QUASI-EUCLIDEAN TILINGS OVER 2–DIMENSIONAL ARTIN GROUPS AND THEIR APPLICATIONS

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Abstract. We describe the structure of quasiflats in two-dimensional Artin groups. We rely on the notion of metric systolicity developed in our previous work. Using this weak form of non-positive curvature and analyzing in details the combinatorics of tilings of the plane we describe precisely the building blocks for quasiflats in all two-dimensional Artin groups – atomic sectors. This allows us to provide useful quasi-isometry invariants for such groups – completions of atomic sectors, stable lines, and the intersection pattern of certain abelian subgroups. These are described combinatorially, in terms of the structure of the graph defining an Artin group. As an important tool, we introduce an analogue of the curve complex in the context of two-dimensional Artin groups – the intersection graph. We show quasi-isometric invariance of the intersection graph under natural assumptions.

As immediate consequences we present a number of results concerning quasi-isometric rigidity for the subclass of CLTTF Artin groups. We give a necessary and sufficient condition for such groups to be strongly rigid (self quasi-isometries are close to automorphisms), we describe quasi-isometry groups, we indicate when quasi-isometries imply isomorphisms for such groups. In particular, there exist many strongly rigid large-type Artin groups. In contrast, none of the right-angled Artin groups are strongly rigid by a previous work of Bestvina, Kleiner and Sageev.

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1. Introduction

Overview. Let Γ be a finite simple graph with each edge labeled by an integer ≥ 2. An *Artin group with defining graph* Γ, denoted A_Γ, is given by the following presentation. Generators of A_Γ are in one to one correspondence with vertices of Γ, and there is a relation of the form

$$\underbrace{aba\cdots}_{m} = \underbrace{bab\cdots}_{m}$$

whenever two vertices a and b are connected by an edge labeled by m.

Despite the seemingly simple presentation, the most basic questions (torsion, center, word problem and cohomology) on Artin groups remain open, though partial results are obtained by various authors. We refer to the survey papers by Godelle and Paris [GP12], and McCammond [McC17]. Other fundamental and natural questions for Artin groups which are equally exciting and difficult can be found in Charney [Cha16].

One common feature of various classes of Artin groups studied so far is that they all exhibit features of non-positive curvature of certain form:

- small cancellation [AS83, App84, Pri86, Pei96];
- bi-automaticity, automaticity, or some form of combing [Cha92, Cha95, CP03, Dig06, HR12, Dig12, McC15];
- other notions of combinatorial non-positive curvature [Bes99, HO20];
- $CAT(0)$ [CD95a, BM00a, BC02, Bel05, BM10, HKS16];
- hierarchical hyperbolicity (hence coarse median) [CC05, BHS17a, Gor04, BHS19];
- relative hyperbolicity [KS04, CC07];
- acylindrical hyperbolicity [CW17, CMW19] and hyperbolicity in a statistical sense [Cum19, Yan18].

Conjecturally, all Artin groups should be non-positively curved in an appropriate sense, and this is intertwined with understanding many fundamental questions about Artin groups.

As most of the Artin groups have many abelian subgroups intersecting in a highly non-trivial way [DH17], it is natural to compare them with other "higher rank spaces with non-positive-curvature features" like symmetric spaces of non-compact type of rank > 1, Euclidean buildings, mapping class groups and Teichmuller spaces of surfaces etc., and ask how much of the properties on geometry and rigidity of these spaces still hold for Artin groups and what are the new phenomena for Artin groups.

Motivated by such considerations, we study Gromov's program of understanding quasi-isometric classification and rigidity of groups and metric spaces in the realm of 2–*dimensional* Artin groups. We build upon a previous result [HO19], where it was shown that all 2–dimensional Artin groups satisfy a form of non-positive curvature called metric systolicity. In the current paper, we will show that certain "0–curvature chunks" of the group have a very specific structure, which gives rise to rigidity results.

Background. Previous works on quasi-isometric rigidity and classification of Artin groups fall into the following three classes, listed from the most rigid to the least rigid situation.

(1) Some affine type Artin groups are commensurable to mapping class groups of surfaces [CC05]. Hence the quasi-isometric rigidity results for mapping class groups [BKMM12, Ham05] apply to them.
(2) Atomic right-angled Artin groups [BKS08] and their right-angled generalizations beyond dimension 2 [Hua17a, HK18, Hua18, Hua16].
(3) Artin groups whose defining graphs are trees [BN08] (they are fundamental groups of graph manifolds) and a right-angled generalization beyond dimension 2 [BJN10].

Most of the mapping class groups of surfaces enjoy the strong quasi-isometric rigidity property that any quasi-isometry of the group to itself is uniformly close to an automorphism. However, it follows implicitly from [BKS08, Section 11] that none of the right-angled Artin groups satisfies such form of rigidity (see [Hua17a, Example 4.14] for a more detailed explanation).

One is more likely to obtain stronger rigidity result when the complexity of the intersection pattern of flats in the space is higher. Thus we are motivated to study Artin groups which are not necessarily right-angled, whose structure is generally more intricate than the one of right-angled ones. One particular case is the class of *large-type* Artin group, where the label of each edge in the defining graph is ≥ 3. A commensuration rigidity result was proved by Crisp [Cri05] for certain class of large-type Artin groups.

An Artin group is *n–dimensional* if it has cohomological dimension n. One-dimensional Artin groups are free groups. The class of two-dimensional Artin groups is much more abundant – all large-type Artin groups have dimension ≤ 2. By Charney and Davis [CD95b], A_Γ has dimension ≤ 2 if and only if for any triangle $\Delta \subset \Gamma$ with its sides labeled by p, q, r, we have $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1$. A model example to keep in mind is A_Γ with Γ being a complete graph with all its edges labeled by 3.

Structure of quasiflats. Let A_Γ be a two-dimensional Artin group. We study 2–dimensional quasiflats in A_Γ, since the structure of top-dimensional quasiflats often plays a fundamental role in quasi-isometric rigidity of non-positively curved spaces, see the list of references after Theorem 1.1.

Let X^*_Γ be the universal cover of the standard presentation complex of A_Γ. Modulo some technical details, quasiflats can be viewed as subcomplexes of X^*_Γ which are homeomorphic to $\mathbb{R}^2$ and are quasi-isometric to $\mathbb{E}^2$ with the induced metric. Such subcomplexes are called *quasi-Euclidean tilings over X^*_Γ* and studying such tilings of $\mathbb{R}^2$ is of independent interests. This viewpoint is motivated by both the geometric aspects of quasiflats explored in [BKS16] and diagrammatic aspects studied in [AS83, Pri86, OS19].

To construct such a tiling, one can first search for tilings of Euclidean sectors appearing in X^*_Γ (see Figure 1 for some examples), then glue these sectors along their boundaries in a cyclic fashion to form a quasi-Euclidean tiling. The following theorem says that this is essentially the only way one can obtain a quasiflat. Inside X^*_Γ one can build a list of sectors which naturally arise from certain abelian subgroups, or centralizers of certain elements in A_Γ. We call them *atomic sectors* (cf. Section 5.2, in particular, Table 2 on page 41) since they are building blocks of quasiflats. Some of them are shown in Figure 1.

Theorem 1.1 (=Theorem 9.2)**.** *Suppose A_Γ is two-dimensional. Then any 2–dimensional quasiflat Q of X^*_Γ is at finite Hausdorff distance away from a union of finitely many atomic sectors in X^*_Γ.*

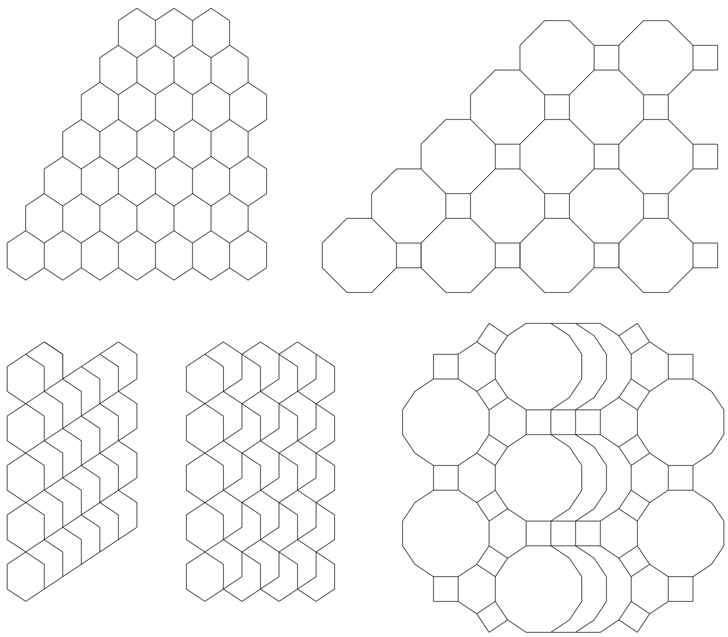

FIGURE 1. Some atomic sectors.

A simpler but less concise restatement of the above theorem is that every 2-dimensional quasiflat is made of "fragments" of subgroups of A_Γ of form $F_n \times F_m$ $(n, m \geq 1)$, where F_n denotes the free group with n generators.

Previously, structure theorems for quasiflats were proved for Euclidean buildings and symmetric spaces of non-compact type [KL97b, EF97, Wor06], universal covers of certain Haken manifolds [KL97a], certain CAT(0) complexes [BKS16, Hua17b] and hierarchically hyperbolic spaces [BHS17b].

Our viewpoint of tilings is convenient for, simultaneously, analyzing relevant group theoretic information and studying geometric aspects of quasiflats. The combinatorics of the group structure make up a substantial part. This is quite different from the more geometric situation in [BKS16, Hua17b, BHS17b]. Some of the atomic sectors are actually half-flats (see Definitions 5.13–5.16 below). This might seems strange compared to several previous quasiflat results, however, these sectors arise naturally when considering the group structure and they are convenient for our later applications.

Atomic sectors are not necessarily preserved under quasi-isometries. However, atomic sectors have natural *completions* (cf. Section 5.4) which are preserved under quasi-isometries (see Corollary 10.6 for a precise statement).

The following theorem says that certain $\mathbb{Z}$–subgroups corresponding to boundaries of atomic sectors are preserved under quasi-isometries. These $\mathbb{Z}$–subgroups act on what we call *stable lines*, and they are analogues of $\mathbb{Z}$–subgroups generated by Dehn twists in mapping class groups.

Theorem 1.2 (=Theorem 10.11)**.** *Suppose A_{Γ_1} and A_{Γ_2} are two-dimensional Artin groups. Let $q\colon X^*_{\Gamma_1} \to X^*_{\Gamma_2}$ be an (L,A)–quasi-isometry. Then there exists a constant D such that for any stable line $L_1 \subset X^*_{\Gamma_1}$, there is a stable line $L_2 \subset X^*_{\Gamma_2}$ with $d_H(q(L_1), L_2) < D$.*

For each 2–dimensional Artin group A_Γ, we define an *intersection graph* $\mathcal{I}_\Gamma$ (Definition 10.13), which describes the intersection pattern of certain abelian subgroups. This object can be viewed as an analogue of the spherical building at infinity for symmetric spaces of non-compact type, or the curve graph in the case of mapping class groups. When Γ is a triangle with its edges labeled by 3, the group A_Γ is commensurable to the mapping class group of the 5–punctured sphere [CC05], and $\mathcal{I}_\Gamma$ is isomorphic to the curve complex of the 5–punctured sphere.

Theorem 1.3. *Let A_Γ and $A_{\Gamma'}$ be two 2–dimensional Artin groups and let $q\colon A_\Gamma \to A_{\Gamma'}$ be a quasi-isometry. Then q induces an isomorphism from the stable subgraph of $\mathcal{I}_\Gamma$ onto the stable subgraph of $\mathcal{I}_{\Gamma'}$. In particular, if both A_Γ and $A_{\Gamma'}$ have finite outer automorphism group and Γ has more than two vertices, then q induces an isomorphism between their intersection graphs.*

We refer to Theorem 10.16 for a more general version of Theorem 1.3.

Analogously, quasi-isometries of higher rank symmetric spaces of non-compact type and Euclidean buildings induce automorphisms of their spherical buildings at infinity [KL97b, EF97]; and quasi-isometries of most mapping class groups induce automorphisms of their curve complexes [Ham05, BKMM12, BHS17b].

Recall that a finitely generated group H is *virtually* A_Γ if there exists a finite index subgroup $H' \leq H$ and a homomorphism $\phi\colon H' \to A_\Gamma$ with finite kernel and finite index image.

Corollary 1.4. *Let A_Γ be a 2–dimensional Artin group with finite outer automorphism group. Suppose Γ has more than two vertices. If $q\colon A_\Gamma \to A_\Gamma$ is an (L,A)–quasi-isometry inducing the identity map on $\mathcal{I}_\Gamma$, then there exists $C = C(L,A,\Gamma)$ such that $d(q(x),x) \leq C$ for any $x \in A_\Gamma$. Thus the map $\mathrm{QI}(A_\Gamma) \to \mathrm{Aut}(\mathcal{I}_\Gamma)$ described in Theorem 1.3 is an injective homomorphism.*

Thus if we know in addition that the homomorphism $A_\Gamma \to \mathrm{Aut}(\mathcal{I}_\Gamma)$ induced by the action $A_\Gamma \curvearrowright \mathrm{Aut}(\mathcal{I}_\Gamma)$ has finite index image, then any finitely generated groups quasi-isometric to A_Γ is virtually A_Γ.

We refer to Corollary 10.17 for a more general version of Corollary 10.17.

Unlike the case of mapping class groups, $\mathrm{Aut}(\mathcal{I}_\Gamma)$ could be much larger than $\mathrm{QI}(A_\Gamma)$ in the context of Corollary 1.4. This happens for example in the right-angled case [Hua17a, Corollary 4.20]. However, we expect more rigidity to occur outside the right-angled world – see the next subsection for some special cases.

Rigidity of certain large-type Artin groups. We now discuss some immediate consequences of the results from the previous subsection. We will restrict ourselves to the class of *CLTTF* Artin groups introduced by Crisp [Cri05] in order to use his results directly. Nevertheless, we believe that similar goals for more general classes of Artin groups can be achieved using the invariants described above. See the list of questions in Section 12 for possible further directions.

Definition 1.5. An Artin group A_Γ is CLTTF if all of the following conditions are satisfied:

- (C) Γ is connected and has at least three vertices;
- (LT) A_Γ is of large type;
- (TF) Γ is triangle-free, i.e. Γ does not contain any triangles.

In particular, CLTTF Artin groups are 2–dimensional. As pointed out by Crisp, (C) serves to rule out 2–generator Artin groups, which are best treated as a separate case. Here, we improve the commensurability rigidity of [Cri05, Theorem 3] to quasi-isometric rigidity results and point out some new phenomena in the setting of quasi-isometries.

Theorem 1.6 (=Corollary 11.8)**.** *Let A_Γ and $A_{\Gamma'}$ be CLTTF Artin groups. Suppose Γ does not have separating vertices and edges. Then A_Γ and $A_{\Gamma'}$ are quasi-isometric if and only if Γ and Γ' are isomorphic as labeled graphs.*

By [Cri05, Theorem 1], a CLTTF Artin group has finite outer automorphism group if and only if its defining graph has no separating vertices and edges. So Theorem 1.6 can be compared to [Hua17a, Theorem 1.1].

Theorem 1.7 (=Theorem 11.9)**.** *Let A_Γ be a CLTTF Artin group such that Γ does not have separating vertices and edges. Let $\mathrm{QI}(A_\Gamma)$ be the quasi-isometry group of A_Γ. Let $\mathrm{Isom}(A_\Gamma)$ be the isometry group of A_Γ with respect to the word distance for the standard generating set. Then the following hold.*

(1) Any quasi-isometry from A_Γ to itself is uniformly close to an element in $\mathrm{Isom}(A_\Gamma)$.
(2) There are isomorphisms $\mathrm{QI}(A_\Gamma) \cong \mathrm{Isom}(A_\Gamma) \cong \mathrm{Aut}(D_\Gamma)$, where $\mathrm{Aut}(D_\Gamma)$ is the simplicial automorphism group of the Deligne complex D_Γ of A_Γ.

There are counterexamples if we drop the condition that Γ does not have separating vertices and edges, see [Cri05, Lemma 42].

If A_Γ is right-angled then $\mathrm{QI}(A_\Gamma)$ is much smaller than $\mathrm{Aut}(D_\Gamma)$, see [Hua17a, Corollary 4.20]. This suggests that the Artin groups in Theorem 1.7 are more rigid than right-angled Artin groups. On the other hand, the Artin groups in Theorem 1.7 may not be as rigid as mapping class groups, since elements in $\mathrm{Isom}(A_\Gamma)$ are not necessarily uniformly close to automorphisms of A_Γ. A group G is *strongly rigid* if any element in $\mathrm{QI}(G)$ is uniformly close to an element in $\mathrm{Aut}(G)$. Now we characterize all strongly rigid members of the class of large-type and triangle-free Artin groups.

Theorem 1.8 (=Theorem 11.10)**.** *Let A_Γ be a large-type and triangle-free Artin group. Then A_Γ is strongly rigid if and only if Γ satisfies all of the following conditions:*

(1) Γ is connected and has ≥ 3 vertices;
(2) Γ does not have separating vertices and edges;
(3) any label preserving automorphism of Γ which fixes the neighborhood of a vertex is the identity.

Moreover, if a large-type and triangle-free Artin group A_Γ satisfies all the above conditions and H is a finitely generated group quasi-isometric to A_Γ, then H is virtually A_Γ.

Remark 1.9. We note that the above quasi-isometric rigidity results do not use the full strength of Theorem 1.1 and Theorem 1.2, since proving the two theorems in the special cases of triangle-free Artin groups and Artin groups of hyperbolic type in the sense of [Cri05] are much easier (though still non-trivial). Moreover, if an Artin group is triangle-free, then it acts geometrically on a 2–dimensional CAT(0) complex [BM00b], and it is C(4)-T(4) with respect to the standard presentation [Pri86]. There are alternative starting points of studying quasiflats in such complexes, either by using [BKS16], or using [OS19]. However, several new ingredients are needed to deal with the general 2–dimensional case. We present the above quasi-isometric rigidity results in order to demonstrate the potential of using Theorem 1.1 and Theorem 1.2 to obtain similar results for more general classes of Artin groups.

Comments on the proof. First we discuss the proof of Theorem 1.1. Let X^*_Γ be the universal cover of the standard presentation complex of a 2–dimensional Artin group A_Γ. To avoid technicalities, we assume the quasiflat is a subcomplex Q of X^*_Γ homeomorphic to $\mathbb{R}^2$ and quasi-isometric to $\mathbb{E}^2$, and we would like to understand the tiling of Q.

Step 0: The general idea is to use geometry of A_Γ to control the tiling of Q. We start by showing that A_Γ is non-positively curved in an appropriate sense. We would like to use CAT(0) geometry, however, at the time of writing, it is not known whether all 2–dimensional Artin groups are CAT(0). Moreover, [BC02] implies that if certain 2–dimensional Artin groups act geometrically on CAT(0) complexes, the dimension of the complexes is ≥ 3. There are quite non-trivial technicalities in studying 2–quasiflats in higher dimensional complexes and relating them to the combinatorics of groups, which we would like to bypass.

In [HO19], we built a geometric model X_Γ for A_Γ with features of both CAT(0) geometry and two-dimensionality. More precisely, X_Γ is a thickening of X^*_Γ. We equip the 2–skeleton $X^{(2)}_\Gamma$ with a metric and it turns out that X_Γ becomes non-positively curved in the sense that any 1–cycle can be filled by a CAT(0) disc in $X^{(2)}_\Gamma$. Such a complex X_Γ is an example of a *metrically systolic complex*.

Step 1. We study the local structure of Q and show that outside a compact set, Q is locally flat in an appropriate combinatorial sense.

Step 1.1. We approximate Q by a CAT(0) subcomplex Q' in X_Γ such that Q' is homeomorphic to $\mathbb{R}^2$. To do that, we pick larger and larger discs in Q, and we replace them by minimal discs in $X_\Gamma^{(2)}$, which are CAT(0), then we take a limit. By a version of Morse Lemma proved in [HO19], Q' is at bounded Hausdorff distance from Q. An argument on area growth implies that Q' is flat outside a compact set. In fact, we obtain the following result in the more general setting of all metrically systolic complexes (see Theorem 2.7 for the detailed statement).

Theorem 1.10. *(cf. Theorem 2.7) Every quasiflat in a metrically systolic complex X can be uniformly approximated by a simplicial map $q' \colon Y \to X$ from a CAT(0) triangulation Y of $\mathbb{R}^2$ being flat outside a compact set such that q' is injective on neighbourhoods of flat vertices.*

Recently, independently, Elsner [Els19] has obtained an analogous result for systolic complexes. Theorem 1.10 applies to a larger class of spaces, however, Elsner's result provides better control on the structure of quasiflats in the systolic setting.

Step 1.2. Though Q' is CAT(0) and flat outside a compact set, the cell structure on Q' is not ideal. A priori there may be triangles in Q' such that their angles are irrational multiples of π, and the cell structure is not compatible with the intersection pattern of quasiflats of A_Γ. So we go back to X^*_Γ, whose combinatorial structure is simpler than the one of X_Γ. By the design of X_Γ, there is a partial retraction ρ from X_Γ (defined on a subcomplex of X_Γ) to X^*_Γ. We argue that outside a compact subset, ρ is well-defined on Q'. Let $Q'' = \rho(Q')$. Then Q'' is a subcomplex with a "hole" (since ρ may not be defined on all of Q'). The map ρ transfers information on local structure of Q' to information on local structure of Q'' (cf. Lemma 4.11). This step is discussed in Section 3 and Section 4.

Step 2. So far, we have a subcomplex Q'' about which we know that its local structure belongs to one of several types (as in Lemma 4.11). The next goal is to run a local-to-global argument in order to produce the sectors as required.

Step 2.1. We first produce a boundary ray for the sector, which is easier than producing the whole sector. To guess what kind of ray it could be, we look at a motivating example $X = \mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3,\mathbb{R})$. Quasiflats in X are Hausdorff close to a finite union of Weyl cones [KL97b, EF97]. The boundary of a Weyl cone is a singular ray, i.e. this ray is contained in the intersection of two flats. Analogously, we search for *singular rays* in X_Γ, which are possibly contained in the intersection of two free abelian subgroups of rank 2. A list of plausible singular rays is given in Section 5.

Step 2.2. We show that there is at least one singular ray in Q''. The strategy is to start at one vertex of Q'', then walk away from this vertex and collect information about local landscape from Step 1.2 along the way to decide where to go further. Note that there is a "hole" in Q'' we want to avoid. This can be handled in the following way. Paths in Q'' have shadows in Q'. Thus we can use the CAT(0) geometry in Q' to control where to go so we can avoid the "hole". Section 6, Section 7 and Section 8 are devoted to the proof of the existence of singular rays in Q''. Another thing we need to be careful about is that the local structure discussed in Step 1.2 concerns not just the shapes of the cells around one vertex, but also the orientation of the edges around that vertex. In Section 7 we deal with the orientation issues.

Step 2.3. Now we know there is at least one singular ray in Q''. We start with this singular ray, and use the local characterization from Step 1.2 to "sweep out" a sector which ends in another singular ray. This uses a development argument. Now we iterate this process. It turns out that each sector in Q'' corresponds to a CAT(0) sector in Q' with angle bounded below by a uniform number. Since Q' has quadratic growth, after finitely many steps we produce enough sectors to fill Q'', which finishes the proof of Theorem 1.1. This step is discussed in Section 9.

Finally, we discuss how to deduce the quasi-isometric rigidity theorems for CLTTF Artin groups from Theorem 1.1. Again, we proceed in several steps. The first two steps can be generalized in an appropriate way to all 2–dimensional Artin groups (cf. Theorem 10.11 and Theorem 10.16).

Step 1: Let A_Γ be a CLTTF Artin. We consider the collection of maximal cyclic subgroups of A_Γ with centralizers commensurable to $F_2 \times \mathbb{Z}$. We show that these cyclic subgroups are preserved under quasi-isometries.

Step 2: Let Θ_Γ be a graph whose vertices correspond to cyclic subgroups from the previous step, and two vertices are adjacent if the associated cyclic subgroups generate a free abelian subgroup of rank 2. This graph was introduced by Crisp [Cri05]. We show that any quasi-isometry from A_Γ to itself induces an automorphism of Θ_Γ.

