Geometric and Asymptotic Group Theory II

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http://www.mat.univie.ac.at/∼dosaj/GGTWien/Course.html
Dienstag, 11:00–12:00, Raum D1.07 UZA 4

Blatt 6
Banach-Tarski Paradox

Banach-Tarski Paradox. The unit ball in $\mathbb{R}^3$ can be split up into a finite number of pieces and then reassembled to obtain two copies of the unit ball.

Paradoxical sets

A set X acted upon a group G is *G–paradoxical* if there are pairwise disjoint subsets of X: $A_1, \ldots, A_n$, $B_1, \ldots, B_n$, and elements of G: $g_1, \ldots, g_n$, $h_1, \ldots, h_m$ such that $X = \bigcup_{i=1}^{n} g_i A_i = \bigcup_{j=1}^{m} h_j B_j$. A group G is *paradoxical* if it is G–paradoxical with respect to its action on itself by left translations.

(1) Show that if a group G is paradoxical and acts freely on a set X, then X is G–paradoxical.
(2) Show that the free group F_2 is paradoxical.

Free subgroups of $SO(3)$

(3) Let A be the rotation through $\arccos\left(\frac{1}{3}\right)$ about the z–axis and B be the rotation through the same angle about the x–axis. Show that A and B generate a free subgroup of $SO(3)$.
(4) (*Hausdorff Paradox.*) Prove that there is a countable subset D of S^2 such that $S^2 \setminus D$ is $SO(3)$–paradoxical.
Hint: Omit axes.

Banach-Tarski Paradox

(5) Show that S^2 is $SO(3)$–paradoxical.
(6) Prove the Banach-Tarski Paradox

Relations to amenability

(7) Let X be G–paradoxical. Show that there is no finitely additive probabilistic G–invariant measure on X.