Combinatorial negative curvature

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Problem list 2

(1) How to construct a Cayley graph of $G_1 \times G_2$ using Cayley graphs $\mathrm{Cay}(G_1, S_1)$ and $\mathrm{Cay}(G_2, S_2)$?

(2) Let a finite group G act on a set X. Show Burnside's Lemma:
$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

(3) Let T be a finite (as a graph) tree. Show that $\mathrm{Aut}(T)$ acting on T always fixes a vertex or an edge. What about infinite trees?

(4) For a tree T, let $G < \mathrm{Aut}(T)$. Show that the subgraph spanned (induced) by vertices fixed by G is a tree.

(5) Let X be a quasi-tree. Show that there is a constant $C > 0$ such that for any finite group of isometries of X there exists a nonempty invariant set of diameter at most C in X.

(6) Show the *Helly property* for a tree: Any family of pairwise intersecting connected sets have an intersection.

(7) Let H be a finite index subgroup of G. Show that there exists a finite index normal subgroup of G contained in H.
Hint: Consider the action of G on the cosets.

(8) Does every Cayley graph have to be edge-transitive?

(9) Why the Petersen graph is not a Cayley graph?

(10) Show that every Cayley graph of an infinite group contains an isometric copy of $\mathbb{R}$.

(11) Draw a Cayley graph of the *Heisenberg group*:
$$H = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} | a, b, c \in \mathbb{Z} \right\}.$$

(12) Draw a Cayley graph of the Baumslag-Solitar group $BS(1, n)$, that is the subgroup of $GL(n, \mathbb{Q})$ generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}$.

(13) Prove the Sabidussi Theorem: A connected simplicial graph Γ is a Cayley graph of a group G iff there exists an action of G on Γ by graph automorphisms, which is simply transitive on the set of vertices of Γ.