Step 3: Crisp [Cri05] showed that any automorphism of Θ_Γ is induced by a canonical bijection from A_Γ to itself, under appropriate additional conditions. In such way we approximate quasi-isometries by maps which preserve more combinatorial structure, and then deduce the quasi-isometric rigidity results listed above, see Section 11.

On the structure of the paper. In Section 2 we analyze the structure of quasiflats in metrically systolic complexes and we prove Theorem 1.10 above (Theorem 2.7 in the text). In Sections 3– 9 we describe the structure of complexes approximating quasiflats in two-dimensional Artin groups. In Section 9 we prove Theorem 1.1 (Theorem 9.2). In Section 10 we prove Theorem 1.2 (Theorem 10.11) and Theorem 1.3 (Theorem 10.16). In Section 11 we provide proofs of the results concerning CLTTF Artin groups: Theorem 1.6 (Corollary 11.8), Theorem 1.7 (Theorem 11.9), and Theorem 1.8 (Theorem 11.10).

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2. Quasiflats in metrically systolic complexes

We start with several notations. Let X be a combinatorial cell complex. For a subset $Y \subset X$, the *carrier* of Y in X is the union of closed cells in X which contain at least one point of Y in their interior. For a vertex $x \in X$, the *closed star* of v in X, denoted by $\mathrm{St}(x, X)$, is the union of closed cells in X which contain x. When X is a piecewise Euclidean polyhedral complex, i.e. X is obtained by taking the disjoint union of a family of convex polyhedra in Euclidean spaces of various dimensions and gluing them along isometric faces, see [BH99, Definition I.7.37], the ϵ–sphere around each vertex of X inherits a natural polyhedral complex structure from X, for ϵ small (see [BH99, Definition I.7.15]). This polyhedral complex is called the *link* of x in X, and is denoted by $\mathrm{lk}(x, X)$. If X is a 2–dimensional simplicial complex, then we always identify $\mathrm{lk}(x, X)$ with the full subgraph of the one-skeleton $X^{(1)}$ of X spanned by vertices adjacent to x.

For a metric space Z and a point $z \in Z$, we use $B(z, R)$ to denote the R–ball in Z centered at z. For a subset $Y \subset Z$, we use $N_R(Y, R)$ to denote the R–neighborhood of Y in Z.

2.1. Preliminaries on metrically systolic complexes.

Let X be a flag simplicial complex. We put a piecewise Euclidean structure on $X^{(2)}$ (the 2–skeleton of X) in the following way. Since $X^{(2)}$ can be viewed as a disjoint collection of simplices with identifications between their faces, we assume every 2–simplex (triangle) is isometric to a non-degenerate Euclidean triangle and all the identifications are isometries. This gives a length metric on $X^{(2)}$, which we denote by d. Since we will work with the 2–skeleton of X, for a vertex $v \in X$, we define its *link* to be the full subgraph of $X^{(1)}$ spanned by all vertices adjacent to v as explained as above. Every link is equipped with an *angular metric*, defined as follows. For an edge $\overline{v_1 v_2}$, we define the *angular length* of this edge to be the angle $\angle_v(v_1, v_2)$ with the apex v. This turns the link into a metric graph, and the angular metric, which we denote by $d_\angle$, is the path metric of this metric graph (note that a priori we do not know $\angle_v(v_1, v_2) = d_\angle(v_1, v_2)$ for adjacent vertices v_1 and v_2). The *angular length* of a path ω in the link, which we denote by $\text{length}_\angle(\omega)$, is the summation of angular lengths of edges in this path. In this paper we assume that the following weak form of triangle inequality holds for angular lengths of edges in X: for each $v \in X$ and every three pairwise adjacent vertices v_1, v_2, v_3 in the link of v we have that $\angle_v(v_1, v_3) \leq \angle_v(v_1, v_2) + \angle_v(v_2, v_3)$. Then we call X (with metric d) a *metric simplicial complex.*

Remark 2.1. We allow that the above inequality becomes equality – intuitively, it corresponds to degenerate 2–simplices in a link, which correspond to degenerate 3–simplices in X.

For $k = 4, 5, 6, \ldots$, a simple k–cycle C in a simplicial complex is 2*–full* if there is no edge connecting any two vertices in C having a common neighbor in C (that is, there are no *local diagonals*).

Definition 2.2 (Metrically systolic complex)**.** A link in a metric simplicial complex is 2π*–large* if every 2–full simple cycle in the link has angular length at least 2π. A metric simplicial complex X is *locally* 2π*–large* if every its link is 2π–large. A simply connected locally 2π–large metric complex is called a *metrically systolic* complex.

In Theorem 2.3 below we state a fundamental property of metrically systolic complexes. It concerns filling diagrams for cycles, and it is a main tool used in proofs of various results about such complexes in subsequent sections and, previously, in [HO19].

Let X be a simplicial complex. A *cycle* in X is a simplicial map from a triangulated circle to X which is injective on each edge. A *singular disc* D is a simplicial complex isomorphic to a finite connected and simply connected subcomplex of a triangulation of the plane. There is the (obvious) *boundary cycle* for D, that is, a map from a triangulation of 1–sphere (circle) to the boundary of D, which is injective on edges. More precisely, we can view D as a subset of the 2–sphere $\mathbb{S}^2$. Then $\mathbb{S}^2 \setminus D$ is an open cell, and the boundary of this open cell gives rise to the boundary cycle of D. For a cycle $C\colon K \to X$ in a simplicial complex X, a *singular disc diagram for* C is a simplicial map $f\colon D \to X$ from a singular disc D to X such that $C\colon K \to X$ factors through the boundary cycle of D. By the relative simplicial approximation theorem [Zee64], for every cycle in a simply connected simplicial complex there exists a singular disc diagram (cf. also van Kampen's lemma e.g. in [LS01, pp. 150-151]). It is an essential feature of metrically systolic complexes that singular disc diagrams may be modified to ones with some additional properties; see Theorem 2.3 below.

A singular disc diagram is called *nondegenerate* if it is injective on all simplices. It is *reduced* if distinct adjacent triangles (i.e., triangles sharing an edge) are mapped into distinct triangles. For a metric simplicial complex

X and a nondegenerate singular disc diagram $f: D \to X$ we equip D with a metric in which $f|_\sigma$ is an isometry onto its image, for every simplex σ in D. Then, f is a *CAT(0) singular disc diagram* if D is CAT(0), that is, if the angular length of every link in D being a cycle (that is, the link of an interior vertex in D) is at least 2π. An internal vertex in D is *flat* when the angular length of its link is exactly 2π.

Theorem 2.3 (CAT(0) disc diagram)**.** [HO19, Theorem 2.8] *Let $f: D \to X$ be a singular disc diagram for a cycle C in a metrically systolic complex X. Then there exists a singular disc diagram $f': D' \to X$ for C such that*

(1) $f': D' \to X$ is a CAT(0) nondegenerate reduced disc diagram;
(2) f' does not use any new vertices in the sense that there is an injective map i from the vertex set of D' to the vertex set of D such that $f = f' \circ i$ on the vertex set of D;
(3) if $v \in D'$ is a flat interior vertex, then f' is injective on the closed star of v in D'.

The number of 2–simplices in D' is at most the number of 2–simplices in D.

The following result is immediate.

Corollary 2.4. *Suppose X is a metrically systolic complex with finitely many isometry types of cells. Then there exists a constant $L > 0$ such that for each cycle C with $\leq n$ edges, there is a singular disc diagram for C with $\leq Cn^2$ triangles in the disc diagram and the image of the singular disc diagram is contained in the $(L \cdot n)$–neighborhood of C.*

Note that whenever there is a statement regarding metric on X, we always mean the length metric d with respect to the piecewise Euclidean structure on $X^{(2)}$. If there are finitely many isometry types of cells in $X^{(2)}$, then $(X^{(2)}, d)$ is a complete geodesic metric space [BH99, Theorem I.7.19]. Moreover, d is quasi-isometric to the path metric on $X^{(1)}$ such that each edge has length 1 [BH99, Proposition I.7.31].

We also need the following version of the Morse Lemma for discs in metrically systolic complexes. See [HO19, Theorem 3.9] for the proof.

Lemma 2.5. *[Morse Lemma for 2–dimensional quasi-discs] Let D be a combinatorial ball in the Euclidean plane tiled by equilateral triangles. Let $f: D \to X$ be a disc diagram for a cycle C in X being an (L, A)–quasi-isometric embedding. Let $g: D' \to X$ be a singular disc diagram for C. Then $\mathrm{im}(f) \subseteq N_a(\mathrm{im}(g))$, where $a > 0$ is a constant depending only on L and A.*

2.2. **The structure of quasiflats.** Recall that $B_X(x, r)$ denotes the ball of radius r centered at x in a metric space X. Here and elsewhere we use the notation d_H for the Hausdorff distance. The following is a consequence of [BKS16, Theorem 4.1].

Theorem 2.6. *Suppose X is a 2–dimensional piecewise Euclidean CAT(0) complex such that X is homeomorphic to $\mathbb{R}^2$ and X has finitely many isometry types of cells. If there is a constant $L > 0$ and a base point $x \in X$ such that $\operatorname{Area}(B_X(x, r)) \leq Lr^2$, for any $r > 0$, then X is flat outside a compact subset.*

Theorem 2.7. *Let X be a locally finite metrically systolic complex with finitely many isometry types of cells. Let $q \colon \mathbb{E}^2 \to X$ be an (L, A)–quasi-isometric embedding. Then there exist a constant M_0 depending only on L, A and X, a simplicial complex Y, and a reduced nondegenerate simplical map $q' \colon Y \to X$ such that*

(1) Y is a 2–dimensional simplicial complex homeomorphic to $\mathbb{R}^2$;
(2) if we endow Y with the piecewise Euclidean structure such that each simplex in Y is isometric to its q'–image, then Y is CAT(0);
(3) Y is flat outside a compact subset;
(4) for any flat vertex $v \in Y$, q' is injective on $\operatorname{St}(v, Y)$; and if we view $\operatorname{lk}(v, Y)$ as a simple close cycle in X, then it does not bound a disc diagram without interior vertices;
(5) $d_H(\operatorname{Im} q, \operatorname{Im} q') < M_0$.

Proof. Let $q \colon \mathbb{E}^2 \to X$ be an (L, A)–quasi-isometric embedding. We view $\mathbb{E}^2$ as a simplicial complex which is tiled by equilateral triangles. Using Corollary 2.4 and a skeleton by skeleton approximation argument, we can assume q is also L'–Lipschitz. Since X has finitely many isometry types of cells, it follows from q being L'–Lipschitz and [BH99, Lemma I.7.54] that if the diameter of equilateral triangles in $\mathbb{E}^2$ is small enough, then the q–image of the closed star of a vertex in $\mathbb{E}^2$ is contained in the closed star of a vertex in X. By a standard simplicial approximation argument, we can assume in addition that q is a simplicial map. The new quasi-isometric constants of q depend only on the old constants and X.

Pick a base vertex $x \in \mathbb{E}^2$. Let $D_n \subset \mathbb{E}^2$ be the full subcomplex spanned by vertices with combinatorial distance $\leq n$ from x. Up to attaching a thin annulus to D_n along ∂D_n, we assume $q_n = q|_{D_n}$ maps each edge in ∂D_n to an edge in X. As in Theorem 2.3, for each n, we modify $q_n \colon D_n \to X$ to obtain a reduced nondegenerate CAT(0) singular disc diagram $q'_n \colon D'_n \to X$ such that q'_n and q_n have the same boundary cycle. Moreover, we assume $q'_n \colon D'_n \to X$ has the least number of triangles among all singular disc diagrams satisfying the conditions of Theorem 2.3 and has the same boundary cycle as q_n. Then

- $d_H(\operatorname{Im} q'_n, \operatorname{Im} q_n) < M$ with M depending only on L and A;
- Theorem 2.7 (4) holds for flat interior vertices in D'_n.

The first property follows from Lemma 2.5. For the second property, if $\operatorname{lk}(v, D'_n)$ bounds a singular disc diagram without interior vertices for some flat interior vertex $v \in D'_n$, then we can replace $\operatorname{St}(v, D'_n)$ with such diagram to obtain a singular disc diagram $\bar{q}_n \colon \bar{D}_n \to X$ with fewer triangles. Now, use Theorem 2.3 to modify $\bar{q}_n$ further to obtain a singular disc diagram $\hat{q}_n\colon \widehat{D}_n \to X$. Then $\widehat{D}_n$ contradicts the minimality of D'_n.

Let $A_0 = \max\{L, A, L', M\}$. For $R \geq 100A_0$ and $n \geq 100A_0R$, let $K_{R,n} \subset D'_n$ be the largest possible subcomplex such that $q'_n(K_{R,n})$ is contained in the R–ball $B_X(q(x), R)$ of X centered at $q(x)$. Then the following hold:

(a) $K_{R,n}$ is locally CAT(0);
(b) there is M' depending only on L and A such that $q(\mathbb{E}^2) \cap B_X(q(x), \frac{R}{2})$ is contained in the M'–neighborhood of $q'_n(K_{R,n})$;
(c) $\mathrm{lk}(v, K_{R,n})$ is a circle for any vertex $v \in K_{\frac{R}{2},n}$;

(b) follows from the inequality $d_H(\mathrm{Im}\, q'_n, \mathrm{Im}\, q_n) < M$, and (c) follows from the fact that $q'_n(v)$ is far away from $q'_n(\partial D'_n)$. Since q'_n does not use new vertices in the sense of Theorem 2.3, the cardinality of $K^{(0)}_{R,n}$ is $\leq$ the number of vertices in $\mathbb{E}^2$ whose q–images are contained in $B_X(q(x), R)$. Moreover, X is locally finite by our assumption. Thus by passing to a subsequence, we assume for any n and m, there is a simplicial isomorphism $\phi_{n,m}\colon K_{R,n} \to K_{R,m}$ such that $q'_n = q'_m \circ \phi_{n,m}$ on $K_{R,n}$. Now we let $R \to \infty$ and use a diagonal argument to produce $q'\colon Y \to X$ such that for any R and any subcomplex $K \subset Y$ such that $q'(K)$ is contained in the R–ball of x, there exists n and a simplicial embedding $\phi_K\colon K \to D'_n$ such that $q' = q'_n \circ \phi_K$ on K.

Now we show that q' satisfies all the requirements. First we show Y is simply-connected. Take a closed curve $C \subset Y$ and take R such that $q'(C) \subset B_X(q(x), \frac{R}{100})$. Let $K \subset Y$ be the maximal subcomplex such that $q(K) \subset B_X(q(x), R)$ and let D'_n be as above. Since D'_n is CAT(0), we can find a geodesic homotopy in D'_n that contracts C to a point in C. Note that q'_n is Lipschitz since it is simplicial. Thus the q–image of this homotopy is contained in $B_X(q(x), R)$, and the homotopy actually happens inside K. This together with property (a) above implies that Y is CAT(0). By (c), Y is homeomorphic to $\mathbb{R}^2$. By Theorem 2.3, $q'(Y)$ is contained in the full subcomplex of X spanned by $q(\mathbb{E}^2)$. This and property (b) above imply $d_H(\mathrm{Im}\, q', \mathrm{Im}\, q) < M_0$ for $M_0 = M_0(L, A, X)$. Also Theorem 2.7 (4) follows from Theorem 2.3 (3) and the properties of q'_n discussed before.

It remains to show Y is flat outside a compact set. By Theorem 2.6, it suffices to estimate the area of balls in Y. Pick a base point $y \in Y$ such that $d(q'(y), q(x)) < M_0$. Let B_R be the R–ball in Y centered at y with respect to the CAT(0) metric and let K_R be the union of faces of Y that intersect B_R. Then there exists n and a simplicial embedding $\phi_R\colon K_R \to D'_n$ such that $q' = q'_n \circ \phi_R$ on K_R. Thus we also view y, B_R and K_R as subsets of D'_n. Since $B_R \subset B_{D'_n}(y, R)$, it suffices to show $\mathrm{Area}(B_{D'_n}(y, R)) \leq L_3R^2$ for L_3 independent of R and n.

Note that there exists a constant $0 < \delta < 1$ independent of n and R such that $B_{D'_n}(y, \delta n)$ does not touch the boundary of D'_n. This uses the fact that q_n and q'_n agree on the boundary, q_n is a quasi-isometry and q'_n is simplicial (hence Lipschitz). We assume n is large enough so that $R < \delta n$.

For $r < \delta n$, let $h_r : B_{D'_n}(y, \delta n) \to B_{D'_n}(y, r)$ be the map which moves every point $p \in B_{D'_n}(y, \delta n)$ towards y along the geodesic $\overline{py}$ by a factor of $\frac{r}{\delta n}$. Then h_r is $\frac{r}{\delta n}$–Lipschitz and we have

$$\frac{\text{Area}(B_{D'_n}(y, \delta n))}{(\delta n)^2} \geq \frac{\text{Area}(B_{D'_n}(y, r))}{r^2}$$

On the other hand,

$$\text{Area}(B_{D'_n}(y, \delta n)) \leq \text{Area}(D'_n) \leq L_1(\ell(\partial D'_n))^2 = L_1(\ell(\partial D_n))^2 \leq L_2 n^2.$$

Here L_1 is the isoperimetric constant for CAT(0) space and $\ell(\partial D'_n)$ denotes the length of $\partial D'_n$. Thus there exists L_3 independent of n and R such that $\text{Area}(B_{D'_n}(y, r)) \leq L_3 r^2$ for any $r < \delta n$. This finishes the proof. □

3. The complexes for dihedral Artin groups

In this section we recall the local structure of the complexes for dihedral Artin groups constructed in [HO19] and study disc diagrams over such complexes.

3.1. **The complex for dihedral Artin groups.** Let DA_n be the 2–generator Artin group presented by $\langle a, b \mid \underbrace{aba\cdots}_{n} = \underbrace{bab\cdots}_{n} \rangle$.

Let P_n be the standard presentation complex for DA_n. Namely the 1–skeleton of P_n is the wedge of two oriented circles, one labeled a and one labeled b. Then we attach the boundary of a closed 2–cell C to the 1–skeleton with respect to the relator of DA_n. Let $C \to P_n$ be the attaching map. Let X_n^* be the universal cover of P_n. Then any lift of the map $C \to P_n$ to $C \to X_n^*$ is an embedding (cf. [HO20, Corollary 3.3]). These embedded discs in X_n^* are called *precells*. Figure 2 depicts a precell Π^*. X_n^* is a union of copies of Π^*'s. We pull back the labeling and orientation of edges in P_n to obtain labeling and orientation of edges in X_n^*. We label the vertices of Π^*

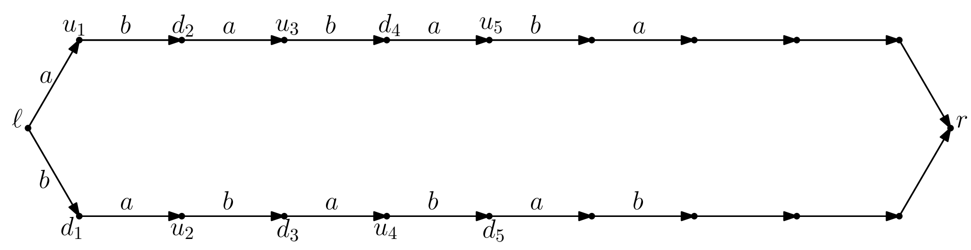


Figure 2. Precell Π^*.

as in Figure 2. The vertices ℓ and r are called the *left tip* and the *right tip* of Π^*. The boundary $\partial\Pi^*$ is made of two paths. The one starting at ℓ, going along $\underbrace{aba\cdots}_{n}$ (resp. $\underbrace{bab\cdots}_{n}$), and ending at r is called the *upper half* (resp. *lower half*) of $\partial\Pi^*$. The orientation of edges inside one half is consistent, thus each half has an orientation.

We summarize several facts on how precells intersect each other.

Lemma 3.1. [HO20, Corollary 3.4] *Let Π_1^* and Π_2^* be two different precells in X_n^*. Then*

(1) either $\Pi_1^* \cap \Pi_2^* = \emptyset$, or $\Pi_1^* \cap \Pi_2^*$ is connected;
(2) if $\Pi_1^* \cap \Pi_2^* \neq \emptyset$, $\Pi_1^* \cap \Pi_2^*$ is properly contained in the upper half or in the lower half of Π_1^* (and Π_2^*);
(3) if $\Pi_1^* \cap \Pi_2^*$ contains at least one edge, then one end point of $\Pi_1^* \cap \Pi_2^*$ is a tip of Π_1^*, and another end point of $\Pi_1^* \cap \Pi_2^*$ is a tip of Π_2^*, moreover, among these two tips, one is a left tip and one is a right tip.

Lemma 3.2. [HO19, Corollary 4.3] *Let Π_1^* and Π_2^* be two different precells in X_n^*. If $\Pi_1^* \cap \Pi_2^*$ contains at least one edge, and $\Pi_3^* \cap \Pi_2^* = \Pi_1^* \cap \Pi_2^*$, then $\Pi_3^* = \Pi_1^*$.*

Lemma 3.3. *Let $\{\Pi_i^*\}_{i=1}^3$ be three different precells in X_n^*. Suppose*

(1) $\Pi_1^* \cap \Pi_2^*$ contains an edge;
(2) $\Pi_1^* \cap \Pi_3^*$ contains an edge;
(3) $(\Pi_1^* \cap \Pi_2^*) \cap (\Pi_1^* \cap \Pi_3^*)$ is either one point or empty.

Then $\Pi_2^* \cap \Pi_3^*$ is either one point or empty.

Proof. By Lemma 3.1, there are two cases to consider. Either $\Pi_1^* \cap \Pi_2^*$ and $\Pi_1^* \cap \Pi_3^*$ are in the same half of Π_1^*, or they are in different halves. The latter case follows from [HO20, Corollary 3.5]. For the former case, we assume without loss of generality that $\Pi_1^* \cap \Pi_2^*$ and $\Pi_1^* \cap \Pi_3^*$ are contained in the upper half of Π_1^*. By Lemma 3.1 (3) and Lemma 3.3 (3), we assume $\Pi_1^* \cap \Pi_2^*$ (resp. $\Pi_1^* \cap \Pi_3^*$) contains the left tip (resp. right tip) of Π_1^*, see Figure 3. Assume by contradiction that $\Pi_2^* \cap \Pi_3^*$ contains an edge. By [HO20, Corollary 3.5], if $\Pi_2^* \cap \Pi_3^*$ and $\Pi_2^* \cap \Pi_1^*$ are in different halves of Π_2^*, then $\Pi_1^* \cap \Pi_3^*$ can not contain any edge, which yields a contradiction. Thus $\Pi_2^* \cap \Pi_3^*$ and $\Pi_2^* \cap \Pi_1^*$ are in the same half of Π_2^*. It follows that P is not an embedded path, where P travels from the left tip of Π_2^* to the left tip of Π_1^*, then travel to the right tip of Π_1^* along the upper half of Π_1^*, then travel to the right tip of Π_3^*. On the other hand, since P represents a word in the positive Artin monoid, P is an embedded path by the injectivity of positive Artin monoid [Del72, BS72], which leads to a contradiction. $\square$

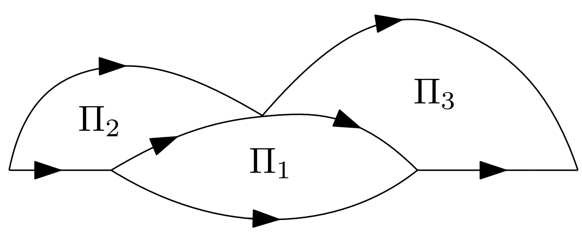


FIGURE 3.

We subdivide each precell Π^* in X_n^* into a simplicial complex by placing a vertex in the middle of the precell, and adding edges which connect this new vertex and vertices in the boundary $\partial\Pi^*$. This subdivision turns X_n^* into a simplicial complex $X_n^{\triangle}$. A *cell* of $X_n^{\triangle}$ is defined to be a subdivided precell, and we use the symbol Π for a cell. The original vertices of X_n^* in $X_n^{\triangle}$ are called the *real vertices*, and the new vertices of $X_n^{\triangle}$ after subdivision are called *fake vertices*. The fake vertex in a cell Π is denoted o.

For each pair of cells (Π_1, Π_2) in $X_n^{\triangle}$ such that $\Pi_1 \cap \Pi_2$ contains at least two edges, we add an edge between the fake vertex of Π_1 and the fake vertex of Π_2. Let X_n be the flag completion of the complex obtained by adding these edges to $X_n^{\triangle}$. There is a simplicial action $DA_n \curvearrowright X_n$.

