for a given set S all free groups over S are isomorphic.

(8) Modify the problem above replacing "free group" by "abelian free group" everywhere, and replacing "group H" by "abelian group H". Solve it.

(9) Show that the fundamental group of a graph is free. Show that a subgroup of a free group is free.

(10) Show that every group is a quotient of a free group.

(11) Show that every short exact sequence $1 \to K \to G \to F(S)$ splits.

(12) Show that the free group F_n is not solvable for $n \geqslant 2$.

(13) Prove the following Ping-pong Lemma.

Let G be a group acting on a set X. Suppose there exist disjoint nonempty subsets $A^+, A^-, B^+, B^- \subset X$, and two elements a, b of G with the following properties:

a) $A^+ \cup A^- \cup B^+ \cup B^- \subsetneq X$;
b) $a(X - A^-) \subseteq A^+$, $a^{-1}(X - A^+) \subseteq A^-$;
b) $b(X - B^-) \subseteq B^+$, $b^{-1}(X - B^+) \subseteq B^-$.

Then $\langle a, b\rangle < G$ is a free subgroup generated by a and b.

(14) Let $\langle S \mid R\rangle$ be a presentation of a group G. Let $\varphi\colon S \to H$ be a map to a group H, such that $\varphi(r) = 1_H$, for every $r \in R$. Show that φ extends to a group homomorphism $G \to H$.

(15) Let $\langle S \mid R\rangle$ be a finite presentation of a group G. Let S' be a finite generating set of G, and let $\langle S' \mid R'\rangle$ be a presentation of G. Show that there exists a finite subset $R'_0 \subset R'$ such that $\langle S' \mid R'_0\rangle$ is a presentation of G.

(16) Show that $\langle a, b \mid a^4 = b^3 = 1,\ a^2b = ba^2\rangle$ is a presentation of $SL(2, \mathbb{Z})$.

(17) Find presentations of the following groups:
  (a) The infinite dihedral group D_∞.
  (b) The Heisenberg group $H_3(\mathbb{Z})$.
  (c) The group of isometries of the Euclidean plane generated by reflections wrt three lines intersecting pairwise at angles $\frac{\pi}{3}$ without triple intersection.
  (d) The lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$.
  (e) The Baumslag-Solitar group $BS(1, n)$.