Combinatorial negative curvature

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Problem list 1

(1) Show that a vertex set of a connected graph with the path metric is a discretely geodesic metric space.
(2) For arbitrarily n construct Cayley graphs Γ_1, Γ_2 of $\mathbb{Z}, \mathbb{Z}^2$, respectively, with the following property. For any vertices $v \in \Gamma_1$, and $w \in \Gamma_2$ the balls $B_{\Gamma_1}(v,n)$, and $B_{\Gamma_2}(w,n)$ are isomorphic.
(3) Show that a composition of coarse/quasi-isometric/bilipschitz embeddings is a coarse/quasi-isometric/bilipschitz embedding. Show that a composition of coarse equivalences/quasi-isometries/bilipschitz eqivalences is a coarse equivalence/quasi-isometry/bilipschitz eqivalence.
(4) Find an example of a coarse embedding that is not a quasi-isometric embedding. Find an example of a coarse equivalence that is not a quasi-isometry.
(5) Find an example of quasi-isometric spaces that are not bi-Lipschitz equivalent.
(6) Show that a metric space is coarsely equivalent to a connected graph iff it is coarsely connected.
(7) Show that a map $f\colon X \to Y$ is a coarse embedding iff f is a coarse equivalence between X and its image $f(X) \subseteq Y$.
(8) Show that two discretely geodesic metric spaces are coarsely equivalent iff they are quasi-isometric.
(9) Show that the map $f\colon (X, d_X) \to (Y, d_Y)$ is a coarse embedding if for every two sequences $(x_n) \subset X$, $(x'_n) \subset X$ we have $d_X(x_n, x'_n) \to \infty$ iff $d_Y(f(x_n), f(x'_n)) \to \infty$.
(10) Show that an inclusion of a subgroup is a coarse embedding. Does it have to be a quasi-isometric embedding?
(11) Find a quasi-isometric embedding of a half-line $\mathbb{R}^+ = [0, +\infty)$ into the Euclidean plane $\mathbb{E}^2$, which is not at a finite distance from a geodesic ray in $\mathbb{E}^2$. Can one find a similar embedding into a tree of valence at least 3, with the metric in which every edge has length 1?
(12) Show that the following spaces are not quasi-isometric one to another (with their natural metrics). The line $\mathbb{R}$, the Euclidean plane $\mathbb{E}^2$ and the regular tree of valence 4.
(13) Show that the regular tree of valence m is quasi-isometric to the regular tree of valence n (with their natural metrics), for every $n, m > 2$.
(14) Show that a finitely generated group is quasi-isometric to the quotient by its finite normal subgroup.
(15) Show that a finitely generated group is quasi-isometric to its finite index subgroup. Conclude that virtually free groups are quasi-isometric to trees.