Definition 3.4. We assign lengths to edges of X_n. Edges connecting a real vertex to a fake vertex have length 1. Edges between two real vertices have length equal to the distance between two adjacent vertices in a Euclidean regular $(2n)$–gon with radius 1.

Now we assign lengths to edges between two fake vertices. First define a function $\phi\colon [0,\pi) \to \mathbb{R}$ as follows. Let $\triangle(ABC)$ be a Euclidean isosceles triangle with length of AB and AC equal to 1, and $\angle_A(B,C) = \alpha$. Then $\phi(\alpha)$ is defined to be the length of BC. Suppose Π_1 and Π_2 are cells such that $\Pi_1 \cap \Pi_2$ contains i edges ($i \geq 2$). Then the edge between the fake vertex of Π_1 and the fake vertex of Π_2 has length $= \phi(\frac{n-i}{2n}2\pi)$.

Remark 3.5 (Intuitive explanation of the construction of X_n)**.** A natural way to metrize X_n^* to declare each 2–cell in X_n^* is a regular polygon in the Euclidean plane. However, if we take Π_1 and Π_2 (say, two n–gons) such that $P = \Pi_1 \cap \Pi_2$ has ≥ 2 edges, then any interior vertex of P is not non-positively curved. Let o_i be the fake vertex in Π_i and let the two endpoints of P be v_1 and v_2. Let K be the region in $\Pi_1 \cup \Pi_2$ bounded by the 4–gon whose vertices are o_1, o_2, v_1 and v_2. Those positively curved cone points are contained in K. Now we replace K by something flat as follows. Add a new edge e between o_1 and o_2 and add two new triangles $\{\Delta_i\}_{i=1}^2$ such that the three sides of Δ_i are e, $\overline{o_1 v_i}$ and $\overline{o_2 v_i}$. We replace K by $\Delta_1 \cup \Delta_2$, which is flat. Moreover, we would like $\angle_{o_1}(v_1, o_2) = \angle_{o_1}(o_2, v_2) = \frac{|P|\pi}{2n}$ so that o_1 is still flat ($|P|$ is the number of edges in P). That is why we assign the length of e as in Definition 3.4.

By [HO19, Lemma 4.7], the lengths of the three sides of each triangle in $X_n^{(1)}$ satisfy the strict triangle inequality. Thus we can treat $X_n^{(2)}$ as a piecewise Euclidean complex such that each 2–simplex is a Euclidean triangle whose lengths of sides coincide with the assigned lengths on $X_n^{(1)}$.

3.2. **Local structure of X_n.** For a vertex $v \in X_n$, define $\Lambda_v = \mathrm{lk}(v, X_n^{(2)})$. In this subsection we study the structure of Λ_v.

Pick an identification between real vertices of X_n and elements of DA_n via the action $DA_n \curvearrowright X_n$. Let Π be a base cell in X_n with its vertices and edges labeled as in Figure 2. We assume $u_0 = d_0 = \ell$, $u_n = d_n = r$, and ℓ is identified with the identity element of DA_n.

We first look at the case when v is a real vertex. Up to action of DA_n, we assume $v = \ell$. Vertices of Λ_v consist of two classes. (1) Real vertices a^i, a^o, b^i and b^o, where a^i and a^o are the vertices in Λ_v which corresponds to the incoming and outgoing a–edge containing v (b^i and b^o are defined similarly). (2) Fake vertices. There is a 1-1 correspondence between such vertices and cells in X_n containing ℓ. Then the fake vertices of Λ_v are of the form $w^{-1}o$ where w is a vertex of $\partial\Pi$ and o is the fake vertex in Π ($w^{-1}o$ means the image of o under the action of w^{-1}), that is $\{\ell^{-1}o, r^{-1}o, d_1^{-1}o, d_2^{-1}o, \ldots, d_{n-1}^{-1}o, u_1^{-1}o, u_2^{-1}o, \ldots, u_{n-1}^{-1}o\}$.

Edges of Λ_v can be divided into two classes. Edges of *type I* in Λ_v are edges between a real vertex and a fake vertex. These edges are drawn in Figure 4 (we use the convention that the real vertices are drawn as solid points and the fake vertices as circles). By [HO19, Lemma 5.2], each edge of type I has angular length $= \frac{n-1}{4n}2\pi$.

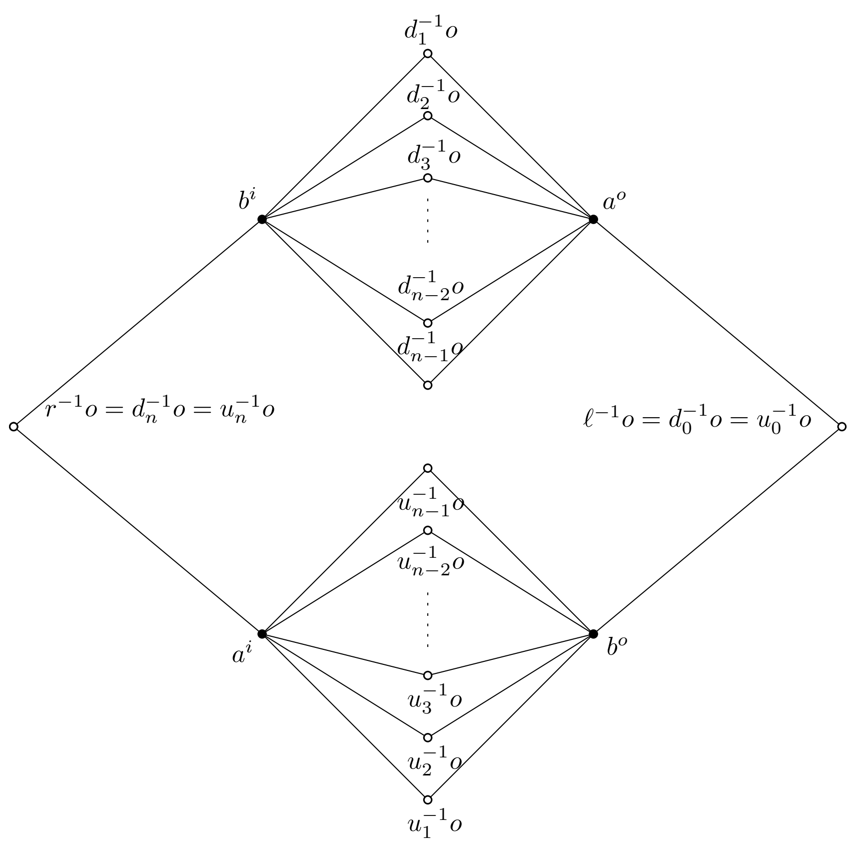


FIGURE 4. Edges of type I in the link of a real vertex.

Edges of *type II* in Λ_v are edges between two fake vertices. They are characterized by Lemma 3.6. There do not exist edges of Λ_v which are between two real vertices. We write $t \sim s$ (resp. $t \nsim s$) if vertices t and s are connected (resp. are not connected) by an edge.

Lemma 3.6.

(1) $d_i^{-1}o \sim d_j^{-1}o$ if and only if $1 \leq |j-i| \leq n-2$, in this case, the edge between $d_i^{-1}o$ and $d_j^{-1}o$ has angular length $= \frac{j-i}{2n}2\pi$. A similar statement holds with d replaced by u.

(2) If $1 \leq i \leq n-1$ and $1 \leq j \leq n-1$, then $d_i^{-1}o \nsim u_j^{-1}o$.

This lemma is deduced from the fact that $d_i^{-1}o \sim d_j^{-1}o$ if and only if $d_i^{-1}\Pi \cap d_j^{-1}\Pi$ has ≥ 2 edges. We refer to [HO19, Lemma 5.3] for a proof.

Now we study cycles in Λ_v. Let Λ_v^+ be the full subgraph of Λ_v spanned by $\{b^i, a^o, d_0^{-1}o, d_1^{-1}o, \dots, d_n^{-1}o\}$. Let Λ_v^- be the full subgraph of Λ_v spanned by $\{b^o, a^i, u_0^{-1}o, u_1^{-1}o, \dots, u_n^{-1}o\}$.

Lemma 3.7. [HO19, Lemma 5.8] *Suppose $\omega \subset \Lambda_v$ is a simple cycle. Then at least one of the following two situations happen:*

(1) $\omega \subset \Lambda_v^+$ or $\omega \subset \Lambda_v^-$;

(2) $\ell^{-1}o \in \omega$ and $r^{-1}o \in \omega$.

Thus $\{\ell^{-1}o, r^{-1}o\}$ is called the *necks* of Λ_v. The next two lemmas give more detailed description of situations (1) and (2) of Lemma 3.7 respectively.

Lemma 3.8. *Suppose ω is a simple cycle in Λ_v^+ or Λ_v^-. Then ω is not 2–full.*

This follows from [HO19, Lemma 5.6] and [HO19, Lemma 5.7].

Lemma 3.9. [HO19, Lemma 5.11] *Suppose v is real and ω is an edge path from $r^{-1}o$ to $\ell^{-1}o$. Then ω has angular length $\geq \pi$.*

If ω has angular length $= \pi$, then either $\omega \subset \Lambda_v^+$ or $\omega \subset \Lambda_v^-$, and the following are the only possibilities of ω when $\omega \subset \Lambda_v^+$:

(1) $\omega = r^{-1}o \to b^i \to d_1^{-1}o \to d_0^{-1}o$;

(2) $\omega = d_n^{-1}o \to d_{n-1}^{-1}o \to a^o \to \ell^{-1}o$;

(3) $\omega = d_{i_1}^{-1}o \to d_{i_2}^{-1}o \to \cdots \to d_{i_k}^{-1}o$ where $n = i_1 > i_2 > \cdots > i_k = 0$.

A similar statement holds for $\omega \subset \Lambda_v^-$.

Now we turn to the case when v is a fake vertex. Up to the action of DA_n, we assume $v = o$. Vertices of Λ_v consists of two classes. (1) Real vertices: these are the vertices in $\partial\Pi$. (2) Fake vertices: these are the fake vertices of cells Π' such that $\Pi' \cap \Pi$ contain at least two edges.

By Lemma 3.1, $\Pi' \cap \Pi$ is a connected path such that it contains exactly one of the tips of Π and it is properly contained in a half of $\partial\Pi$. If $\Pi' \cap \Pi$ is a path in the upper half (resp. lower half) of $\partial\Pi$ that has m edges and contains the left tip, then we denote the fake vertex in Π' by L_m (resp. L'_m), and write $\Pi' = \Pi_{L_m}$ (resp. $\Pi' = \Pi_{L'_m}$). Note that such Π' is unique by Lemma 3.2. Similarly, we define R_m, R'_m, Π_{R_m} and $\Pi_{R'_m}$ respectively; see Figure 5.

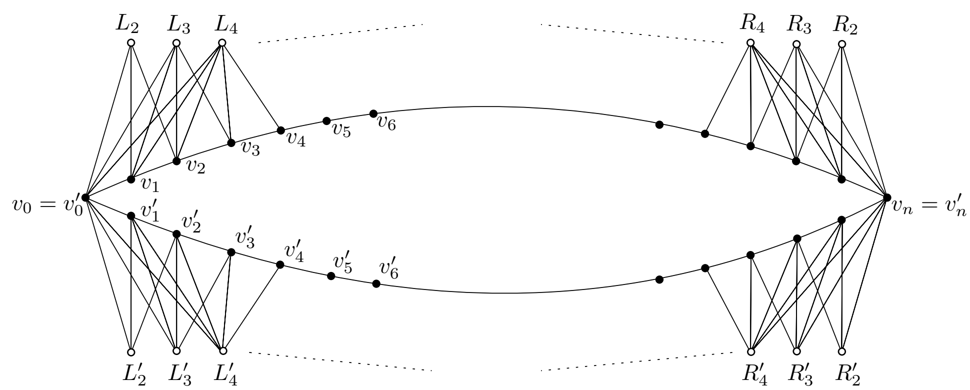


FIGURE 5. The link of a fake vertex.

The structure of Λ_v will be easier to describe if we label vertices of $\partial\Pi$ differently as follows. The vertices in the upper half (resp. lower half) of $\partial\Pi$ are called $v_0, v_1, \ldots, v_n$ (resp. $v'_0, v'_1, \ldots, v'_n$) from left to right. Note that $v_0 = v'_0$ and $v_n = v'_n$.

Edges of Λ_v consist of three classes:

(1) *Edges of type I.* They are edges between real vertices of Λ_v. Hence they are edges in $\partial\Pi$. Each of them has angular length $= \frac{1}{2n}2\pi$.
(2) *Edges of type II.* They are edges between a real vertex and a fake vertex, and they are characterized by Lemma 3.10 below.
(3) *Edges of type III.* They are edges between fake vertices of Λ_v. We will not need information about them in this paper. The interested reader can find a description of them in [HO19, Lemma 5.13].

We refer to Figure 5 for a picture of Λ_v. Edges of type I and some edges of type II are drawn. Edges of type III are not drawn in the picture.

Lemma 3.10.

(1) The collection of vertices in $\partial\Pi$ adjacent to L_i (resp. L'_i) is $\{v_0, v_1, \ldots, v_i\}$ (resp. $\{v'_0, v'_1, \ldots, v'_i\}$).
(2) The collection of vertices in $\partial\Pi$ adjacent to R_i (resp. R'_i) is $\{v_n, v_{n-1}, \ldots, v_{n-i}\}$ (resp. $\{v'_n, v'_{n-1}, \ldots, v'_{n-i}\}$).
(3) The angular length of any edge between L_i and a real vertex of Λ_v is $\frac{i}{4n}2\pi$. The same holds with L_i replaced by L'_i, R_i and R'_i.

Here (1) and (2) follow from the definition of L_i and R_i. (3) follows from Definition 3.4.

Let Λ_v^+ be the full subgraph of Λ_v spanned by

$$\{v_0, v_1, \ldots, v_n\} \cup \{L_2, L_3, \ldots, L_{n-1}\} \cup \{R_2, R_3, \ldots, R_{n-1}\}.$$

Let Λ_v^- be the full subgraph of Λ_v spanned by

$$\{v'_0, v'_1, \dots, v'_n\} \cup \{L'_2, L'_3, \dots, L'_{n-1}\} \cup \{R'_2, R'_3, \dots, R'_{n-1}\}.$$

The following is proved in [HO19, Lemma 5.15] and [HO19, Lemma 5.16]. It says that Lemma 3.7 and Lemma 3.8 continue to hold also in the case of v being fake ($r^{-1}o$ and $\ell^{-1}o$ in the statement of Lemma 3.7 should be replaced by v_0 and v_n).

Lemma 3.11. *Suppose $\omega \subset \Lambda_v$ is a simple cycle. Then at least one of the following two situations happen:*

(1) $\omega \subset \Lambda_v^+$ or $\omega \subset \Lambda_v^-$;
(2) $v_0, v_n \in \omega$.

If ω is a simple cycle in Λ_v^+ or Λ_v^- then ω is not 2–full.

We call $\{v_0, v_n\}$ the *necks* of Λ_v. The following is parallel to Lemma 3.9.

Lemma 3.12. [HO19, Lemma 5.18] *Suppose v is fake and ω is an edge path in Λ_v from v_0 to v_n. Then ω has angular length $\geq \pi$.*

If ω has angular length $= \pi$, then either $\omega \subset \Lambda_v^+$ or $\omega \subset \Lambda_v^-$, and the following are the only possibilities of ω when $\omega \subset \Lambda_v^+$:

(1) ω does not contain fake vertices, i.e. $\omega = v_0 \to v_1 \to \cdots \to v_n$;
(2) $\omega = v_0 \to v_1 \to \cdots \to v_{n-i_1} \to R_{i_1} \to \cdots \to R_{i_m} \to v_n$, where $i_1 > \cdots > i_m \geq 2$;
(3) $\omega = v_0 \to L_{i_1} \to \cdots \to L_{i_m} \to v_{i_m} \to v_{i_m+1} \to \cdots \to v_n$, where $2 \leq i_1 < \cdots < i_m$;
(4) $\omega = v_0 \to L_{i_1} \to \cdots \to L_{i_m} \to v_{i_m} \to v_{i_{m+1}} \to \cdots \to v_{n-i'_1} \to R_{i'_1} \to \cdots \to R_{i'_{m'}} \to v_n$, where $2 \leq i_1 < \cdots < i_m$, $i'_1 > \cdots > i'_{m'} \geq 2$, and $i_m \leq n - i'_1$.

A similar statement holds when $\omega \subset \Lambda_v^-$.

Remark 3.13. Note that in all the cases (1)–(4) of Lemma 3.12, for any two non-neck vertices $u_1, u_2 \in \omega$ with u_1 real and u_2 fake, either u_1 is a tip of the cell Π_{u_2} that contains u_2, or $u_1 \notin \Pi_{u_2}$. This statement can be deduced from Lemma 3.10 and Lemma 3.1.

It follows from Lemma 3.7, Lemma 3.8, Lemma 3.9, Lemma 3.11 and Lemma 3.12 that X_n is locally 2π–large (cf. Definition 2.2). Moreover, we showed in [HO19, Lemma 6.4] that X_n is simply-connected.

Theorem 3.14. *X_n is metrically systolic.*

3.3. **Flat vertices in disc diagrams over X_n.** In this subsection, we explore links of certain flat vertices in reduced disc diagrams.

Definition 3.15. A flat interior vertex in a disc diagram whose all neighbours are flat interior vertices is called a *deep flat vertex.*

Lemma 3.16. *Suppose $q' \colon Y \to X_n$ satisfies all the conditions in Theorem 2.7. Let $y \in Y$ be a flat interior vertex and let $v = q'(y)$. Then*

$q'(\mathrm{lk}(y, Y))$ is a simple cycle in Λ_v that is made of two paths of angular length $= \pi$ connecting the two necks of Λ_v, one in Λ_v^+ and one in Λ_v^-.

Proof. Since q' is injective on $\mathrm{St}(y, Y)$, to simplify notation, we identify $\mathrm{lk}(y, Y)$ and $q'(\mathrm{lk}(y, Y))$. By Lemma 3.7, Lemma 3.9 and Lemma 3.12, it suffices to rule out the cases $\mathrm{lk}(y, Y) \subset \Lambda_v^+$ and $\mathrm{lk}(y, Y) \subset \Lambda_v^-$. To rule out the former case, note that $\mathrm{lk}(y, Y)$ is a simple cycle in Λ_v^+, then by Lemma 3.8 (or Lemma 3.11), we can add a local diagonal to $\mathrm{lk}(y, Y)$ to cut it into a triangle and a cycle σ with smaller number of edges. Now we apply same procedure to σ (note that $\sigma \subset \Lambda_v^+$ since Λ_v^+ is a full subgraph of Λ_v by definition). By repeating this process finitely many times, we know that $\mathrm{lk}(y, Y)$ bounds a disc diagram without interior vertices, which contradicts Theorem 2.7 (4). Similarly, we rule out the case $\mathrm{lk}(y, Y) \subset \Lambda_v^-$. □

Lemma 3.17. *Suppose $q' \colon Y \to X_n$ satisfies all the conditions in Theorem 2.7. Let $v \in Y$ be a deep flat vertex. Then there do not exist two adjacent non-neck fake vertices in $q'(\mathrm{lk}(v, Y))$.*

To simply notation, denote $q'(v)$ by v and identify $\mathrm{lk}(v, Y)$ and $q'(\mathrm{lk}(v, Y))$.

Proof. Arguing by contradiction we assume there are two adjacent non-neck fake vertices o_1, o_2 in $\mathrm{lk}(v, Y)$. We assume that $v = o$ when v is fake (recall that o is the fake vertex in the base cell Π), and $v = \ell$ when v is real. For $i = 1, 2$, let Π_i be the cell containing o_i.

Case 1: $v = \ell$. By Lemma 3.16, $\mathrm{lk}(v, Y)$ consists of two paths σ_1, σ_2 of angular length $= \pi$ connecting the two necks of Λ_v. By Lemma 3.9, we can assume that $o_1, o_2 \in \sigma_1 \subset \Lambda_v^+$. Then σ_1 must satisfy Lemma 3.9 (3), moreover, we can assume $d_{i_2}^{-1} o = o_1$ and $d_{i_3}^{-1} o = o_2$. See Figure 6.

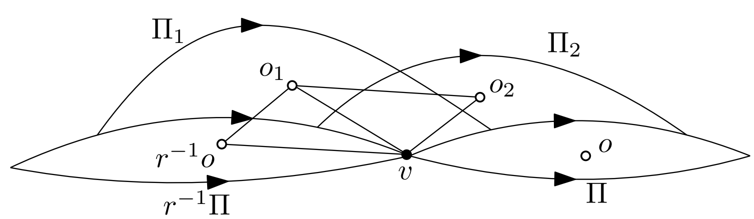


FIGURE 6.

Now we consider the path $P = r^{-1}o \to v \to o_2$ in $\mathrm{lk}(o_1, Y)$. Since v is not a tip of Π_1, v is not a neck of Λ_{o_1}. By Lemma 3.16, there is a path σ of angular length $= \pi$ connecting the two necks of Λ_{o_1} such that $P \subset \sigma \subset \mathrm{lk}(o_1, Y)$. Since P has a real vertex between two fake vertices, σ satisfies Lemma 3.12 (4). It follows from Lemma 3.1 that $r^{-1}\Pi \cap \Pi_1 \cap \Pi_2$ contains at least one edge. However, by Lemma 3.10, in case (4) of Lemma 3.12, $\Pi_{L_{i_m}} \cap \Pi \cap \Pi_{R_{i'_1}}$ is either empty (when $i_m < n - i'_1$) or one point (when $i_m = n - i'_1$). This leads to a contradiction.

Case 2: $v = o$. By Lemma 3.16, o_1 and o_2 are consecutive non-neck fake vertices in a path of angular length π between the two necks of Λ_o. By Lemma 3.12, we can assume either $o_1 = L_{i_1}$, $o_2 = L_{i_2}$, or $o_1 = R_{i'_{m'-1}}$, $o_2 = R_{i'_{m'}}$. Now we study the former case. See Figure 7. Consider the path

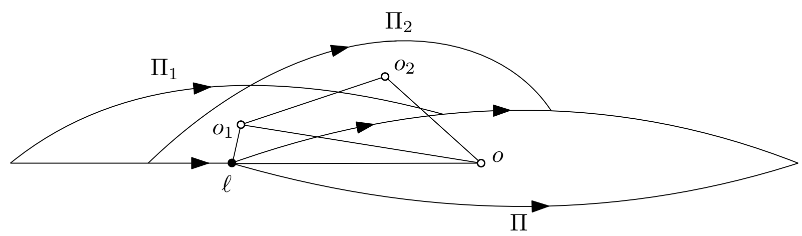


FIGURE 7.

$P = \ell \to o \to o_2$ in $\mathrm{lk}(o_1, Y)$. Since ℓ is not a tip of Π_1, ℓ is not a neck of Λ_{o_1}. It follows that P is contained in a path $\sigma \subset \Lambda_{o_1}$ of angular length $= \pi$ connecting the two necks of Λ_{o_1}. Note that $\ell \in \Pi_2$, but ℓ is not a tip of Π_2 (by applying Lemma 3.1 (3) to $\Pi_2 \cap \Pi$). This contradicts Remark 3.13. The case $o_1 = R_{i'_{m'-1}}$ and $o_2 = R_{i'_{m'}}$ can be treated similarly. □

4. LOCAL STRUCTURE OF QUASI-EUCLIDEAN DIAGRAMS

In this section we recall the construction of the metrically systolic complexes for two-dimensional Artin groups, and use these complexes as a tool to study the local structure of quasi-Euclidean diagrams over the universal covers of standard presentation complexes of two-dimensional Artin groups.

4.1. **The complex for 2–dimensional Artin groups.** Let A_Γ be an Artin group with defining graph Γ. Let $\Gamma' \subset \Gamma$ be a full subgraph with induced edge labeling and let $A_{\Gamma'}$ be the Artin group with defining graph Γ'. Then there is a natural homomorphism $A_{\Gamma'} \to A_\Gamma$. By [vdL83], this homomorphism is injective. Subgroups of A_Γ of the form $A_{\Gamma'}$ are called *standard subgroups*. Let P_Γ be the standard presentation complex of A_Γ, and let X^*_Γ be the universal cover of P_Γ. We orient each edge in P_Γ and label each edge in P_Γ by a generator of A_Γ. Thus edges of X^*_Γ have induced orientation and labeling. There is a natural embedding $P_{\Gamma'} \hookrightarrow P_\Gamma$. Since $A_{\Gamma'} \to A_\Gamma$ is injective, $P_{\Gamma'} \hookrightarrow P_\Gamma$ lifts to various embeddings $X^*_{\Gamma'} \to X^*_\Gamma$. Subcomplexes of X^*_Γ arising in such way are called *standard subcomplexes*.

