Geometric and Asymptotic Group Theory II

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Blatt 4

Mineyev's proof of Hanna Neumann Conjecture

(as simplified by Warren Dicks)

Hanna Neumann Conjecture. Let H, N be finitely generated subgroups of a free group F. Then $\overline{rk}(H \cap N) \leq \overline{rk}(H) \cdot \overline{rk}(N)$, where $\overline{rk}(K) = \max\{0, rk(K) - 1\}$.

Free group is orderable

(1) (*Magnus Embedding.*) Show that the embedding $F(a,b) \to \mathbb{Z}[[t,u]]^*$ given by $a \mapsto 1+t$, $b \mapsto 1+u$ is a monomorphism.

(2) Define a left-invariant linear order $\preccurlyeq$ on $F = F(a,b)$ induced by a lexicographic order on $\mathbb{Z}[[t,u]]$.

Bridges and Islands

(3) Define a left invariant order $\preccurlyeq$ on the set $E\Gamma$ of edges of the Cayley graph $\Gamma = Cay(F, \{a,b\})$ of F.

(4) For a finitely generated subgroup $G \leqslant F$, let $T(G)$ be a minimal G–invariant subtree of Γ. Show that $T(G)$ is unique and that $T(G) = \bigcup_{g \in G \setminus \{1\}} Axis(g)$, where $Axis(g)$ consists of vertices with the minimal displacement wrt g.

An edge $e \in E\Gamma$ is called a *G–bridge* if there is a biinfinite geodesic γ in $T(G)$ such that e is the $\preccurlyeq$–largest edge in γ. The set of G–bridges in $T(G)$ is denoted by $B(G)$. Connected components of $T(G) \setminus B(G)$ are called *G–islands*.

(5) Show that for every G–bridge e and for every $g \in G$, the edge ge is again a G–bridge.

(6) Show that for every G–island T and for every $g \in G$, the set gT is again a G–island.

(7) Show that if T, T' are G–islands then either $T = T'$ or $T \cap T' = \emptyset$.

Island Theorem

Let $Y = Y(G)$ be the tree obtained from $T(G)$ by contracting G–islands to points, i.e. $VY = \{T_0 \mid T_0 \text{ is a } G\text{–island}\}$ and $EY = B(G)$. Let T_0 be a G–island with nontrivial stabilizer $G_0 \leqslant G$.

(8) Show that G acts on Y without edge inversions and with trivial edge-stabilizers.
(9) Let $g \in G_0$, and let e be the $\preccurlyeq$–largest edge in a segment $[g^{-1}v, gv] \subseteq Axis(g)$, for some vertex v. What is the $\preccurlyeq$–largest edge in a given (infinite) subray of $Axis(g)$?
(10) Let $g, h \in G_0$ be two elements with disjoint axes, and let p be the geodesic connecting $Axis(g)$ with $Axis(h)$. Find the $\preccurlyeq$–largest edge in a (biinfinite) geodesic in $Axis(g) \cup p \cup Axis(h)$.
(11) (*Island Theorem.*) Show that G_0 is cyclic.

Bridge Theorem

Let $\mathbb{A} = G\backslash Y$ be the quotient graph of groups. Let $I(G)$ be the set of G–islands with trivial stabilizers.

(12) Show that:
  (a) the underlying graph A of $\mathbb{A}$ is finite;
  (b) edge groups in $\mathbb{A}$ are trivial;
  (c) vertex groups in $\mathbb{A}$ are trivial or cyclic;
  (d) $|EA| = |G\backslash B(G)|$.

(13) Show that the fundamental group $\pi_1(\mathbb{A})$ of the graph of groups $\mathbb{A}$ is $G = \pi_1(A) * F_m$, where m is the number of vertices in $\mathbb{A}$ with cyclic vertex groups.
(14) Prove that $I(G) = \emptyset$:
  Assume that $I(G) \neq \emptyset$. Let $Stab_G(T_0) = \{1\}$.
  (a) Assume T_0 is finite. Consider the $\preccurlyeq$–smallest bridge adjacent to T_0. Conclude that its existence leads to a contradiction.
  (b) Assume T_0 is infinite. Consider the projection of T_0 to the finite graph $G\backslash T(G)$. Show that there exists $g \in G$ with $gT_0 \cap T_0 \neq \emptyset$, and that this leads to a contradiction.
(15) (*Bridge Theorem.*) Show that $\overline{rk}(G) = |EA| - |VA| + m = |G\backslash B(G)|$.

Final step

(16) Using a theorem by Howson (saying that $H \cap N$ is finitely generated), define a (diagonal) map

$$j\colon (H \cap N)\backslash B(H \cap N) \to (H\backslash B(H)) \times (N\backslash B(N)).$$

(17) Show that j is injective.
(18) Prove the Hanna Neumann Conjecture!