Grupy i kompleksy

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(1) Find a metric space that is not quasi-isometric to a vertex set of a connected graph with the path metric.

(2) Show that every geodesic space is quasi-isometric to a vertex set of a connected graph with the path metric.

(3) Show that vertex sets X, Y of graphs (with the path metric) are quasi-isometric iff there exist functions $\varphi\colon X \to Y$, $\psi\colon Y \to X$, and constants a, b, c, d, such that for any $x, x' \in X$, $y, y' \in Y$:
- $d_Y(\varphi(x), \varphi(x')) \leqslant a \cdot d_X(x, x')$;
- $d_X(\psi(y), \psi(y')) \leqslant b \cdot d_Y(y, y')$;
- $d_X(\psi(\varphi(x)), x) \leqslant c$;
- $d_Y(\varphi(\psi(y)), y) \leqslant d$.

(4) Show that $\operatorname{diam} \partial^{\text{ext}} C < \operatorname{diam} \partial^{\text{int}} C + 2$, and $\operatorname{diam} \partial^{\text{int}} C < \operatorname{diam} \partial^{\text{ext}} C + 2$.

(5) Show that for every vertex o of a graph the structure graph T_o of radial cuts centered at o is a tree.

(6) For the vertex set X of a graph let $\varphi_o\colon X \to T_o$ and $\psi_o\colon T_o^{(0)} \to X$ be the canonically defined maps. Show that:
- φ_0 is surjective;
- ψ_0 is injective;
- $d_{T_o}(\varphi_o(x), \varphi_o(x')) \leqslant d_X(x, x')$;
- $d_X(\psi_o(y), \psi_o(y')) \leqslant \lambda \cdot d_{T_o}(y, y')$, where $\lambda = \sup\{\operatorname{diam} \partial^{\text{int}} C \mid C \in \mathcal{C}_o\}$.

(7) Prove the *Bottleneck Criterion* of Manning:

The vertex set X of a connected graph (with the path metric) is a quasi-tree iff the following holds: There exists $\delta > 0$ such that for every $x, y \in X$ there exists $m = m(x, y) \in X$, with $|d(x, m) - d(y, m)| \leqslant 1$, such that every path between x and y passes δ–close m.

(8) Show that a finitely generated group is quasi-isometric to the quotient by its finite normal subgroup.

(9) Show that a finitely generated group is quasi-isometric to its finite index subgroup. Conclude that virtually free groups are quasi-isometric to trees.

(10) Let X be a quasi-tree. Show that there is a constant $C > 0$ such that for any finite groups of isometries of X there exists a nonempty invariant set of diameter at most C in X.