A *block* of X^*_Γ is a standard subcomplex which comes from an edge in Γ. This edge is called the *defining edge* of the block. Two blocks with the same defining edge are either disjoint, or identical. A block is *large* if the label of the corresponding edge is at least 3.

We define precells of X^*_Γ as in Section 3.1. Each precell is embedded. We subdivide each precell as in Section 3.1 to obtain a simplicial complex $X^\triangle_\Gamma$. Fake vertices and real vertices of $X^\triangle_\Gamma$ are defined in a similar way.

Within each block of $X^{\triangle}_{\Gamma}$, we add edges between fake vertices as in Section 3.1. Then we take the flag completion to obtain X_Γ. The action $A_\Gamma \curvearrowright X^{\triangle}_{\Gamma}$ extends to a simplicial action $A_\Gamma \curvearrowright X_\Gamma$, which is proper and cocompact. A *block* in X_Γ is defined to be the full subcomplex spanned by vertices in a block of $X^{\triangle}_{\Gamma}$. Intersection of two different blocks of X_Γ does not contain fake vertices. Pick a block $B^{\Delta} \subset X^{\triangle}_{\Gamma}$, and let $B \subset X_\Gamma$ be the block containing B^{Δ}. Let n be the label of the defining edge of B^{Δ}. Then by [HO19, Lemma 6.3], the natural isomorphism $B^{\Delta} \to X^{\triangle}_{n}$ extends to an isomorphism $B \to X_n$.

Next we assign lengths to edges of X_Γ. Let $B \subset X_\Gamma$ be a block and let $B \to X_n$ be the isomorphism in the previous paragraph. We first rescale the edge lengths of X_n defined in Section 3.1 by a uniform factor such that any edge between two real vertices has length $= 1$. Then we pull back these edge lengths to B by the above isomorphism. We repeat this process for each block of X_Γ. Note that if an edge of X_Γ belongs to two different blocks, then this edge is between two real vertices, hence it has a well-defined length (which is 1). The action of A_Γ preserves edge lengths.

Theorem 4.1. [HO19, Theorem 6.1] *If A_Γ has dimension ≤ 2, then X_Γ with its piecewise Euclidean structure is metrically systolic.*

From now on we will assume A_Γ has dimension ≤ 2.

Now we consider local structure of X_Γ. If $v \in X_\Gamma$ is a fake vertex, then there is a unique block $B \ni v$. Moreover, $\mathrm{lk}(v, X^{(2)}_{\Gamma}) = \mathrm{lk}(v, B^{(2)})$ by our construction, which reduces to discussion in Section 3.2. Links of real vertices are more complicated since they can travel through several blocks. However, cycles of angular length $\leq 2\pi$ in the link have a relatively simple characterization as follows.

Lemma 4.2. [HO19, Lemma 6.7] *Let $v \in X_\Gamma$ be a real vertex and let ω be a simple cycle in $\mathrm{lk}(v, X^{(2)}_{\Gamma})$ with angular length $\leq 2\pi$ in the link of v. Then exactly one of the following four situations happens:*

(1) ω is contained in one block;
(2) ω travels through two different blocks B_1 and B_2 such that their defining edges intersect in a vertex a, and ω has angular length $= \pi$ inside each block, moreover, there are exactly two vertices in $\omega \cap B_1 \cap B_2$ and they corresponding to an incoming a–edge and an outgoing a–edge based at v;
(3) ω travels through three blocks B_1, B_2, B_3 such that the defining edges of these blocks form a triangle $\triangle(abc) \subset \Gamma$ and $\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = 1$ where n_1, n_2 and n_3 are labels of the edges of this triangle, moreover, ω is a 6–cycle with its vertices alternating between real and fake such that the three real vertices in ω correspond to an a–edge, a b–edge and a c–edge based at v;

(4) ω travels through four blocks such that the defining edges of these blocks form a full 4–cycle in Γ, moreover, ω is a 4–cycle with one edge of angular length $\pi/2$ in each block.

Note that in cases (2), (3) and (4), ω actually has angular length $= 2\pi$.

4.2. **Classification of flat points in diagrams over X_Γ.** Let $q' : Q' \to X_\Gamma$ be a map which satisfies all the requirements of Theorem 2.7. We study the local properties of q' and Q' in this subsection.

We define a partial retraction ρ from $X^{(2)}_\Gamma$ (defined on its subset) to $X^{\triangle}_\Gamma$ as follows. The map is the identity on the 2–skeleton of $X^{\triangle}_\Gamma$. Let $\overline{o_1o_2}$ be an edge of $X^{(2)}_\Gamma$ not in $X^{\triangle}_\Gamma$, where o_i is the fake vertex in some cell Π_i for $i = 1, 2$. Let m be the middle point of $\partial\Pi_1 \cap \partial\Pi_2$ and let x_1, x_2 be the two endpoints of $\partial\Pi_1 \cap \partial\Pi_2$. We map $\overline{o_1o_2}$ homeomorphically to the concatenation of $\overline{o_1m}$ and $\overline{mo_2}$, and map the triangle $\triangle(o_1o_2x_1)$ (resp. $\triangle(o_1o_2x_2)$) homeomorphically to the region in $\Pi_1 \cup \Pi_2$ bounded by $\overline{o_1x_1}$, $\overline{x_1o_2}$, $\overline{o_2m}$ and $\overline{mo_1}$ (resp. $\overline{o_1x_2}$, $\overline{x_2o_2}$, $\overline{o_2m}$ and $\overline{mo_1}$). This finishes the definition of ρ. Note that ρ is not defined on the whole space $X^{(2)}_\Gamma$. That is why we call it a partial retraction. More precisely, for a triangle $\Delta \subset X^{(2)}_\Gamma$, $\rho(\Delta)$ is defined if and only if Δ satisfies one of the following two cases:

(1) Δ has two real vertices and one fake vertices, in this case $\rho(\Delta) = \Delta$;
(2) Δ has one real vertex v and two interior vertices o_1 and o_2, moreover, v is a tip of Π_1 or Π_2, where Π_i is the cell containing o_i for $i = 1, 2$.

Lemma 4.3. *Let $q = \rho \circ q'$. If $\Delta \subset Q'$ is a triangle such that all of its vertices are flat and deep, then $q(\Delta)$ is defined.*

Proof. If all vertices of $q'(\Delta)$ are fake, then for any vertex $x \in \Delta$, the cycle $q'(\mathrm{lk}(x, Q'))$ in $\mathrm{lk}(q'(x), X^{(2)}_\Gamma)$ contains two consecutive fake vertices, hence contradicts Lemma 3.17. If $q'(\Delta)$ has one real vertex u and two interior vertices o_1 and o_2, but u is neither the tip of Π_1 nor the tip of Π_2 (Π_i is the cell containing o_i), then $u \in \Pi_1$ and u is not a neck of $\mathrm{lk}(o_2, X^{(2)}_\Gamma)$, which contradicts Remark 3.13 (let the non-neck real vertex and the fake vertex in Remark 3.13 be u and o_1 respectively). $\square$

In what follows we use the map q as in Lemma 4.3 (wherever it is well-defined). A vertex $x \in Q'$ is *real* or *fake* if $q'(x) = q(x)$ is, respectively, real or fake.

Lemma 4.4. *Suppose $x \in Q'$ is flat, deep and fake, then the restriction of q to $\mathrm{St}(x, Q')$ is an embedding. Moreover, $q(\mathrm{St}(x, Q'))$ contains the cell of $X^{\triangle}_\Gamma$ containing $q(x)$.*

Proof. By Theorem 2.7, the restriction of q' to the closed star $\mathrm{St}(x, Q')$ is an embedding. Let $v = q'(x)$ and let $\omega = q'(\mathrm{lk}(x, Q'))$. Then

(1) ω is a concatenation of two paths ω^+ and ω^- such that each of them connects the two necks of $\Lambda_v = \mathrm{lk}(v, X^{(2)}_\Gamma)$, and $\omega^+ \subset \Lambda^+_v$, $\omega^- \subset \Lambda^-_v$;

(2) ω^+ and ω^- do not contain consecutive fake vertices.

(1) follows from Lemma 3.16, (2) follows from Lemma 3.17. By (2) and Lemma 3.12, there are only four possibilities for ω^+:

(1) ω^+ does not contain fake vertices, i.e. $\omega = v_0 \to v_1 \to \cdots \to v_n$;
(2) $\omega^+ = v_0 \to v_1 \to \cdots \to v_{n-i} \to R_i \to v_n$, where $i \geq 2$;
(3) $\omega^+ = v_0 \to L_i \to v_i \to v_{i+1} \to \cdots \to v_n$, where $2 \leq i$;
(4) $\omega^+ = v_0 \to L_i \to v_i \to v_{i+1} \to \cdots \to v_{n-j} \to R_j \to v_n$, where $2 \leq i$, $j \geq 2$, and $i \leq n - j$.

A similar statement holds for ω^-. Then the lemma follows from Lemma 3.10 and the definition of ρ (see Figure 8 for an example). □

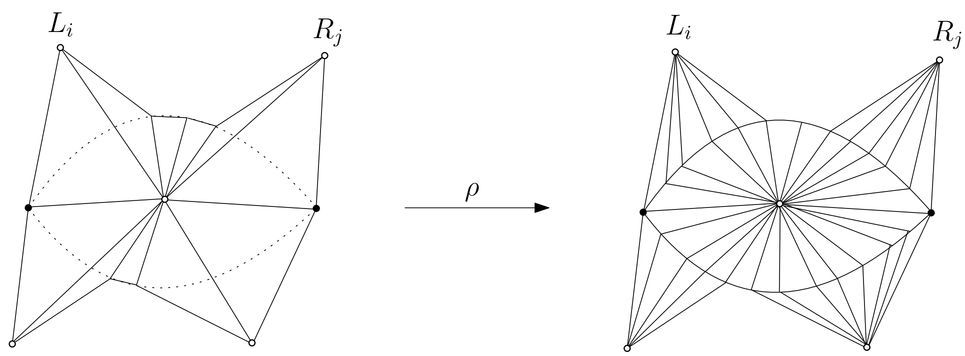


FIGURE 8.

Let x be as in Lemma 4.4. Then for any fake vertex $x' \in \mathrm{lk}(x, Q')$, there are exactly two triangles in $\mathrm{St}(x, Q')$ containing x'. The union of these two triangles is called an *ear* of $\mathrm{St}(x, Q')$. Note that $\mathrm{St}(x, Q')$ can have at most four ears. By Lemma 4.4 again, there is a subset of $\mathrm{St}(x, Q')$ which is mapped to the cell containing $q'(x)$ homeomorphically by q. We denote this subset by C_x. Note that the boundary ∂C_x cuts through the ears of $\mathrm{St}(x, Q')$ and follows the edges in $\mathrm{lk}(x, Q')$ that are between two real vertices.

Lemma 4.5. *Suppose $x \in Q'$ is flat, deep and real. Let $v = q'(x)$. Suppose the cycle $\omega = q'(\mathrm{lk}(x, Q'))$ in $\mathrm{lk}(v, X^{(2)}_\Gamma)$ satisfies Lemma 4.2 (1), then the restriction of q to $\mathrm{St}(x, Q')$ is an embedding. Moreover, there are exactly four fake vertices in ω, and the four cells containing these four fake vertices can be ordered as $\{\Pi_i\}_{i=1}^4$ such that (see Figure 9)*

(1) each Π_i contains v, and $\Pi_1, \Pi_2, \Pi_3, \Pi_4$ are in the same block;
(2) v is the right tip of Π_2 and is the left tip of Π_1;
(3) v is not a tip of Π_3 or Π_4;
(4) each of $\Pi_3 \cap \Pi_2$, $\Pi_3 \cap \Pi_1$, $\Pi_4 \cap \Pi_2$ and $\Pi_4 \cap \Pi_1$ contains at least one edge, hence $\Pi_i \cap (\Pi_1 \cup \Pi_2)$ is a half of Π_i for $i = 3, 4$ by Lemma 3.1;
(5) if in addition each vertex in $\mathrm{lk}(x, Q')$ is deep and flat, then $\mathrm{St}(x, Q')$ is contained in $\cup_{i=1}^4 C_{x_i}$, where x_i is the fake vertex in $\mathrm{lk}(x, Q')$ such that $q(x_i)$ is the fake vertex in Π_i;

(6) under the assumption of (5), $x \in C_{x_i}$ for each i.

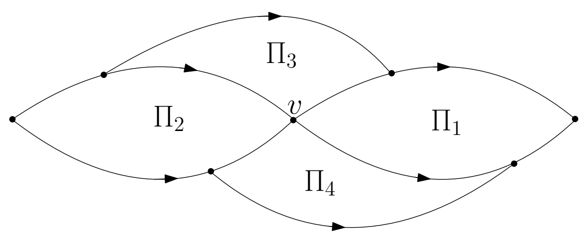


FIGURE 9.

Proof. Let B be the block containing ω. We argue as before to deduce ω is a concatenation of two paths σ_1 and σ_2 connecting two necks of $\mathrm{lk}(v, B^{(1)})$ such that each of them satisfies one of the three possibilities in Lemma 3.9. Moreover, by Lemma 3.17, $k = 3$ in Lemma 3.9 (3). Thus there are four fake vertices in ω. We can assume $\Pi_1 = \Pi$ is the base cell of X_n. Then the conclusions (1)-(4) of the lemma follows by letting $\Pi_2 = r^{-1}\Pi$, $\Pi_3 = d_i^{-1}o$ and $\Pi_4 = u_j^{-1}o$ for some $1 \leq i, j \leq n-1$. For (5), it follows from our assumption that the link of each fake vertex in $\mathrm{lk}(x, Q')$ is as described in the proof of Lemma 4.4. Thus (5) follows from the definition of ρ and C_{x_i}. To see (6), note that for each C_{x_i}, x is either a real vertex in an ear of $\mathrm{St}(x_i, Q')$, or x is inside an edge of $\mathrm{lk}(x_i, Q')$ whose two end points are real. Thus $x \in C_{x_i}$. We refer to Figure 10 for a particular example. □

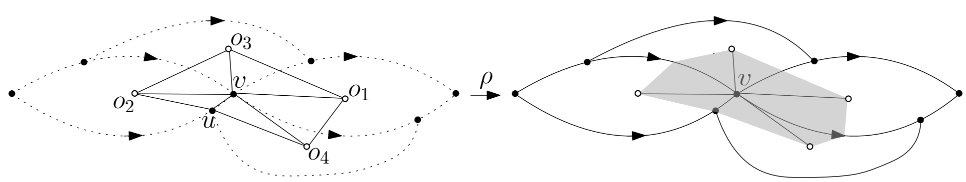


FIGURE 10. The path $o_2 \to o_3 \to o_1$ satisfies Lemma 3.9 (3), and the path $o_2 \to u \to o_4 \to o_1$ satisfies Lemma 3.9 (1).

Lemma 4.6. *Let x, v, ω be as in Lemma 4.5. If ω satisfies Lemma 4.2 (2), then the restriction of q to $\mathrm{St}(x, Q')$ is an embedding.*

Let $\overline{ab}, \overline{ac} \subset \Gamma$ be the defining edges of B_1 and B_2 as in Lemma 4.2 (2). For $i = 1, 2$, let the label of the defining edge of B_i be n_i. Then there are exactly four fake vertices in ω, and the four cells contains these four fake vertices can be ordered as $\{\Pi_i\}_{i=1}^4$ (see Figure 11) such that

(1) each Π_i contains v;
(2) Π_1 and Π_2 are in B_1, and Π_3 and Π_4 are in B_2;
(3) $\Pi_1 \cap \Pi_2$ has $n_1 - 1$ edges, $\Pi_3 \cap \Pi_4$ has $n_2 - 1$ edges, $\Pi_1 \cap \Pi_3$ intersects along an a–edge, and $\Pi_2 \cap \Pi_4$ intersects along an a–edge;

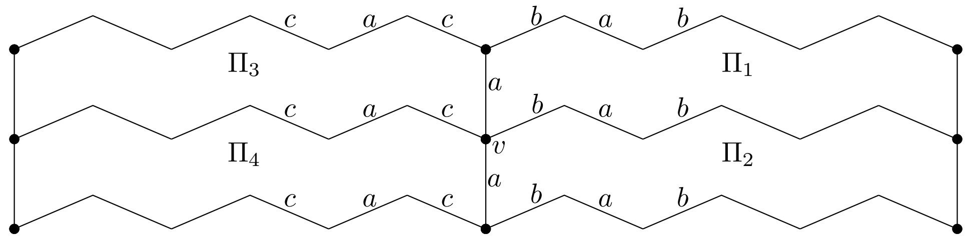


FIGURE 11.

(4) the analogous statements of Lemma 4.5 (5) and (6) hold.

Remark 4.7. We did not specify orientations of edges in Figure 11. However, it follows from Lemma 4.6 (3) that all vertical edges in Figure 11 have orientations pointing towards the same direction (all up or all down). All non-vertical edges in Figure 11 that are in B_1 have orientations pointing towards the same direction (all left or all right), and a similar statement holds with B_1 replaced by B_2.

Proof of Lemma 4.6. We denote the two vertices in $\omega \cap B_1 \cap B_2$ by a^i and a^o respectively. For $i = 1, 2$, let σ_i be the sub-path of ω from a^o to a^i in $\mathrm{lk}(v, B_i^{(1)})$. We now study the possibilities for σ_1. We identify B_1 with X_{n_1} and assume $v = \ell$ in the cell Π of X_{n_1}.

By the discussion in Section 3.2, all edges of type II in $\mathrm{lk}(v, B_1^{(1)})$ (see Figure 4) are between two interior vertices, and there are no edges between real vertices. Thus to travel from one real vertex to another real vertex in $\mathrm{lk}(v, B_1^{(1)})$, one has to go through at least two edges of type I. However, only two edges of type I do not bring one from a^i to a^o. So we need at least one another edge. By Lemma 3.6, an edge in $\mathrm{lk}(v, B_1^{(1)})$ has angular length at least $\frac{1}{2n}2\pi$. Thus σ is made of two edges of type I and one edges of type II with minimal angular length. Thus $\sigma_1 = a^o \to u_0^{-1}o \to u_1^{-1}o \to a^i$ or $\sigma_1 = a^o \to d_{n_1-1}^{-1}o \to d_n^{-1}o \to a^i$. Note that $u_0^{-1}\Pi \cap u_1^{-1}\Pi$ has $n_1 - 1$ edges, and $d_{n-1}^{-1}\Pi \cap d_n^{-1}\Pi$ has $n_1 - 1$ edges (see Figure 12). A similar statement holds for σ_2. Now the lemma follows from the definition of ρ. $\square$

Lemma 4.8. *Let x, v, ω be as in Lemma 4.5. If ω satisfies Lemma 4.2 (3), then the restriction of q to $\mathrm{St}(x, Q')$ is an embedding. Moreover, let $\{B_i\}_{i=1}^3$ be as in Lemma 4.2 (3). Then there are three cells $\{\Pi_i \subset B_i\}_{i=1}^3$ as in Figure 13 (left) such that*

(1) $v \in \Pi_i$ for $1 \le i \le 3$;
(2) each of $\Pi_1 \cap \Pi_2$, $\Pi_2 \cap \Pi_3$, and $\Pi_3 \cap \Pi_1$ consists of one edge;
(3) the analogous statements of Lemma 4.5 (5) and (6) hold.

A similar statement holds when ω satisfies Lemma 4.2 (4), where we have four squares around v, see Figure 13 right.

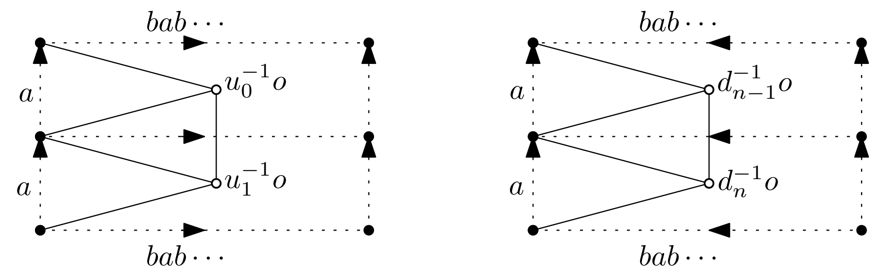


FIGURE 12.

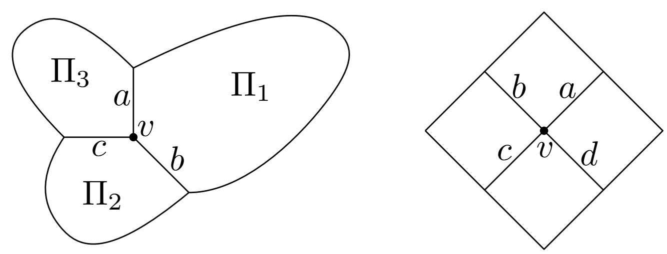


FIGURE 13.

Proof. Take $\omega = \sigma_1 \cup \sigma_2 \cup \sigma_3$ where each σ_i is a sub-path of ω made of two edges such that $\sigma_i \subset B_i$. For $1 \leq i \leq 3$, we take Π_i to be the cell that contains the fake vertex in σ_i. Then $\sigma_i \subset \Pi_i$. Thus $q'(\mathrm{St}(x, Q')) \subset X^{\triangle}_{\Gamma}$. Consequently, ρ is the identity on $q'(\mathrm{St}(x, Q'))$. Hence the lemma follows. $\square$

4.3. **A new cell structure on the quasiflat.** It follows from Lemma 4.3, Lemma 4.4, Lemma 4.5, Lemma 4.6 and Lemma 4.8 that if $x \in Q'$ is flat and deep, then the restriction of q to $\mathrm{St}(x, Q')$ is well-defined (in particular, it is well-defined outside a compact set) and it is an embedding. Now we think of the range of q as X^*_Γ. In this subsection we want to "pull back" the cell structure on X^*_Γ to an appropriate subset of Q' via q.

Choose a base point $x_0 \in Q'$. For $R > 0$, let Q'_R be the smallest subcomplex of Q' that contains $Q' \setminus B(x_0, R)$. Recall that $B(x_0, R)$ is the ball of radius R centered at x_0. Let Q'_{R^+} be the subcomplex made of all triangles of Q' which have non-trivial intersection with Q'_R. By Theorem 2.7, we choose R large enough such that each vertex in Q that has combinatorial distance ≤ 5 from Q'_{R^+} is flat and deep.

Lemma 4.9. *Suppose $C_{x_1} \cap C_{x_2} \neq \emptyset$ for two fake, deep and flat vertices $x_1, x_2 \in Q'$. We also assume each vertex in $\mathrm{St}(x_1, Q')$ is flat and deep. Let Π_i be the cell containing $o_i = q(x_i)$. Then q maps $C_{x_1} \cup C_{x_2}$ homeomorphically onto $\Pi_1 \cup \Pi_2$. In particular, $C_{x_1} \cap C_{x_2}$ is a connected interval (possibly degenerate).*

Proof. Note that x_1 and x_2 have combinatorial distance ≤ 2 in Q'. We claim that if x_1 and x_2 are not adjacent, then o_1 and o_2 are not adjacent. To see this, note that there exists $x \in \mathrm{lk}(x_1, Q')$ such that $x_1, x_2 \in \mathrm{lk}(x, Q')$. Since x is deep and flat, the claim follows from the descriptions of $\mathrm{lk}(x, Q')$ in Lemma 4.4, Lemma 4.5, Lemma 4.6 and Lemma 4.8.

If x_1 and x_2 are adjacent, then so are o_1 and o_2. It is clear that $q(C_{x_1} \cap C_{x_2}) \subset \Pi_1 \cap \Pi_2$. Let ω be the arc on ∂C_{x_1} which is mapped homeomorphically to $\Pi_1 \cap \Pi_2 \subset \partial \Pi_1$. Then $C_{x_1} \cap C_{x_2} \subset \omega$ since $q|_{C_{x_1}}$ is an embedding. However, ω cuts through an ear of $\mathrm{St}(x_1, Q')$ thus, by definition of C_{x_i}, we have $\omega \subset C_{x_1} \cap C_{x_2}$. Hence $C_{x_1} \cap C_{x_2} = \omega$ and the lemma follows.

Suppose x_1 and x_2 are not adjacent. Since $C_{x_1} \cap C_{x_2} \neq \emptyset$, the intersection $\mathrm{St}(x_1, Q') \cap \mathrm{St}(x_2, Q')$ consists of at least one vertex. If there is a fake vertex $x \in \mathrm{St}(x_1, Q') \cap \mathrm{St}(x_2, Q')$, then let Π_x be the cell containing $q(x)$. Since x_1 and x_2 are fake vertices in $\mathrm{lk}(x, Q')$, by the description of $\mathrm{lk}(x, Q')$ in Lemma 4.4, we know $\Pi_1 \cap \Pi_x$ contains an edge, $\Pi_2 \cap \Pi_x$ contains an edge and $(\Pi_1 \cap \Pi_x) \cap (\Pi_2 \cap \Pi_x)$ is at most one point. Thus $\Pi_1 \cap \Pi_2$ is one point by Lemma 3.3 (note that Π_1, Π_2 and Π_x are in the same block). Thus the lemma follows by the discussion in the previous paragraph. Now suppose there are no fake vertices in $\mathrm{St}(x_1, Q') \cap \mathrm{St}(x_2, Q')$. If there is an edge e in $\mathrm{St}(x_1, Q') \cap \mathrm{St}(x_2, Q')$, then $e \subset C_{x_1} \cap C_{x_2}$ since both endpoints of e are real. Thus $q(e)$ is an edge in $\Pi_1 \cap \Pi_2$. However, $\Pi_1 \cap \Pi_2$ has at most one edge since o_1 and o_2 are not adjacent. Thus $e = C_{x_1} \cap C_{x_2}$ and the lemma follows. If there are no edges in $\mathrm{St}(x_1, Q') \cap \mathrm{St}(x_2, Q')$, then let x be a vertex in this intersection. Since x_1 and x_2 are two fake vertices in $\mathrm{lk}(x, Q')$, in all cases of Lemma 4.5, Lemma 4.6 and Lemma 4.8, $\Pi_1 \cap \Pi_2$ is a point whenever $\mathrm{St}(x_1, Q') \cap \mathrm{St}(x_2, Q')$ is a real vertex. Hence the lemma follows. $\square$

Lemma 4.10. *Q'_R is contained in the union of C_x with x varying among vertices of Q'_{R+} that are flat, deep and fake.*

Proof. Recall that there are no triangles with three fake vertices in X_Γ, thus the same is true for Q'. Thus Q'_R is contained in the union of $\mathrm{St}(x, Q')$ with x ranging over real vertices in Q'_R. By Lemma 4.5, Lemma 4.6, Lemma 4.8 and our choice of R, for any real $x \in Q'_R$, $\mathrm{St}(x, Q')$ is contained in the union of C_y with y varying among fake vertices in $\mathrm{lk}(x, Q')$. Note that $y \in Q'_{R+}$. Then the lemma follows. $\square$

Let Q_R be the union of C_x with x varying among fake vertices of Q'_{R+}. By Lemma 4.9 and Lemma 4.10, Q_R has a well-defined cell structure whose closed 2–cells are the C_x, and whose edges (resp. vertices) are arcs (resp. points) in the boundary of C_x which are mapped to edges (resp. vertices) in X^*_Γ by q. Also, Lemma 4.9 implies that we can pullback the orientation and labeling of edges of X^*_Γ to orientation and labeling of edges of Q_R.

A vertex of Q_R is *interior* if it has a neighborhood in Q_R which is homeomorphic to an open disc. Let $\mathrm{St}(x, Q_R)$ be the union of cells of Q_R that contain x. Now we look at the structure of $\mathrm{St}(x, Q_R)$.

Lemma 4.11. *Let $x \in Q_R$ be an interior vertex. Then the following are the only possibilities for* $\mathrm{St}(x, Q_R)$.

(1) $\mathrm{St}(x, Q_R)$ *is a union of two 2–cells.*
(2) The point x corresponds to a real vertex in Q', and q maps $\mathrm{St}(x, Q_R)$ *homeomorphically onto the union of the Π_i in X^*_Γ described in Lemma 4.5, Lemma 4.6, or Lemma 4.8.*

Proof. Suppose x is not a real vertex of Q'. There are at least two 2–cells C_{x_1}, C_{x_2} in Q_R that contain x. Thus x is in the interior of an ear of $\mathrm{St}(x_i, Q')$ for $i = 1, 2$. Hence x_1 and x_2 are adjacent in Q', and $\mathrm{St}(x_1, Q')$ and $\mathrm{St}(x_2, Q')$ share an ear. Now (1) follows.

Suppose x is a real vertex of Q'. Then each vertex of $\mathrm{St}(x, Q')$ is flat and deep by our choice of R. If x satisfies the assumptions of Lemma 4.5, then $\cup_{i=1}^4 C_{x_i} \subset \mathrm{St}(x, Q')$, by Lemma 4.5 (6). By Lemma 4.5 (5), $\cup_{i=1}^4 C_{x_i}$ contains a disc neighborhood of x, thus $\cup_{i=1}^4 C_{x_i} = \mathrm{St}(x, Q')$. Since, by Lemma 4.9, q maps $C_{x_i} \cup C_{x_j}$ homeomorphically onto $\Pi_i \cup \Pi_j$, q maps $\cup_{i=1}^4 C_{x_i}$ homeomorphically onto $\cup_{i=1}^4 \Pi_i$. The cases of Lemma 4.6 and Lemma 4.8 are similar. □

Definition 4.12. An interior vertex $v \in Q_R$ is of *type O* if it satisfies Lemma 4.11 (1), and is of *type I, II, or III* if it satisfies Lemma 4.5, Lemma 4.6, or Lemma 4.8, respectively. The *support* of a 2–cell in Q_R is the defining edge of the block of X^*_Γ that contains the q–image of this 2–cell. The *support* of a vertex of Q_R is the union of the supports of 2–cells in Q_R that contain this vertex. For a type III vertex v, its support is either a triangle or a square, and v is called either a *△–vertex* or *a □–vertex*, respectively. (See Table 1 on page 33.) By Lemma 4.2 (3) and (4), the Coxeter group whose defining graph is the support of v acts on the Euclidean plane.

Note that $\mathrm{St}(v_1, Q_R) \cap \mathrm{St}(v_2, Q_R)$ contains two 2–cells for two adjacent vertices v_1, v_2 of type III, thus we have the following result.

Lemma 4.13. *If two vertices of type III of Q_R are adjacent, then they have the same support.*

Since $Q'_R \subset Q_R$ by Lemma 4.10, we will assume the quasiflat is represented by $q\colon Q_R \to X^*_\Gamma$. Each 2–cell in Q_R corresponds to a fake vertex in Q'_R, which is called the vertex *dual* to this 2–cell.

5. Singular lines, singular rays and atomic sectors

Throughout this section Γ will the defining graph of an Artin group A_Γ with dimension ≤ 2.

5.1. **Singular lines and singular rays.**

Definition 5.1. A *diamond line* is a locally injective cellular map $k\colon U \to X^*_\Gamma$ satisfying

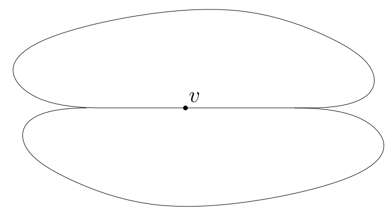
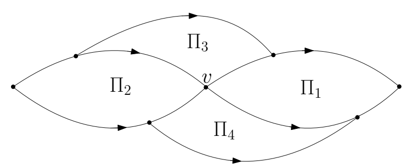
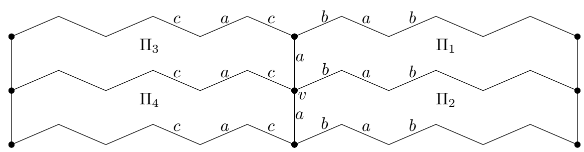
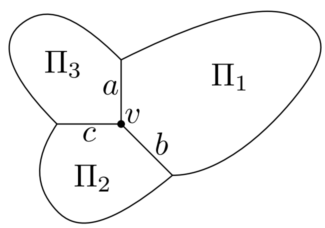
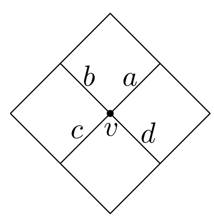
type O	
type I	
type II	
type III $\triangle$–vertex	
type III $\square$–vertex	

TABLE 1. Types of flat vertices.

(1) $U = \cup_{i=-\infty}^{\infty} C_i$ such that each C_i is a 2–cell whose boundary is a $2n$–gon for $n \geq 3$, and each C_i intersects C_{i+1} and C_{i-1} in opposite vertices (and has empty intersection with other 2–cells), see Figure 14 for the $n = 3$ case;
(2) $k(U)$ is contained in a block;
(3) $k(C_i \cap C_{i+1})$ is a tip of both $k(C_i)$ and $k(C_{i+1})$.

We define a *diamond ray* in a similar way by replacing $\cup_{i=-\infty}^{\infty} C_i$ by $\cup_{i=0}^{\infty} C_i$.

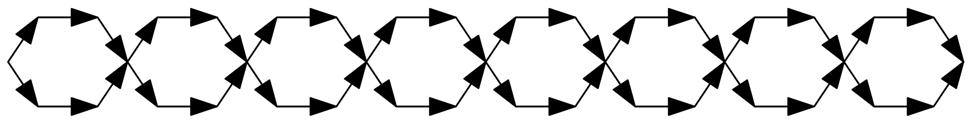


FIGURE 14. A part of a diamond line.

Let k, U be as in Definition 5.1 and let $\overline{ab} \subset \Gamma$ be the defining edge of the block containing $k(U)$. Then the two lines in the 1–skeleton of U corresponding to the bi-infinite alternating word $\cdots ababa \cdots$ are called the *boundary lines* of U.

Remark 5.2. Roughly speaking, each diamond line corresponds to the centralizer of the stabilizer of the block of X^*_Γ that contains this diamond line (note that the stabilizer of a block is a conjugate of the standard subgroup associated with the defining edge of this block).

Each diamond line is embedded and quasi-isometrically embedded in X^*_Γ. This is clear when Γ is an edge, and the general case follows from a result by Charney and Paris [CP14, Theorem 1.2]. Note that each large block is a union of diamond lines.

Before we state the next definition, recall that each Coxeter group with defining graph Γ gives rise to its *Davis complex* $\mathbb{D}_\Gamma$, whose 1–skeleton is the Cayley graph of the associated Coxeter group with bigons collapsed to single edges. In particular, each edge of $\mathbb{D}_\Gamma$ is labeled by a vertex of Γ. A *wall* in $\mathbb{D}_\Gamma$ is the fixed point set of a reflection in the associated Coxeter group.

Definition 5.3. Suppose the defining graph Γ of the Artin group A_Γ contains a triangle $\Delta \subset \Gamma$ such that the labels m, n, r of the three sides of Δ satisfy $\frac{1}{m} + \frac{1}{n} + \frac{1}{r} = 1$. Let $\mathbb{D}_\Delta$ be the Davis complex of the Coxeter group with the defining graph Δ. Then $\mathbb{D}_\Delta$ is isometric to $\mathbb{E}^2$ and edges of $\mathbb{D}_\Delta$ are labeled by vertices of Δ. Let U be the *carrier* of a wall in $\mathbb{D}_\Delta$ (i.e. U is the union of cells that intersect this wall). See Figure 15 for an example when $(m, n, r) = (2, 3, 6)$. A *Coxeter line* of X^*_Γ is a locally injective cellular map

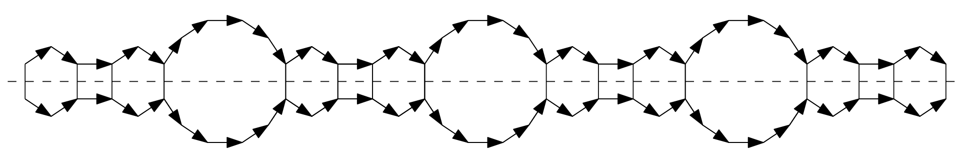


FIGURE 15. A part of a Coxeter line.

$k \colon U \to X^*_\Gamma$ such that

(1) k preserves the label of edges;
(2) if we pull back the orientation of edges of X^*_Γ to U, then there are no orientation reversing vertices in the boundary of U.

If we restrict k to a subcomplex of U homeomorphic to $[0,\infty)\times[0,1]$ (resp. $[0,k]\times[0,1]$), we obtain a *Coxeter ray* (resp. *Coxeter segment*).

Let V be the carrier (in $\mathbb{D}_\Delta$) of the region bounded by two different parallel walls $\mathcal{W}_1,\mathcal{W}_2$ in $\mathbb{D}_\Delta$. A *thickened Coxeter line* is a locally injective cellular map $k'\colon V\to X^*_\Gamma$ such that k' preserves the label of edges and k' restricted to the carriers of $\mathcal{W}_1$ and $\mathcal{W}_2$ forms two Coxeter lines.

The maps k and k' in Definition 5.3 are injective and they are quasi-isometric embeddings. This can be deduced by considering the quotient homomorphism from A_Γ to the Coxeter group with the defining graph Γ.

For a set of vertices V in Γ, let $V^\perp$ be the collection of vertices of Γ that are adjacent to each element in V along an edge labeled by 2. If both $V\neq\emptyset$ and $V^\perp\neq\emptyset$, as A_Γ is assumed to have dimension ≤ 2, the subgroup of A_Γ generated by $V^\perp$ has to be free, and so is the subgroup of A_Γ generated by V. In particular, neither $V^\perp$ nor V contain a pair of adjacent vertices of Γ.

Definition 5.4. A *plain line* is a line L in the 1–skeleton of X^*_Γ such that the collection V_L of labels of edges satisfies either V_L is a singleton or $V_L^\perp\neq\emptyset$. A *plain ray* is defined in a similar fashion. A plain line or ray is *single-labeled* if V_L is a singleton. A plain ray is *chromatic* if V_r is not a singleton for any sub-ray r of this plain ray.

Let $U=\cup_{i=-\infty}^{\infty}C_i$ be a union of 2–cells such that
1. ∂C_i is a $2n$–gon for each i ($n\geq 2$);
2. $C_i\cap C_{i-1}$ and $C_i\cap C_{i+1}$ are two disjoint connected paths in ∂C_i and each of them has $n-1$ edges;
3. $C_i\cap C_j=\emptyset$ for $|i-j|\geq 2$.

A *thickened plain line* is a cellular embedding $k\colon U\to X^*_\Gamma$ such that k restricted to the two boundary lines of U are plain lines.

Again it follows from [CP14, Theorem 1.2] that each plain line is quasi-isometrically embedded for some uniform quasi-isometric constants independent of the plain line.

Definition 5.5. A *singular ray* is either a diamond ray, or a Coxeter ray, or a plain ray. We define *singular line* analogously.

5.2. **Flats, half-flats and sectors.** Since the left action $A_\Gamma\curvearrowright X^*_\Gamma$ is simply transitive on the vertex set of X^*_Γ, we choose an identification between elements of A_Γ and vertices of X^*_Γ.

Definition 5.6. Let L be a diamond line containing the identity element of A_Γ. A *diamond-plain flat* is a subcomplex of X^*_Γ of the form $g\cup_{i=-\infty}^{\infty}a^iL$ or $g\cup_{i=-\infty}^{\infty}b^iL$, where a and b are the labels of edges in L, a^iL means the left translation of L under the group element a^i and g is an element in A_Γ. Note that each diamond-plain flat can be naturally realized as the image of a locally injective cellular map $f\colon U\to X^*_\Gamma$ where U is a union of subcomplexes isomorphic to diamond lines; see Figure 16.

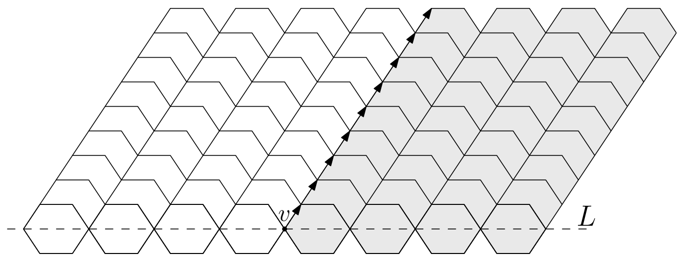


Figure 16. A part of a diamond-plain flat and a diamond-plain sector (shaded).

Lemma 5.7. *The map f above is an embedding.*

Proof. Since the image of f is contained in a block, it suffices to consider the case when Γ is an edge. One can deduce the injectivity of f from the solution of the word problem for spherical Artin groups [BS72, Del72]. Here we provide a geometric proof depending on a complex constructed by Jon McCammond [McC10]. The construction of such complex was presented in the proof of [HJP16, Theorem 5.1].

Suppose the edge of Γ is labeled by n. Let K_n be the cube complex described in the Figure 17 below. On the left we see part of the 1–skeleton

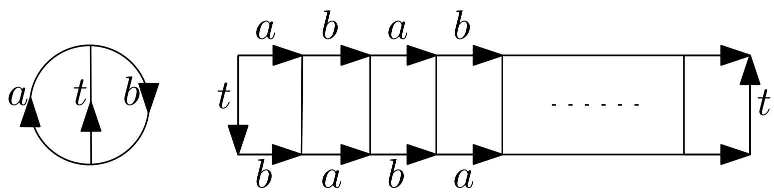


Figure 17.

of K_n consisting of three edges labelled by a, b, t, and the right side indicates how to attach a rectangle (subdivided into n squares) along its boundary path $\underbrace{ab\dots}_{n}t^{-1}\underbrace{b^{-1}a^{-1}\dots}_{n}t^{-1}$. It is easy to check that the link of each of the two vertices in K_n is isomorphic to the spherical join of two points with n points, hence K_n is nonpositively curved and its universal cover $\widetilde{K}_n$ is isometric to tree times $\mathbb{E}^1$. There is a homotopy equivalence $h\colon K_n \to P_\Gamma$ by collapsing the t–edge. This induces a map $\widetilde{K}_n \to X^*_\Gamma$. Note that h gives a one to one correspondence between lifts of t–edges in $\widetilde{K}_n$ and vertices in X^*_Γ; as well as one to one correspondence between vertical flat strips in $\widetilde{K}_n$ and diamond lines in X^*_Γ. Now the lemma follows from the CAT(0) geometry on $\widetilde{K}_n$. $\square$

Thus each diamond-plain flat F is homeomorphic to $\mathbb{R}^2$. Moreover, there is a subgroup $A_F \leq A_\Gamma$ isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ acting cocompactly on F

(for example, when $F = \cup_{i=-\infty}^{\infty} a^i L$, then A_F is generated by a and the centralizer of the standard subgroup of A_Γ generated by a and b).

Definition 5.8. Choose a diamond-plain flat F and let $v \in F$ be a vertex of a diamond line L in F being the intersection of two 2–cells of L. Let $L_1 \subset L$ be a diamond ray based at v and let $L_2 \subset F$ be a plain ray based at v. Then the region in F bounded by L_1 and L_2 (including L_1 and L_2) is called a *diamond-plain sector*; see Figure 16. L_1 and L_2 are called the *boundary rays* of the diamond-plain sector.

Definition 5.9. Let $L \subset X_\Gamma^*$ be a Coxeter line containing the identity element of A_Γ and let e be the edge in L such that e is dual to the wall in L and e contains the identity element of A_Γ. Suppose the label of e is a. Then a *Coxeter-plain flat* is a subcomplex of X_Γ^* of the form $g \cup_{i=-\infty}^{\infty} a^i L$, where $g \in A_\Gamma$. Let $L' \subset L$ be a Coxeter ray starting at the edge e. A *Coxeter-plain sector* is a subcomplex of X_Γ^* of the form $g \cup_{i=0}^{\infty} a^i L'$, where $g \in A_\Gamma$; see Figure 18. The Coxeter-plain sector has two *boundary rays*, one is L' and another one is the plain ray containing e.

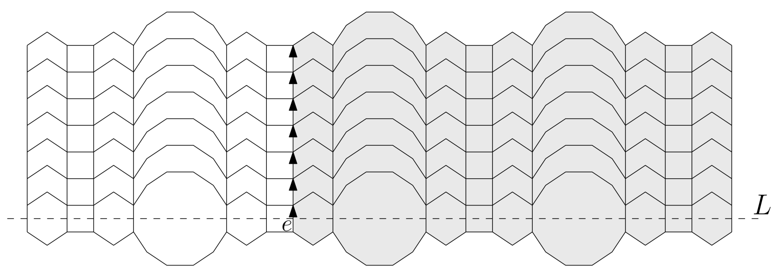


FIGURE 18. A part of a Coxeter-plain flat and a Coxeter-plain sector (shaded).

Since the Artin monoid injects into the Artin group by the work of Paris [Par02], $L \cap a^i L = \emptyset$ for $i \geq 2$ and $L \cap aL$ is a boundary line of L. Thus each Coxeter-plain flat is a subcomplex of X_Γ^* homeomorphic to $\mathbb{R}^2$.

Definition 5.10. Let Δ and $\mathbb{D}_\Delta$ be as Definition 5.3. A *Coxeter flat* is a locally injective cellular map $f\colon \mathbb{D}_\Delta \to X_\Gamma^*$ such that

(1) f preserves the label of edges;
(2) the orientation of edges in $\mathbb{D}_\Delta$ induced by f satisfies the following: if two edges of $\mathbb{D}_\Delta$ are dual to parallel walls in $\mathbb{D}_\Delta$, then they are oriented towards the same direction.

By considering the 1–Lipschitz quotient homomorphism from A_Γ to the Coxeter group with defining graph Γ as before (cf. the discussion after Definition 5.3), we know each Coxeter flat is embedded and quasi-isometrically embedded.

For each Coxeter flat F, there are exactly two families of parallel walls whose carriers in F give rise to Coxeter lines. To see this, choose a 2–cell $C \subset F$ with the maximal number of edges on its boundary and let $e_1, e_2 \subset \partial C$ be the two edges containing a tip $t \in \partial C$. Then by Definition 5.3 (2) and Definition 5.10 (2), the carrier of a wall $\mathcal{W} \subset F$ is a Coxeter line if and only if $\mathcal{W}$ is parallel to the wall of F dual to e_1 or e_2.

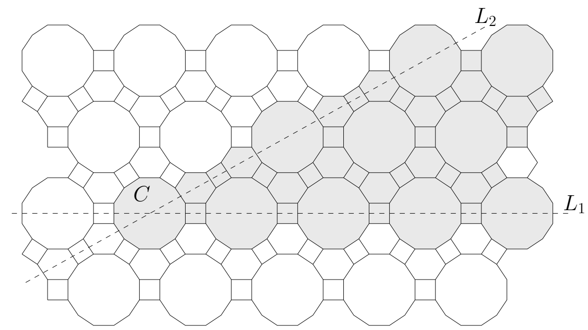


FIGURE 19. A part of a Coxeter flat and a Coxeter sector (shaded).

Definition 5.11. Let F be a Coxeter flat and choose two Coxeter lines L_1, L_2 such that they intersect in a 2–cell $C \in F$ (such Coxeter lines exists by the discussion in the previous paragraph). For $i = 1, 2$, let $L'_i \subset L_i$ be a Coxeter ray starting at C. Then a *Coxeter sector* is defined to be the region in F bounded by L'_1 and L'_2 (including L'_1 and L'_2); see Figure 19. L'_1 and L'_2 are called the *boundary rays* of the Coxeter sector.

Definition 5.12. Let Q be a quarter plane tiled by unit squares in a standard way. A *plain sector* is a locally injective cellular map $f \colon Q \to X^*_\Gamma$ such that f restricted to the two *boundary rays* of Q gives two plain rays.

For each plain sector, there is a full subgraph $\Gamma' \subset \Gamma$ such that Q is contained in a copy of $X^*_{\Gamma'}$ inside X^*_Γ and $A_{\Gamma'}$ is a right-angled Artin group. Note that both $Q \to X^*_{\Gamma'}$ and $X^*_{\Gamma'} \to X^*_\Gamma$ are injective quasi-isometric embeddings (the first one follows from the fact that the Salvetti complexes of right-angled Artin groups are non-positively curved, and the second one follows from [CP14, Theorem 1.2]), thus f is an injective quasi-isometric embedding.

Definition 5.13. A *diamond chromatic half-flat (DCH)* is a locally injective cellular map $f \colon U = \cup_{i=1}^\infty L_i \to X^*_\Gamma$ such that

(1) $f|_{L_i}$ is a diamond line for each i;
(2) $L_i \cap L_{i+1}$ is a boundary line of both L_i and L_{i+1}, moreover, $L_i \cap L_j = \emptyset$ for $|i - j| \geq 2$;
(3) there does not exist $i_0 \geq 1$ such that $f(\cup_{i=i_0}^\infty L_i)$ is contained in a diamond-plain flat.

The *boundary line* of this DCH is defined to be the diamond line $f|_{L_1}$.

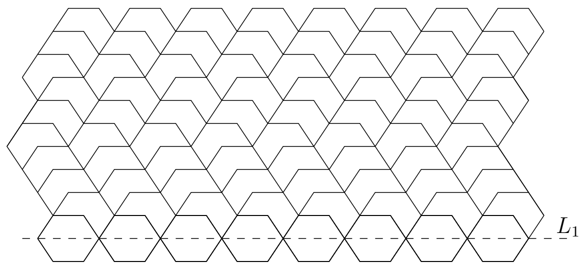


FIGURE 20. A part of a diamond chromatic half-flat (DCH).

By using the CAT(0) cube complex $\widetilde{K}_n$ in Lemma 5.7, one readily deduces that each DCH is embedded and quasi-isometrically embedded.

Definition 5.14. A *Coxeter chromatic half-flat (CCH)* of type I is a locally injective cellular map $f\colon U = \cup_{i=1}^{\infty} L_i \to X_\Gamma^*$ such that

(1) $f|_{L_i}$ is a thickened Coxeter line or a Coxeter line;
(2) $L_i \cap L_j = \emptyset$ for $|i-j| \geq 2$;
(3) $L_i \cap L_{i+1}$ is a boundary component of both L_i and L_{i+1}, moreover, each vertex in $L_i \cap L_{i+1}$ is either of type O, or of type II (cf. Definition 4.12, and Table 1 on page 33);
(4) there does not exist i_0 such that for all $i \geq i_0$, each vertex in $L_i \cap L_{i+1}$ is of type O; and there does not exist i_0 such that for all $i \geq i_0$, each vertex in $L_i \cap L_{i+1}$ is of type II.

The *boundary line* of this CCH is defined to be the Coxeter line containing $f|_{\partial U}$, where ∂U is the boundary of U in the topological sense.

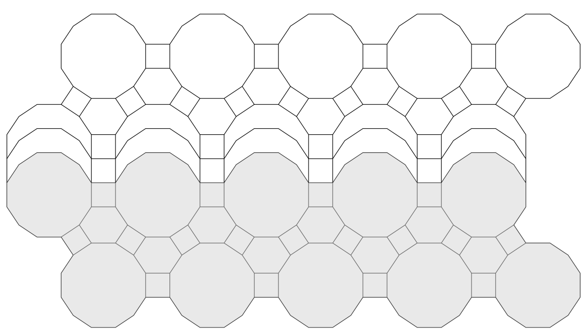


FIGURE 21. A part of a Coxeter chromatic half-flat (CCH) of type I. A thickened Coxeter line L_1 is shaded.

Definition 5.15. Let Δ and $\mathbb{D}_\Delta$ be as Definition 5.3. Let H be the carrier of a halfspace $\mathcal{H} \subset \mathbb{D}_\Delta$ bounded by a wall $\mathcal{W}_1$ of $\mathbb{D}_\Delta$. A *Coxeter chromatic half-flat (CCH)* of type II is a locally injective cellular map $f\colon H \to X_\Gamma^*$ satisfying all the following conditions:

(1) f preserves the label of edges;

(2) f restricted to the carrier of $\mathcal{W}_1$ is a Coxeter line;
(3) there does not exist a halfspace $\mathcal{H}' \subset \mathcal{H}$ such that the image of the carrier of $\mathcal{H}'$ under f is contained in a Coxeter flat; see Figure 22.

The *boundary line* of this CCH is the Coxeter line containing $f|_{\mathcal{W}_1}$.

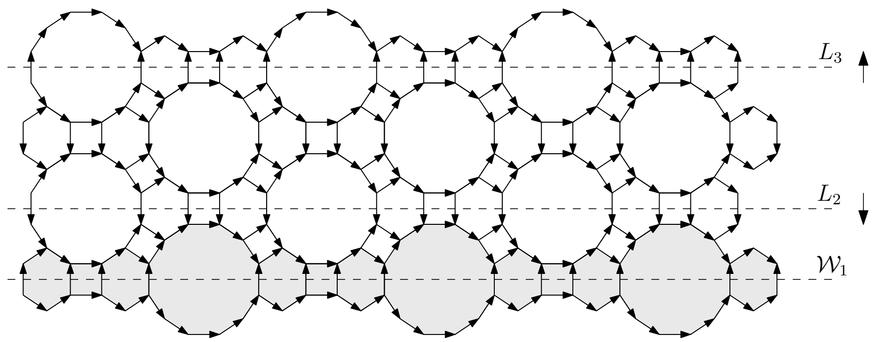


FIGURE 22. A part of a Coxeter chromatic half-flat (CCH) of type II. The Carrier of $\mathcal{W}_1$ (shaded) is a Coxeter line. Orientations of edges along L_2 and L_3 are opposite.

As in the discussion after Definition 5.10, each CCH of type II is embedded and quasi-isometrically embedded.

Definition 5.16. A *plain chromatic half-flat (PCH)* is a locally injective cellular map $f\colon U = \cup_{i=1}^{\infty} L_i \to X_\Gamma^*$ such that

(1) $f|_{L_i}$ is a thickened plain line for each i;
(2) $L_i \cap L_j = \emptyset$ for $|i-j| \geq 2$;
(3) $L_i \cap L_{i+1}$ is a boundary line of both L_i and L_{i+1};
(4) there does not exist $i_0 \geq 1$ such that $f(\cup_{i=i_0}^{\infty} L_i)$ is contained in a block;
(5) there does not exist $i_0 \geq 1$ such that $\cup_{i=i_0}^{\infty} L_i$ is made of squares.

Note that (5) implies that $f(\partial U)$ is a single-labeled plain line; see Figure 23. The *boundary line* of this PCH is $f|_{\partial U}$.

Definition 5.17. An *atomic sector* is one of the objects among diamond-plain sector, Coxeter-plain sector, Coxeter sector, plain sector, DCH, CCH (of type I or II) and PCH. If S is a DCH, CCH or PCH, then a *boundary ray* of S is a singular ray contained in the boundary line of S.

For further reference, we collect all the types of atomic sectors in Table 2 on page 41.

One can build an atomic sector in a local and cell-by-cell fashion. There are plenty of atomic sectors in X_Γ^*. Actually, for any vertex in X_Γ^*, there is an atomic sector of given type whose image contains this vertex.

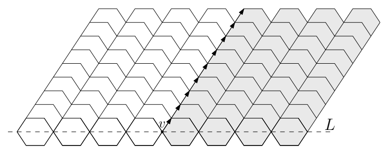
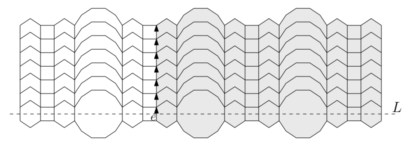
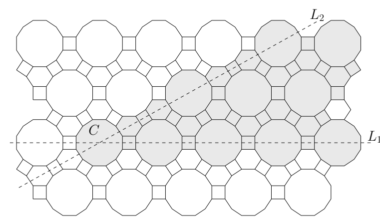
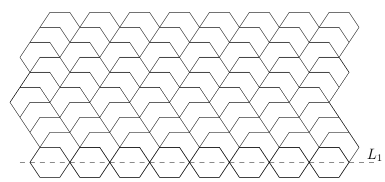
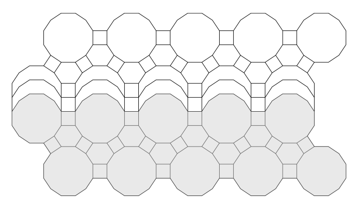
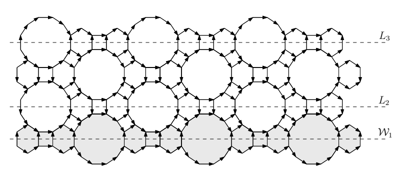
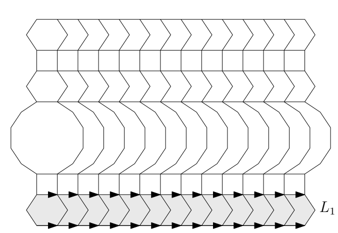
diamond-plain flat and sector	
Coxeter-plain flat and sector	
Coxeter flat and sector	
diamond chromatic half-flat (DCH)	
Coxeter chromatic half-flat (CCH) type I	
Coxeter chromatic half-flat (CCH) type II	
plain chromatic half-flat (PCH)	

TABLE 2. Atomic sectors.

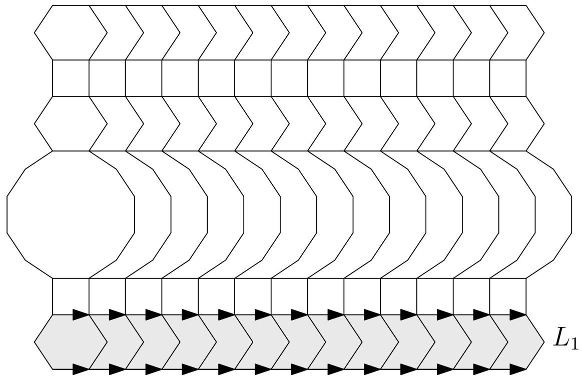


FIGURE 23. A part of a plain chromatic half-flat (PCH). A thickened plain line L_1 is shaded.

5.3. **Properties of atomic sectors.** For each Artin group A_Γ, Charney and Davis [CD95b] defined an associated *modified Deligne complex* D_Γ. Now we recall their construction in the 2–dimensional case, since it will be used for proving certain properties of atomic sectors. The vertex set V of D_Γ is in one to one correspondence with left cosets of the form $gA_{\Gamma'}$, where $g \in A_\Gamma$ and Γ' is either the empty-subgraph of Γ (in which case $A_{\Gamma'}$ is the identity subgroup), or a vertex of Γ, or an edge of Γ. The *rank* of a vertex $gA_{\Gamma'}$ of V is the number of vertices in Γ'. In other words, rank 2 vertices of V correspond to blocks of X^*_Γ, rank 1 vertices of V correspond to single-labeled plain lines in X^*_Γ, and rank 0 vertices of V correspond to vertices of X^*_Γ.

Note that V has a partial order induced by inclusion of sets. A collection $\{v_i\}_{i=1}^k \subset V$ of vertices spans a $(k-1)$–dimensional simplex if $\{v_i\}_{i=1}^k$ form a chain with respect to the partial order. It is clear that D_Γ is a 2–dimensional simplicial complex, and A_Γ acts on D_Γ without inversions, i.e. if an element of A_Γ fixes a simplex of D_Γ, then it fixes the simplex pointwise.

We endow D_Γ with a piecewise Euclidean metric such that each triangle $\triangle(g_1, g_2A_s, g_3A_{\overline{st}})$ is a Euclidean triangle with angle $\pi/2$ at g_2A_s and angle $\frac{\pi}{2n}$ at $g_3A_{\overline{st}}$ with n being the label of the edge $\overline{st}$ of Γ. By [CD95b, Proposition 4.4.5], D_Γ is CAT(0) with such metric. As being observed in [Cri05, Lemma 6], the action $A_\Gamma \curvearrowright D_\Gamma$ is semisimple.

Let X^b_Γ be the barycentric subdivision of X^*_Γ. Then there is a simplicial map $\pi\colon X^b_\Gamma \to D_\Gamma$ which maps the center of each 2–cell (resp. 1–cell) in X^*_Γ to the vertex in D_Γ representing the block (resp. single-labeled plain line) of X^*_Γ that contains this 2–cell (resp. 1–cell), and maps vertices of X^*_Γ to the corresponding rank 0 vertices in D_Γ.

Lemma 5.18. *If two different blocks B and B' of X^*_Γ satisfy that $B \cap B'$ contains an edge, then $B \cap B'$ is a single-labeled plain line.*

Proof. This follows from a more general fact by van der Lek [vdL83] that for any two full subgraphs Γ_1 and Γ_2 of Γ, we have $A_{\Gamma_1\cap\Gamma_2} = A_{\Gamma_1} \cap A_{\Gamma_2}$. □

Lemma 5.19. *Each atomic sector is embedded.*

Proof. It remains to prove the lemma for PCH and CCH of type I.

(PCH case.) Let $f\colon U = \cup_{i=1}^{\infty} L_i \to X_\Gamma^*$ be as in Definition 5.16. We group those consecutive L_i's that are mapped by f to the same block to form a new decomposition $U = \cup_{i=1}^{\infty} L'_i$ so that the block B_i containing $f(L'_i)$ satisfies $B_i \neq B_{i+1}$ for any $i \geq 1$. Each B_i gives rise to a rank 2 vertex in D_Γ, which we denote by v_i, and $B_i \cap B_{i+1}$ gives rise to a rank 1 vertex in D_Γ (cf. Lemma 5.18), which we denote by w_i. Then for each $i \geq 1$, w_i is adjacent to v_i and v_{i+1} in D_Γ. Let $P \to D_\Gamma$ be the edge path starting from v_1 and traveling through $w_1, v_2, w_2, v_3, \ldots$. Note that $v_i \neq v_{i+1}$ (since $B_i \neq B_{i+1}$) and $w_i \neq w_{i+1}$ (since $f(L'_{i+1})$ is embedded in B_{i+1}). Thus $P \to D_\Gamma$ is locally injective.

Let $g \in G$ be the generator of the stabilizer of w_1. Let F_g be the fixed point set of g. Crisp [Cri05] observed that F_g is a tree. We repeat his argument here. Note that F_g is a convex subcomplex of D_Γ. However, g never fixes rank 0 vertices, so F_g lies in the part of the 1–skeleton of D_Γ which is spanned by rank 1 and rank 2 vertices. Thus F_g is a tree.

Note that the edge path P is actually contained in F_g. Since P is locally injective, $P \to F_g$ is an embedding. Now we pick i, j with $|i-j| \geq 2$. Then the segment of P from v_i from v_j is a CAT(0) geodesic segment in D_Γ made of ≥ 4 edges thus, by the way we metrize D_Γ, there does not exist rank 0 vertices which are adjacent to both v_i and v_j. Also there are no rank 1 vertices adjacent to both v_i and v_j, since any such vertex has to be contained in F_g. Let $\pi\colon X_\Gamma^b \to D_\Gamma$ be the simplicial map defined as before. Note that each vertex in $\pi(B_i)$ is adjacent to v_i. Thus $B_i \cap B_j = \emptyset$ whenever $|i-j| \geq 2$. From this, Lemma 5.18, and the fact that $f_{L'_i}$ is injective for each i we deduce that f is injective.

(CCH of type I case.) Now let $f\colon U = \cup_{i=1}^{\infty} L_i \to X_\Gamma^*$ be as Definition 5.14. *Walls* in U are defined to be the walls in the domains of Coxeter lines contained in U. If $\mathcal{W} \subset U$ is a wall, then $\pi(f(\mathcal{W}))$ is a geodesic line in the 1–skeleton of D_Γ. We re-decompose U as $U = \cup_{i=1}^{\infty} L'_i$ such that

1. (1) each L'_i is bounded by two walls;
2. (2) either each vertex of U in L'_i is of type III (we call such L'_i *non-degenerate*), or each vertex of U in L'_i is of type O or type II (we call such L'_i *degenerate*);
3. (3) $\{L'_i\}_{i=1}^{\infty}$ alternates between non-degenerate and degenerate ones.

If L'_i is non-degenerate, then $\pi \circ f|_{L'_i}$ is locally injective, hence injective by CAT(0) geometry. If L'_i is degenerate, then walls of L'_i are mapped by $\pi \circ f$ to the same geodesic line in D_Γ. Since each Coxeter line is periodic, pick an element $g \in A_\Gamma$ acting by translation on a Coxeter line $L \subset f(U)$. Note that $f(U)$ is invariant under the action of g. Moreover, let ℓ_g be the π–image of the wall in L, then g acts by translation on ℓ_g. Let M_g be the collection of points in D_Γ such that $d(x, gx)$ attains its minimum. Then

M_g admits a splitting $M_g = \ell_g \times T$ by [BH99, Theorem II.6.8]. Since D_Γ is 2–dimensional, T is a tree. Note that $\pi \circ f(U) \subset M_g$. For each i, let $s_i \subset T$ be the orthogonal projection of $\pi \circ f(L'_i)$ onto the T–factor. If L'_i is non-degenerate, then s_i is a segment with length > 0. If L'_i is degenerate, then s_i is a star shaped subset, i.e. s_i is an union of segments along a vertex. We assume without loss of generality that L'_1 is non-degenerate. Let $P \to T$ be the concatenation of s_{2i+1} for $i \geq 0$ (note that the endpoint of s_{2i-1} is the starting point of s_{2i+1}). By the argument in the PCH case, we know f restricted to the carrier of L'_i in U is injective for i even. This implies that P is locally injective at $s_i \cap s_{i+2}$ (for i odd). Thus P is an embedding. This and $f|_{L'_i}$ being an embedding for each i imply that f is an embedding. $\square$

Lemma 5.20. *Let S_1 and S_2 be two atomic sectors. If there is a sequence of 2–dimensional discs $\{D_i\}_{i=1}^{\infty} \subset S_1 \cap S_2$ such that D_i contains a ball of radius i in S_1 (with respect to the path metric on the 1–skeleton of S_1), then $d_H(S_1 \cap S_2, S_1) < \infty$ and $d_H(S_1 \cap S_2, S_2) < \infty$. Moreover, S_1, S_2 and $S_1 \cap S_2$ are atomic sectors of the same type.*

Proof. The lemma follows from a case by case inspection. Suppose S_1 and S_2 satisfy the assumption of the lemma. Let $D \subset S_1 \cap S_2$ be a large disc. We only discuss the case when S_1 is a CCH of type I. Then other cases are either similar, or much simpler.

Let $S_1 = \cup_{i=1}^{\infty} L'_i$, $M_{g_1} = \ell_{g_1} \times T \subset D_\Gamma$ and the path $P \subset T$ be as in Lemma 5.19, where $g_1 \in A_\Gamma$ is an element acting on S_1. Since a diamond-plain sector or a DCH is contained in a block, so S_2 can not be one of them. Also S_2 can not be a plain sector since a plain sector is made of squares. If S_2 is a PCH, then D is contained in the carrier of L'_i (in S_1) for some i even. Let $g_2 \subset A_\Gamma$ be the element acting on S_2 and we define the tree F_{g_2} as in Lemma 5.19. The existence of D implies that a sub-segment of F_{g_2} goes along the ℓ_g direction of M_g. The convexity of F_{g_2} and M_{g_1} implies that $F_{g_2} \cap M_{g_2} \subset \ell_g \times \{t\}$ for some $t \in T$. Since $\pi(S_1) \subset M_{g_2}$ and each vertex of $\pi(S_2)$ is either contained in F_{g_2} or adjacent to a vertex in F_{g_2}, we know that $S_1 \cap S_2$ is contained in the carrier of L'_i inside S_1. This rules out the possibility of S_2 being a PCH. The case of S_2 being a Coxeter-plain sector can be ruled out in a similar way. If S_2 is a Coxeter sector, then D is contained in the carrier of L'_i (in S_1) for some i odd and $\pi(S_2) \subset M_{g_1}$. Note that P and the projection of $\pi(S_2)$ on T are two rays which diverge from each other at some point, thus $S_1 \cap S_2$ is contained in the carrier of L'_i inside S_1 for some odd i and S_2 can not be a Coxeter sector.

Now suppose that S_2 is a CCH of type I or II and that S_2 is invariant under $g_2 \in A_\Gamma$. The existence of D implies that there is a Coxeter line $L \subset S_2$ such that $L \cap S_1$ contains a Coxeter segment U (cf. Definition 5.3) which is sufficiently long. Thus we have the following two cases:

(1) if U is contained in a Coxeter line of S_1, then we know L is a Coxeter line of S_1;

(2) if U is not contained in a Coxeter line of S_1, then N is contained in a thickened Coxeter line of S_1, hence L and a Coxeter line of S_1 span a Coxeter flat in X^*_Γ.

Let $\ell \subset D_\Gamma$ be the π–image of the wall in L. In both cases (1) and (2), we have $\ell \subset M_{g_1}$. If ℓ and ℓ_{g_1} are not parallel, then P and the projection of ℓ on T are two rays which diverge from each other at some point (since ℓ is periodic), which implies that $S_1 \cap S_2$ is contained in the carrier of L'_i inside S_1 for some odd i. Thus ℓ and ℓ_{g_1} are parallel. Then $\pi(S_2) \subset M_{g_1}$. The projections of $\pi(S_1)$ and $\pi(S_2)$ must contain a common geodesic ray, otherwise we can reach a contradiction as before. Now one readily deduces that $d_H(S_1 \cap S_2, S_1) < \infty$ and $d_H(S_1 \cap S_2, S_2) < \infty$. □

Now we recall the notion of locally finite homology. We use $\mathbb{Z}/2\mathbb{Z}$ coefficients throughout this paper. Recall that for a topological space X we can consider *locally finite chains* in X, which are formal sums $\Sigma_{\lambda \in \Lambda} a_\lambda \sigma_\lambda$ where $a_\lambda \in \mathbb{Z}/2\mathbb{Z}$, σ_λ are singular simplices, and any compact set in X intersects the images of only finitely many σ_λ with $a_\lambda \neq 0$. This gives rise to *locally finite homology* of X, denoted by $H^{\mathrm{lf}}_*(X)$. We define the relative locally finite homology $H^{\mathrm{lf}}_*(X, Y)$ for a subset $Y \subset X$ in a similar way. For a point $z \in Z \setminus Y$, there is a homomorphism $i\colon H^{\mathrm{lf}}_k(Z, Y) \to H^{\mathrm{lf}}_k(Z, Z \setminus \{z\}) \cong H_k(Z, Z \setminus \{z\})$ induced by the inclusion of pairs $(Z, Y) \to (Z, Z - \{z\})$. For $[\sigma] \in H^{\mathrm{lf}}_k(Z, Y)$, we define the *support set* of $[\sigma]$, denoted $S_{[\sigma],Z,Y}$, to be $\{z \in Z \setminus Y \mid i_*[\sigma] \neq \mathrm{Id}\}$. We will also use $S_{[\sigma]}$ to denote the support set if the underlying spaces Z and Y are clear.

Lemma 5.21. *Each atomic sector is quasi-isometrically embedded.*

Proof. The cases of diamond-plain sector, Coxeter-plain sector and Coxeter sector follow from the fact that any rank 2 abelian subgroup of A_Γ is quasi-isometrically embedded ([HO19, Theorem 7.7]). The case of plain sector has already been discussed after Definition 5.12. It remains to consider chromatic half-flats. Let $U \subset X^*_\Gamma$ be a PCH and let $\{\ell_i\}_{i=1}^\infty$ be the collection of all plain lines in U parallel to ∂U such that $\ell_1 = \partial U$ and ℓ_i is between ℓ_{i-1} and ℓ_{i+1}. Since each ℓ_i is quasi-isometrically embedded with constants independent of i, it suffices to show that there is a constant M such that $d_{A_\Gamma}(\ell_i, \ell_j) \geq M|i - j|$, where d_{A_Γ} denotes the distance in the 1–skeleton of X^*_Γ. Suppose the contrary holds. Then for any given $\epsilon > 0$, there exists ℓ_i, ℓ_j such that $d_{A_\Gamma}(\ell_i, \ell_j) < \epsilon|i - j|$. Suppose $L = d_{A_\Gamma}(\ell_i, \ell_j)$. Then $|i - j| > L/\epsilon$. Let P_0 be a shortest edge path from a vertex $v_0 \in \ell_i$ to a vertex $w_0 \in \ell_j$. Let $g \in A_\Gamma$ be an element which translates ℓ_i by distance L. Let $C_1 = v_0 \to gv_0 \to gw_0 \to w_0 \to v_0$ be the cycle traveling along ℓ_i, gP_0, ℓ_j and P_0. Since C_1 has $\leq 4L$ edges and A_Γ has quadratic Dehn function (by Corollary 2.4 and Theorem 4.1), there is a cellular 2–chain σ such that $\partial\sigma = C_1$, and σ has at most CL^2 cells, where C only depends on A_Γ. Since ℓ_i is quasi-isometrically embedded, $\alpha = \sum_{k=-\infty}^{\infty} g^k \sigma$ gives rise to a locally finite relative homology class $[\alpha] \in H^{\mathrm{lf}}_2(X^*_\Gamma, \ell_i \cup \ell_j)$. On the other hand, let U_{ij} be the region of U bounded by ℓ_i and ℓ_j. The fundamental class of U_{ij} gives another locally finite cellular chain β such that $\partial\beta = \partial\alpha$. Then $[\beta] \in H_2^{\mathrm{lf}}(X_\Gamma^*, \ell_i \cup \ell_j)$ and $S_{[\beta]} = U_{ij} \setminus (\ell_i \cup \ell_j)$. Note that $\alpha \cup \beta$ is a periodic "infinite cylinder". Recall that X_Γ^* is contractible by By [CD95b, Theorem B] and [CD95b, Corollary 1.4.2]. Then X_Γ^* is also uniformly contractible since it is cocompact. Thus we can fill in the periodic infinite cylinder to obtain a locally finite singular 3–chain γ such that $\partial\gamma = \alpha - \beta$. Thus $[\alpha] = [\beta]$ and $S_{[\alpha]} = S_{[\beta]} = U_{ij} \setminus (\ell_i \cup \ell_j)$. It is clear that $S_{[\alpha]} \subset \operatorname{Im}\alpha$, thus $U_{ij} \subset \operatorname{Im}\alpha$. Note that both $\operatorname{Im}\alpha$ and U_{ij} are invariant under the action of the subgroup $\langle g \rangle \cong \mathbb{Z}$. Moreover, $\operatorname{Im}\alpha/\langle g\rangle$ has at most CL^2 2–cells, but $U_{ij}/\langle g\rangle$ has L^2/ϵ 2–cells (since $|i-j| > L/\epsilon$). This will be contradictory to $U_{ij} \subset \operatorname{Im}\alpha$ if we take ϵ small enough. Similarly we deal with other chromatic half-flats, since they all have one periodic direction (the cases of DCH and CCH of type II have already been treated differently in Section 5.2). □

5.4. **Completions of atomic sectors.** A subset A is *coarsely contained* in another subset B of a metric space if A is contained in the R–neighborhood of B for some R. We say A is the *coarse intersection* of a family of subsets $\{B_i\}_{i\in I}$ if there exists R_0 such that for all $R \geq R_0$, the Hausdorff distance between A and $\cap_{i\in I} N_R(B_i)$ is finite.

In the rest of this subsection, we define the *completion* of an atomic sector S, which (roughly speaking) is a "smallest possible subset" that is the coarse intersection of 2-dimensional quasiflats and coarsely contains S.

Lemma 5.22. *Let S be an atomic sector which is not a plain sector or a diamond-plain sector. Then S is a coarse intersection of finitely many 2–dimensional quasiflats.*

Proof. Let $U = \cup_{i=1}^{\infty} L_i$ be a CCH of type I (see Table 2 on page 41). We assume without loss of generality that L_1 is a Coxeter line and L_2 is a thickened Coxeter line. Let F be the Coxeter-plain flat containing L_1. Then the line $L_1 \cap L_2$ divides F into two half-spaces, which we denote by U_1 and U_2. Let $U_3 = \cup_{i=2}^{\infty} L_i$. We glue U_1, U_2 and U_3 along $L_1 \cap L_2$ to form a triplane V. Note that there is a locally injective map $V \to X_\Gamma^*$. By the same argument as in the proof of Lemma 5.19 and Lemma 5.21, we know that $U_i \cup U_j$ is embedded and quasi-isometrically embedded for any $1 \leq i \neq j \leq 3$. Thus U is the coarse intersection of the quasiflat $U_3 \cup U_1$ and the quasiflat $U_3 \cup U_2$. The case of CCH of type II is similar. If U is a PCH, we can assume L_1 and L_2 are not in the same block. Let F be the diamond flat containing L_1. Then we can form a triplane as before and find three quasiflats using the argument of Lemma 5.19 and Lemma 5.21. To deal with DCH, note that each DCH is contained in a large block. Then we can use the cube complex $\tilde{K}_n$ in Lemma 5.7 to conclude the proof.

If S is a Coxeter sector, let F be the Coxeter flat containing S. Let L_1 and L_2 be two Coxeter lines in F that contains the two boundary rays of S. Then there are half-flats H_1 and H_2 of F bounded by L_1 and L_2 respectively such that $S = H_1 \cap H_2$. Since F is quasi-isometrically embedded, S is the coarse intersection of H_1 and H_2. However, each H_i is the coarse intersection of two quasiflats by the argument in the previous paragraph. Thus we are done in the case of Coxeter sector. The case of Coxeter-plain sector is similar. $\square$

We define the *completion* of an atomic sector which is not a plain sector or a diamond-plain sector to be itself. Before discussing completions of diamond-plain sectors and plain sectors, we need the following observation.

For two vertices $a, b \subset \Gamma$, we define $a \sim b$ if a and b are adjacent in Γ along an edge labeled by an odd number. If $a \sim b$, then for each singled-labeled plain line ℓ_a in X^*_Γ labeled by a, there is a thickened plain line L such that

(1) one boundary component of L is ℓ_a;
(2) another boundary component of L is a single-labeled plain line labeled by b.

Now we consider the equivalence relation on the vertex set of Γ generated by $\sim$. Each equivalence class is called an *odd component*. Note that a plain line whose edges are labeled by an element in an odd component can be "parallelly transported" to a plain line with its edges labeled by another element in the same odd component via finitely many thickened plain lines.

If S is a diamond-plain sector, let F be the diamond flat containing S. Then S is the intersection of two half-flats H_1, H_2 of F such that H_1 is bounded by a plain line L_1 and H_2 is bounded by a diamond line L_2. Note that H_2 is the coarse intersection of two 2-dimensional quasiflats. However, this may not be true for H_1 (e.g. when Γ is an edge). Let $a \in \Gamma$ be the vertex corresponding to the label of edges in L_1. If every vertex in the odd component containing a is a leaf in Γ (recall that a leaf in a graph is a vertex which is only adjacent to one another vertex), then we define the *completion* of S to be H_2. Observe that in this case either a is a leaf in Γ contained in an edge with an even label, or the component of a in Γ is an edge (with an odd label). If there exists a vertex c in the odd component containing a which is not a leaf, then the *completion* of S is defined to be itself. In this case, we can form a triplane containing H_1, hence H_1 is the coarse intersection of two quasiflats and S is the coarse intersection of finitely many quasiflats.

Now suppose S is a plain sector. We write $S = r_1 \times r_2$ where r_1 and r_2 are two boundary rays of S. For $i = 1, 2$, let $V_i \subset \Gamma$ be the collection of labels of edges in r_i. Let $V_i^\perp$ be the collection of vertices in Γ which are adjacent to each vertex in V_i along an edge labeled by 2. The set V_i is *good* if either $V_i^\perp$ has ≥ 2 vertices, or V_i is a singleton and there exists a non-leaf vertex in the odd component containing V_i. Since V_1 and V_2 form a join subgraph $\Gamma' \subset \Gamma$, the latter case is equivalent to V_i being a singleton and not a leaf.

If both V_1 and V_2 are good, then the *completion* of S is S itself. In this case, S is the coarse intersection of finitely many 2–dimensional quasiflats. Now we prove this statement. Note that $A_{\Gamma'}$ is a right-angled Artin group. We assume S is contained in a copy of $X^*_{\Gamma'}$ in X^*_Γ. Then there is a flat $F = \ell_1 \times \ell_2$ in $X^*_{\Gamma'}$ such that ℓ_i is a geodesic line extending r_i (such extension may not be unique). We know $S = H_1 \cap H_2$ where H_i is a half-flat in F bounded by ℓ_i. Since F is isometrically embedded in $X^*_{\Gamma'}$ and $X^*_{\Gamma'}$ is quasi-isometrically embedded in X^*_{Γ} ([CP14, Theorem 1.2]), we know S is the coarse intersection of H_1 and H_2. Note that the labels of edges in ℓ_i are in V_i, then we deduce from the fact that V_1 and V_2 are good that each of H_1 and H_2 ia a coarse intersection of two quasiflats.

If only one of V_1, V_2 is good, then S is contained in a canonical half-flat H such that H is made of squares and H is a coarse intersection of two quasiflats. Note that the boundary of H is a single-labeled plain line. H is defined to be the completion of S.

If both V_1 and V_2 are not good, then V_1 and V_2 are singletons. We extend r_i to a single-labeled plain line ℓ_i and define the *completion* of S to be the flat of the form $\ell_1 \times \ell_2$.

Corollary 5.23. *For any atomic sector S, the completion $\bar{S}$ of S is a coarse intersection of finitely many 2–dimensional quasiflats.*

6. Half Coxeter regions in Q' and Q_R

Let $x \in Q_R$ be a vertex of type III (see Table 1 on page 33). Let $K_x \subset Q'$ be the convex hull of fake vertices of Q' that are dual to 2–cells in $\mathrm{St}(x, Q_R)$. Such K_x is called a *chamber* or an *x–chamber*. Note that K_x is either a triangle or a square, and define *edges* and *vertices* of K_x with respect to the standard cell structure on a triangle or a square. K_x is isometric to the standard fundamental domain of the action of a Euclidean Coxeter group on $\mathbb{E}^2$ (the defining graph of this Coxeter group is the support of x). Since $K_x \subset \mathrm{St}(x, Q')$, the intersection of two different chambers is either empty, or a point, or an edge of both of the chambers. Each chamber K_x has a *support* inherited from x.

Let $\mathcal{C}' \subset Q'$ be the union of x–chambers with x ranging over vertices of type III in Q_R (it is possible that $\mathcal{C}' = \emptyset$). A *Coxeter region* of Q' is a connected component of $\mathcal{C}'$, and a *Coxeter region* of Q_R is the union of $\mathrm{St}(x, Q_R)$ such that the associated x–chamber is in a Coxeter region of Q'.

Lemma 6.1. *Let $C \subset Q_R$ be an interior 2–cell, i.e. C is disjoint from the boundary of Q_R. Let V be the collection of type III vertices on ∂C. Suppose that $V \neq \emptyset$ and that ∂C has $2n$ edges. Then all vertices of V have the same support, and one of the following three possibilities happens:*

(1) V is one point;
(2) V is the vertex set of an arc in ∂C made of $n-1$ edges;
(3) each vertex in ∂C is in V.

Proof. Suppose V is not a singleton. Let v_1, v_2 be two vertices of V and let $\omega \subset \partial C$ a path of shortest combinatorial distance from v_1 to v_2. We claim each vertex of ω is in V. Let v_3 be a vertex in ω that is adjacent to v_2. Since v_2 is of type III, let C, C' be the two 2–cells in $\mathrm{St}(v_2, Q_R)$ that contain v_3. By Lemma 4.11 and Lemma 4.8 (2), v_3 is not of type O. Moreover, since

C and C' are mapped to different blocks by q, v_3 is not of type I. If v_3 is of type II, let C_1 and C_1' be other cells in $\mathrm{St}(v_3, Q_R)$. Since $C \cap C'$ is one edge, $C \cap C_1$ has $n-1$ edges by Lemma 4.11 and Lemma 4.6 (3). Thus $v_1 \in C \cap C_1$. By Lemma 4.6 (3), $q(C)$ and $q(C_1)$ are in the same block. This contradicts that v_1 is of type III since different cells containing a vertex of type III have q–images in different blocks. Thus v_3 must be of type III. Moreover, v_3 and v_2 have the same support by Lemma 4.13. Now the claim follows by repeating this argument for other vertices in ω.

It follows from the above claim that either V is the collection of all vertices in ∂C, or V is the collection of vertices in an arc ω of ∂C that has $\leq n-1$ edges. Suppose the number of edges in ω is ≥ 1 and is $< n-1$. Let v_1 and v_2 be two endpoints of ω. For $i = 1, 2$, let $w_i \in (\partial C \setminus \omega)$ be a vertex that is adjacent to v_i. Since w_i is not of type III, by the discussion in the previous paragraph, w_i is of type II. Moreover, there is a cell C_i of Q_R such that $C_i \cap C$ has $n-1$ edges and $(C_i \cap C) \cap \overline{v_i w_i} = w_i$. Since ω has at least one edge, $(C_1 \cap C) \cap (C_2 \cap C)$ contains at least one edge. Since ω has $< n-1$ edges, $(C_1 \cap C) \neq (C_2 \cap C)$, in particular $C_1 \neq C_2$. This contradicts that Q_R is planar. Thus ω is either a point, or an arc with $n-1$ edges. □

The following is a consequence of Lemma 6.1.

Corollary 6.2. *Let $\mathcal{C}' \subset Q'$ be a Coxeter region and let $x \in \mathcal{C}'$ be a fake vertex such that the 2–cell of Q_R associated with x is interior. Let D_ϵ be a disc of radius ϵ around x in Q'. Then for ϵ small enough, exactly one of the following possibilities happens:*

(1) $\mathcal{C}' \cap D_\epsilon = K \cap D_\epsilon$ where K is a chamber of $\mathcal{C}'$ containing x;
(2) $\mathcal{C}' \cap D_\epsilon$ is a half disc of D_ϵ;
(3) $\mathcal{C}' \cap D_\epsilon = D_\epsilon$.

In general, we do not have control of the local structure at each point of $\mathcal{C}'$. This motivates us to pass to a subset of $\mathcal{C}'$ as follows.

Definition 6.3. Let $x \in Q_R$ be a vertex of type III. Suppose there exists an edge $e \subset Q_R$ containing x such that

(1) the geodesic line $\ell \subset Q'$ that is orthogonal to e (we can view e as an edge in Q') and passes through the midpoint of e satisfies that $d(x_0, \ell) > R$, where x_0 is the base point in Q' as in Section 4.3 and d denotes the CAT(0) distance;
(2) x and x_0 are on different sides of ℓ;
(3) let H' be the half-space of Q' that is bounded by ℓ and contains x, then $H' \subset Q_R$ and any 2–cell of Q_R that intersects H' is interior.

Let $\mathcal{C}' \subset Q'$ be the Coxeter region containing x. The *(x, ℓ)–half Coxeter region* of Q' is the connected component of $\mathcal{C}' \cap H'$ that contains x.

Remark 6.4. There is a constant L such that for any vertex $x \in Q_R$ of type III satisfying $d(x, x_0) > LR$ (we can also view x as a vertex in

Q'), there exists an edge of Q_R containing x that satisfies the conditions in Definition 6.3.

By Definition 6.3 (1) and (2), H' is isometric to a half-plane. Let $\mathcal{H}'$ be the (x,ℓ)–half Coxeter region. Then $\mathcal{H}'$ is locally convex in H' by Lemma 6.1 and Definition 6.3 (3). Since $\mathcal{H}'$ is connected, $\mathcal{H}'$ is actually convex in H'.

Lemma 6.5. *$\mathcal{H}'$ is a union of chambers with the same support.*

Proof. Let $x \in Q_R$ be as in Definition 6.3. Let K_x be the x–chamber. Then $K_x \subset \mathcal{H}'$, and $K_x \cap \ell$ is an edge of K_x. Let v_1 be an endpoint of $K_x \cap \ell$ and let ℓ^- be the half-line of ℓ with endpoint v_1 such that $K_x \cap \mathcal{H}' \nsubseteq \ell^-$. Note that v_1 is a fake vertex in Q'. Let $\{K_i\}_{i=1}^k$ be the chambers of $\mathcal{C}'$ that contain v_1. By Definition 6.3 (3) and Lemma 6.1, either $K_i \cap \ell$ is an edge of K_i, or $K_i \cap \ell = \{v_1\}$. If $(\cup_{i=1}^k K_i) \cap \ell = K_x \cap \ell$, then $\ell^- \cap \mathcal{H}' = \{v_1\}$ by the convexity of $\mathcal{H}'$. If $(\cup_{i=1}^k K_i) \cap \ell \neq K_x \cap \ell$, then there exists K_i such that $K_i \cap \ell$ is an edge of K_i and $K_i \cap \ell \neq K_x \cap \ell$. We can assume $K_i \subset H'$ (if K_i and K_x are on different sides of ℓ, then we must be in case (3) of Lemma 6.1). Thus $K_i \subset \mathcal{H}'$. Let v_2 be the endpoint of $K_i \cap \mathcal{H}'$ that is different from v_1. We then repeat previous discussion with v_1 replaced by v_2. Keep running this process to walk along $\ell \cap \mathcal{H}'$ until one has to stop at an endpoint of $\ell \cap \mathcal{H}'$. Then it follows that a small neighborhood of $\ell \cap \mathcal{H}'$ in $\mathcal{H}'$ is contained in a union of chambers inside $\mathcal{H}'$ such that each chamber either intersects ℓ in an edge of this chamber, or intersects ℓ in a point. Thus $\mathcal{H}'$ is a union of chambers. Lemma 6.1 and the connectedness of $\mathcal{H}'$ imply that all chambers in $\mathcal{H}'$ have the same support. $\square$

By Lemma 6.5, $\mathcal{H}'$ has a cell structure such that its vertices are fake vertices of Q' contained in $\mathcal{H}'$ and its 2–cells are chambers with the same support. We define the *(x,ℓ)–half Coxeter region $\mathcal{H} \subset Q_R$ corresponding to* $\mathcal{H}'$ to be the union of 2–cells of Q_R which are associated with fake vertices in $\mathcal{H}'$. Each edge $e \subset \mathcal{H}'$ has a dual edge in Q_R which intersects e in exactly one point. Thus edges of $\mathcal{H}'$ inherit labels from edges of Q_R.

Let $\mathbb{D}$ be the Davis complex of the Coxeter group W whose defining graph is the support of a chamber in $\mathcal{H}'$. The collections of walls of $\mathbb{D}$ cut $\mathbb{D}$ into another complex $\mathbb{D}'$, which is the dual of $\mathbb{D}$. Each edge of $\mathbb{D}'$ inherits a label from its dual edge in $\mathbb{D}$ (edges of $\mathbb{D}$ are labeled by generators of W).

By Lemma 6.1, there exists a label-preserving cellular isometry from the closed star of each vertex of $\mathcal{H}'$ to $\mathbb{D}'$. Since $\mathcal{H}'$ is simply connected (as $\mathcal{H}'$ is convex in H'), by Corollary 6.2 and a developing argument, we can produce a label-preserving cellular local isometry $i' \colon \mathcal{H}' \to \mathbb{D}'$. Note that i is actually an isometric embedding, since both $\mathcal{H}'$ and $\mathbb{D}'$ are CAT(0). By Lemma 4.11 (2), there is an induced cellular embedding $i \colon \mathcal{H} \to \mathbb{D}$ such that $i(\mathcal{H})$ is the union of 2–cells dual to vertices in $i'(\mathcal{H}')$. We view $\mathcal{H}'$ (resp. $\mathcal{H}$) as subcomplexes of $\mathbb{D}'$ (resp. $\mathbb{D}$) via i' (resp. i). A *wall* in $\mathcal{H}'$ (resp. $\mathcal{H}$) is the intersection a wall in $\mathbb{D}'$ (resp. $\mathbb{D}$) with $\mathcal{H}'$ (resp. $\mathcal{H}$). The definition of wall does not depend the choice of the label-preserving map i'. We say $\mathcal{H}$

(or $\mathcal{H}'$) is *irreducible* (resp. *reducible*) if the associated Coxeter group W has defining graph being a triangle (resp. square).

7. Orientation of edges of Coxeter regions

Let x, ℓ and H' be as in Definition 6.3 and let $\mathcal{H}$, $\mathcal{H}'$, $\mathbb{D}$ and $\mathbb{D}'$ be as in the previous section.

A *border* of $\mathcal{H}'$ is a maximal connected subset of $\partial\mathcal{H}'$ which is convex in $\mathcal{H}'$. A border of $\mathcal{H}'$ is *fake* if this border is contained in ℓ. Other borders are *real*. If we think of $\mathcal{H}'$ as of a convex subcomplex of $\mathbb{D}'$, then each border $\mathcal{B}' \subset \mathcal{H}'$ is contained in a wall $\mathcal{W}$ of $\mathbb{D}'$. Let $\mathcal{W}^+$ (resp. $\mathcal{W}^-$) be the half-space of $\mathbb{D}'$ bounded by $\mathcal{W}$ that contains (resp. does not contain) $\mathcal{H}'$. Let $N_{\mathcal{B}'}$ be the carrier of $\mathcal{B}'$ in $\mathbb{D}$ (see the beginning of Section 2 for definition of carrier). Then $\mathcal{B}'$ corresponds to an *outer border* $\mathcal{B}^o$ and an *inner border* $\mathcal{B}^i$ of $\mathcal{H}$, which are defined to be the maximal subcomplexes of $N_{\mathcal{B}'}$ contained in $\mathcal{W}^-$ and $\mathcal{W}^+$, respectively. $\mathcal{B}^o$ or $\mathcal{B}^i$ is *real* or *fake* if $\mathcal{B}'$ is real or fake. Recall that each edge of Q_R inherits an orientation via $q\colon Q_R \to X^*_\Gamma$, so does each edge in $\mathcal{H}$. Now we study the orientation of edges along a border of $\mathcal{H}$.

The borders $\mathcal{B}^o$ and $\mathcal{B}^i$ are made of *pieces*, which are maximal subsegments of the border that are contained in a 2–cell of $N_{\mathcal{B}'}$. Each vertex in a border of $\mathcal{H}$ either belongs to one piece, or belongs to the intersection of two pieces.

7.1. Orientation along the borders.

Lemma 7.1. *The orientation of edges on the same piece on a real outer border of $\mathcal{H}$ is consistent, i.e. there does not exist an interior vertex of a piece such that the orientation is reversed at that vertex. The same is true for real inner border.*

Proof. Let ω be a piece in the lemma. Then there exists one endpoint, say v, of ω such that v belongs to two different cells $C_1, C_2 \subset \mathcal{H}$. Since q maps C_1 and C_2 to cells in different blocks of X^*_Γ, we know v is of type II or type III (see Table 1 on page 33). If v is of type III, then we can enlarge $\mathcal{H}$ (hence $\mathcal{H}'$) since v is in a real border, which contradicts the maximality of $\mathcal{H}'$. Hence v is of type II. Note that if a piece is contained in a cell of $\mathcal{H}$ with $2n$ edges, then the piece has $n-1$ edges. Now, the statement about the outer border follows from Remark 4.7. The statement about the inner border follows from Lemma 7.2 below. □

We will be repeatedly using the following simple observation, which follows from Figure 2.

Lemma 7.2. *If we draw a cell of X^*_Γ as a regular polygon in $\mathbb{E}^2$, then opposite sides of this polygon have orientations pointing towards the same direction.*

Lemma 7.3. *Suppose $\mathcal{H}'$ is irreducible. Suppose one real border $\mathcal{B}'$ of $\mathcal{H}'$ is unbounded. Let $\mathcal{B}^o \subset \mathcal{H}$ be the associated outer border. Then there exists a sub-ray $\mathcal{B}^o_1 \subset \mathcal{B}^o$ such that edges in $\mathcal{B}^o_1$ have consistent orientation.*

Proof. We shall show the inner border $\mathcal{B}^i$ has only finitely many vertices where the orientation is reversed. Then the same holds for $\mathcal{B}^o$ by Lemma 7.2 and the corollary follows. Since $\mathcal{H}'$ is a convex subcomplex of $\mathbb{D}'$ which contains at least one chamber, there exists a wall $\mathcal{W} \subset \mathcal{H}'$ such that $\mathcal{B}' \nsubseteq \mathcal{W}$ and $\mathcal{W}$ is parallel to $\mathcal{B}'$. Note that if $\mathcal{B}'$ is a ray (resp. line), then $\mathcal{W}$ is a ray (resp. line). Suppose $\mathcal{W}$ is closest to $\mathcal{B}'$ among all such walls. Since $\mathcal{B}'$ is a real border, any orientation reserving vertex in the inner border $\mathcal{B}^i$ is the intersection of two pieces by Lemma 7.1.

We now conduct a case study and first assume the associated Coxeter group is $(2, 3, 6)$, i.e. the labels of edges in the defining graph of the Coxeter graph (which is a triangle) are $2, 3, 6$. Depending on the structure of the carrier of $\mathcal{B}$, we consider two sub-cases as follows.

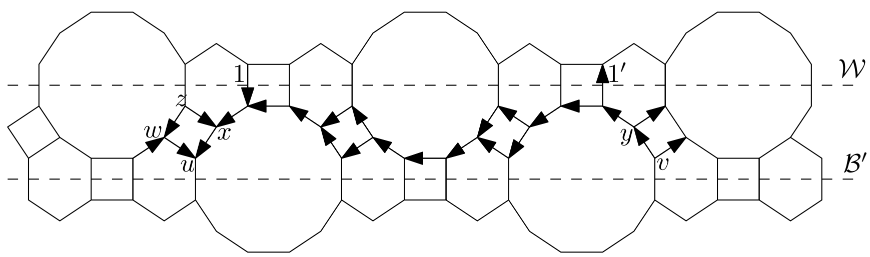


FIGURE 24.

The first sub-case is indicated in Figure 24. Choose orientation reversing vertices $u, v \in \mathcal{B}^i$ such that there are no orientation reversing vertices in $\mathcal{B}^i$ between u and v. We use Lemma 7.2 to find orientation of edges in the boundary of the carrier $N_{\mathcal{W}}$ of $\mathcal{W}$. Note that u may give rise to more than one orientation reversing points (e.g. w, z and x in Figure 24), however, we take the "rightmost" one (e.g. x in Figure 24). Similarly, we pick the "leftmost" orientation reserving vertex y determined by v. Note that

(1) x and y are in the same side of $\mathcal{W}$;
(2) both x and y are contained in only one 2–cell of $N_{\mathcal{W}}$;
(3) there are no orientation reversing interior vertices in ω, where ω is the shortest edge path in $\partial \mathcal{W}_{\mathcal{H}}$ from x to y;

The vertex x completely determines the orientation of edges in the unique 2–cell of $N_{\mathcal{W}}$ that contains x, thus edge 1 in Figure 24 is oriented downwards. Similarly, edge $1'$ is oriented upwards. This contradicts Lemma 7.2.

We now consider the second $(2, 3, 6)$ sub-case. Let u and v be as before. Again we use Lemma 7.2 to find orientation of other edges. When there are at least two 12–gons between u and v, see Figure 25, Figure 26 and Figure 27. When there is only one 12–gon between u and v, see Figure 28. When there are no 12–gons between u and v, see Figure 27 left (u' and v' there play the roles of u and v respectively).

The $(3, 3, 3)$ case and the $(2, 4, 4)$ case are similar and simpler. We omit the proof, but include Figure 29 for the convenience of the reader. Note

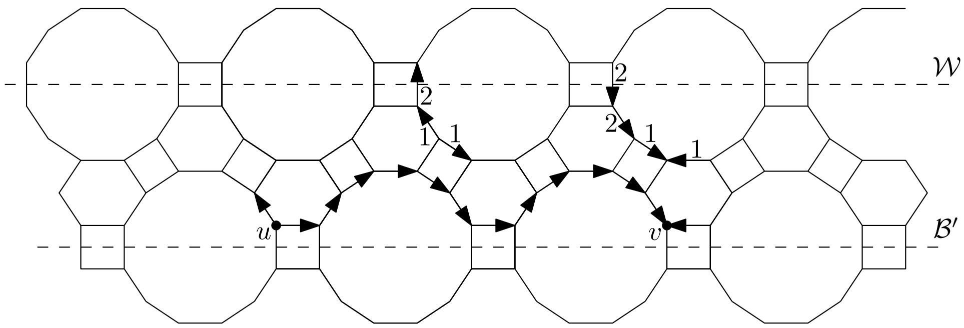


FIGURE 25. The orientations of edges labelled 1 follow from Lemma 7.2. The orientations of edges labelled 2 follow from the form of orientation of edges in the boundary of a cell (cf. Figure 2). The two vertical edges labelled by 2 have opposite orientations, contradicting Lemma 7.2.

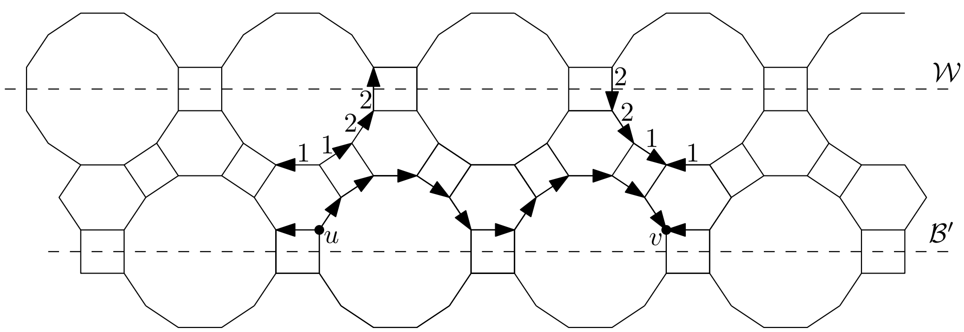


FIGURE 26. The orientations of edges labelled 1 follow from Lemma 7.2. The orientations of edges labelled 2 follow from the form of orientation of edges in the boundary of a cell (cf. Figure 2). The two vertical edges labelled by 2 have opposite orientations, contradicting Lemma 7.2.

that the second $(2,4,4)$ sub-case and the $(3,3,3)$ case are immediate since there is no gap between the carrier of $\mathcal{W}$ and the carrier of $\mathcal{B}'$. $\square$

We record the following result which follows from the proof of Lemma 7.3.

Lemma 7.4. *Suppose $\mathcal{H}'$ is irreducible. Let $\mathcal{B}'$ be a real border of $\mathcal{H}'$ which is homeomorphic to $\mathbb{R}$. Let $\mathcal{B}^o \subset \mathcal{H}$ be the associated outer border. Then there is at most one orientation reversing vertex on $\mathcal{B}^o$.*

Proposition 7.5. *Let $\mathcal{H} \subset Q_R$ be the (x,ℓ)–half Coxeter region. Suppose $\mathcal{H}$ is irreducible and unbounded. Then $\mathcal{H}$ contains a Coxeter ray.*

Proof. Recall that the (x,ℓ)–half Coxeter region $\mathcal{H}'$ associated with $\mathcal{H}$ can be embedded as a convex subcomplex of $\mathbb{D}'$. By our assumption, $\mathcal{H}'$ is also unbounded. Thus either $\mathcal{H}'$ has a real border that is unbounded, or $\mathcal{H}'$ has

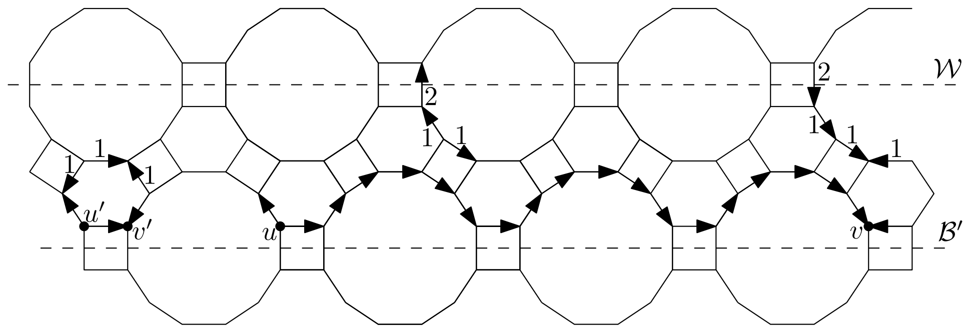


FIGURE 27. The orientations of edges labelled 1 follow from Lemma 7.2. The orientations of edges labelled 2 follow from the form of orientation of edges in the boundary of a cell (cf. Figure 2). The two vertical edges labelled by 2 have opposite orientations, contradicting Lemma 7.2. The orientation of the hexagon on the left contradicts the form of cells (cf. Figure 2).

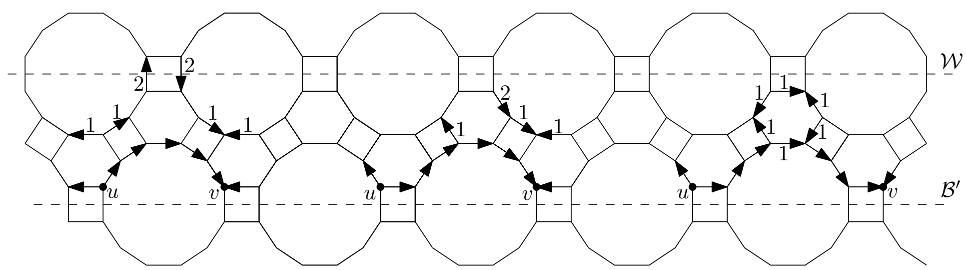


FIGURE 28. The following lead to contradictions: two vertical edges labelled 2 on the left; two edges of the hexagon labelled 1 and 2 in the middle; orientation of the hexagon by edges labelled 1 on the right.

no real border and has only a fake border that is all of ℓ. By Lemma 7.3, it remains to deal with the latter case. We also assume the associated Coxeter group is $(2,3,6)$, since the $(2,4,4)$ and $(3,3,3)$ cases are similar and simpler.

We think of $\mathcal{H}$ as of a subcomplex of $\mathbb{D}$. By Lemma 7.2, edges dual to the same wall of $\mathcal{H}$ have orientation pointing towards the same direction. Thus each wall of $\mathcal{H}$ has a well-defined orientation. Each wall $\mathcal{W}$ has an *associated halfspace*, which is the halfspace bounded by $\mathcal{W}$ such that $\mathcal{W}$ is oriented toward this halfspace.

First we consider the case that there are two different parallel walls $\mathcal{W}_1$ and $\mathcal{W}_2$ of $\mathcal{H}$ with opposite orientations. We assume without loss of generality that $\mathcal{W}_1$ and $\mathcal{W}_2$ are not separated by any other wall of $\mathcal{H}$, and they are oriented as in Figure 30. We define sequences of consecutive points $\{v_i\}_{i=1}^{\infty} \subset \mathcal{W}_1$ and $\{u_i\}_{i=1}^{\infty} \subset \mathcal{W}_2$ such that each v_i (resp. u_i) is the center of a 12–gon C_i (resp. C'_i) and $\angle_{v_i}(u_i, v_{i+1}) = \angle_{u_i}(v_{i+1}, u_{i+1}) = \pi/6$ for $i \geq 1$.

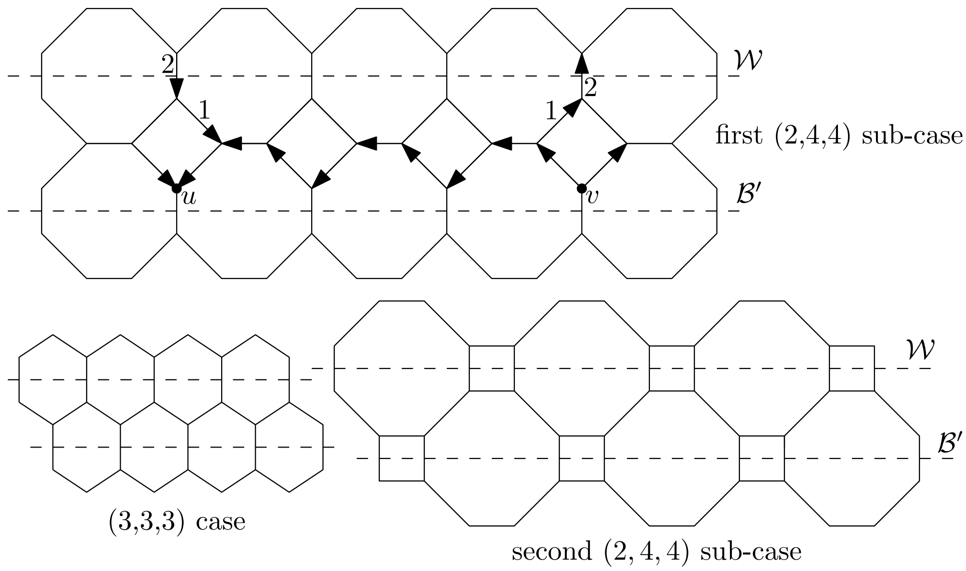


FIGURE 29.

Assume without loss of generality that $v_i, u_i \in \mathcal{H}$ for all i. For each $i \geq 1$, let $\mathfrak{W}_i$ be the collection of walls that are different from $\mathcal{W}_1$ and $\mathcal{W}_2$, and contain at least one of v_j or u_j for $j \geq i$. An element in $\mathfrak{W}_i$ is *positively oriented* if its associated halfspace contains all but finitely many v_i, otherwise it is *negatively oriented*. We claim for i large enough, either each element in $\mathfrak{W}_i$ is positively oriented, or each element in $\mathfrak{W}_i$ is negatively oriented.

Let L be a straight line parallel to $\mathcal{W}_1$ such that L is between $\mathcal{W}_1$ and $\mathcal{W}_2$ (see the dotted line in Figure 30). Then $\mathfrak{W}_1 \cap L$ is a discrete subset of L, which can be naturally identified with the positive integers $\mathbb{Z}^+$. This gives rise to a total order on $\mathfrak{W}_1$. If the smallest element $\mathcal{W}$ of $\mathfrak{W}_1$ is negatively oriented, then the orientation of $\mathcal{W}$ and $\mathcal{W}_1$ gives rise to an orientation-reversing vertex in the boundary of C_1, hence the orientation of each edge in C_1 is determined as in Figure 30. Let $\mathcal{W}_{12}$ be the wall containing u_1 and v_1. Then the orientation of $\mathcal{W}_2$ and $\mathcal{W}_{12}$ gives rise to an orientation reversing vertex in the boundary of C_1', hence the orientation of each edge in C_1' is determined as in Figure 30. By repeating this process, we know the claim holds with $i = 1$. If $\mathcal{W}$ is positively oriented, let $\mathcal{W}'$ be the smallest element in $\mathfrak{W}_1$ that is negatively oriented (if such $\mathcal{W}'$ does not exist, then we are done). Suppose $v = \mathcal{W}' \cap \mathcal{W}_1$. If $\mathcal{W}'$ is the smallest element in $\mathfrak{W}_1$ that contains v, then we finish the proof as before. Otherwise, let $\mathcal{W}''$ be the biggest element in $\mathfrak{W}_1$ that contains v. By considering the orientation of edges in the boundary of the 2–cell containing v, we know $\mathcal{W}''$ is negatively oriented. Since $\mathcal{W}''$ intersects $\mathcal{W}_1$ in a 30 degree angle, we argue as before to show any element of $\mathfrak{W}_1$ which is $\geq \mathcal{W}''$ is negatively oriented.

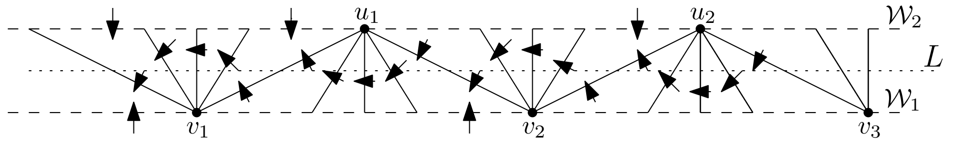


FIGURE 30.

Now we produce a Coxeter ray from the above claim. If $\mathcal{W}_1$ and $\mathcal{W}_2$ are as in Figure 25, then there are two centers of 12–gons in $\mathcal{W}_1$ between v_1 and v_2, which we denote by v_1' and v_1''. Now we define $\mathfrak{W}_1'$ and $\mathfrak{W}_1''$ as in the previous paragraph, and the above claim still holds for them. Since there are infinitely many walls in $\mathfrak{W}_1' \cap \mathfrak{W}_1$ and $\mathfrak{W}_1'' \cap \mathfrak{W}_1$, by deleting finitely many walls from $\mathfrak{W}_1' \cup \mathfrak{W}_1'' \cup \mathfrak{W}_1$, all of the rest are positively oriented (or negatively oriented). This gives rise to a Coxeter ray in the carrier of $\mathcal{W}_1$. If $\mathcal{W}_1$ and $\mathcal{W}_2$ are as in Figure 31, then by the above claim, we assume without loss of generality that the orientation of edges is as in Figure 31. We have yet to determine the orientation of $\overline{st}$. If $\mathcal{W}_2$ is oriented downwards (resp. upwards) and $\mathcal{W}_1$ is oriented upwards (resp. downwards), then the orientation reverses at y (resp. x) if we consider the hexagon containing y (resp. x), hence $\overline{st}$ (resp. $\overline{s't'}$) is oriented from s to t (resp. s' to t'). It follows that there is a Coxeter ray contained in the carrier of $\mathcal{W}_1$.

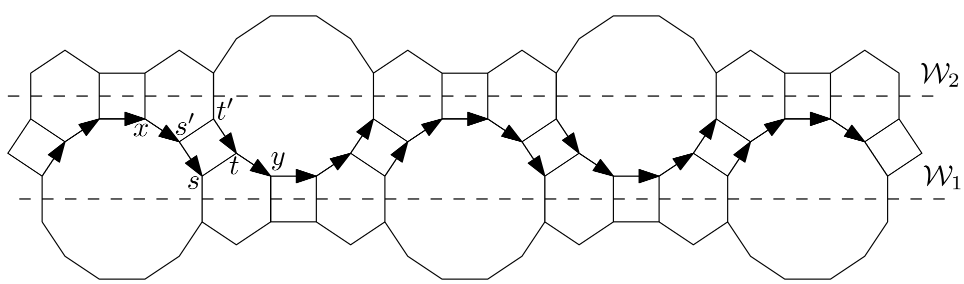


FIGURE 31.

It remains to consider the case where each pair of parallel walls of $\mathcal{H}$ are oriented towards the same direction. However, it is straightforward to produce a Coxeter ray in such case. $\square$

7.2. **Orientation around the corners.** Let $\mathcal{H}' \subset Q', H', \ell$ be as in Definition 6.3. A *corner* of $\mathcal{H}'$ is the intersection of two real borders of $\mathcal{H}'$. Note that if $\mathcal{H}'$ is bounded, then it has at least one corner.

Lemma 7.6. *Suppose $\mathcal{H}'$ is irreducible. Then there does not exist a real border of $\mathcal{H}'$ such that it has two corners of angle $= \pi/2$.*

Proof. By Corollary 6.2, the cell in $\mathcal{H}$ that contains a right-angled corner of $\mathcal{H}'$ must be a square. Thus it suffices to consider the $(2, 3, 6)$ case and the $(2, 4, 4)$ case. The $(2, 3, 6)$ case has two sub-cases. The first sub-case is

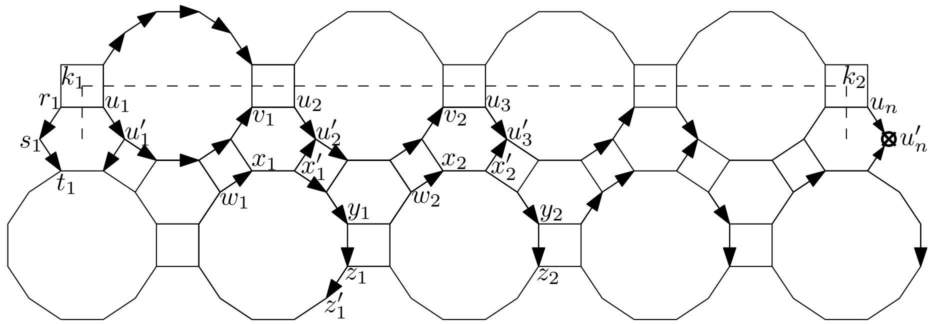


FIGURE 32.

indicated in Figure 32, where k_1 and k_2 are two right-angled corners. By Lemma 7.1, we can assume without loss of generality that both $\overline{r_1s_1}$ and $\overline{s_1t_1}$ are oriented downwards. Thus $\overline{u_1u'_1}$ is oriented downwards since it is parallel to $\overline{s_1t_1}$. Using Lemma 7.1 again, we determine the orientation of each edge in the path ω_1 from u_1 to v_1. Hence we know the orientation of $\overline{w_1x_1}$ and $\overline{u'_2x'_1}$ (since each of them is parallel to one edge in ω_1), and the orientation of $\overline{z_1z'_1}$ (since it is parallel to $\overline{r_1s_1}$). Let ω'_1 be the path made of 6 edges from w_1 to z'_1, containing x_1. The orientation of $\overline{w_1x_1}$ and $\overline{z_1z'_1}$ implies edges in ω'_1 are oriented consistently. In particular, $\overline{y_1z_1}$ is oriented downwards. Using Lemma 7.1 again, we deduce the orientation of each edge in the path ω_2 from u_2 to v_2, since one edge of ω_2 and one edge of ω'_1 are parallel. Hence $\overline{w_2x_2}$ is oriented upwards. Since $\overline{y_2z_2}$ is oriented downwards (it is parallel to $\overline{y_1z_1}$), edges in the path ω'_2 of length 5 from w_2 to z_2 are oriented consistently. Now we can define ω_3 and determine orientation of its edges as before. Since $\overline{x'_2u'_3}$ is oriented upwards, u'_3 is an orientation-reversing point. We can continue this process to deduce that u'_n is an orientation-reversing point, however, this is impossible since u_n is of type II by Lemma 7.1. The second $(2, 3, 6)$ sub-case (see Figure 33) can be studied in a similar way. Note that the orientation of the edges labeled n in Figure 33 can be deduced from orientation of edges whose labels are $< n$, for example, 6 follows from 5 and 2. We deduce that u_n is an orientation-reversing point, which leads to a contradiction.

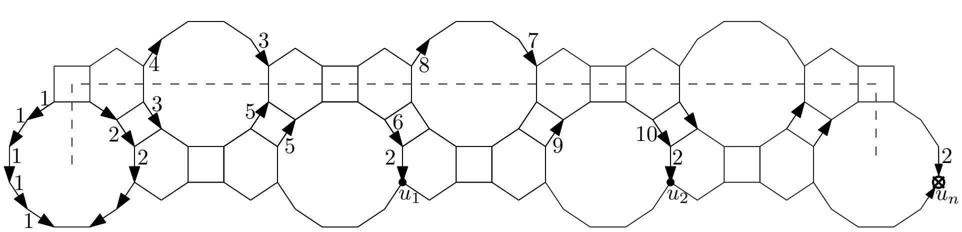


FIGURE 33.

The $(2,4,4)$ case is similar and simpler, so we omit the proof, but we attach Figure 34 for the convenience of the reader. □

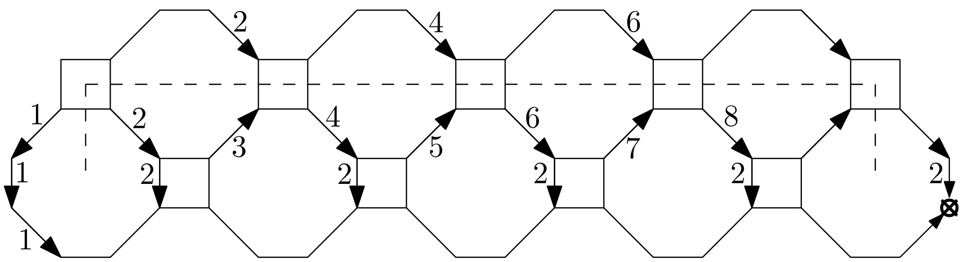


FIGURE 34.

8. Existence of singular rays

The goal of this section is proving Proposition 8.3. Let Q', Q_R and x_0 be as in Section 4.2 and Section 4.3 (recall that x_0 is the base point in Q'). Let x, ℓ and H' be as in Definition 6.3 and let $\mathcal{H}'$ and $\mathcal{H}$ be as in Section 6.

Lemma 8.1. *A corner of $\mathcal{H}'$ with acute angle gives rise to a diamond ray.*

We refer to Figure 35 for the following proof.

Proof. Let $s \in \mathcal{H}'$ be an acute corner. Let $\Pi_1 \subset \mathcal{H}$ be the cell containing s. We are in the situation of Lemma 6.1 (1) and Corollary 6.2 (1). Let K be the unique chamber in $\mathcal{H}'$ containing s. Let $r_K \subset Q'$ be a CAT(0) geodesic ray such that $r_K \cap K = s$ and the angles between r_K and the two sides of K at s are equal. In all cases we have $r_K \subset H'$, in particular $r_K \subset Q_R$.

Let Π, Π' be the 2–cells in $\mathcal{H}$ that intersect Π_1. Then $\Pi_1 \cap \Pi \cap \Pi'$ is a vertex of type III (see Table 1 on page 33), which we denote by x_1. Note that $\Pi_1 \cap \Pi$ (resp. $\Pi_1 \cap \Pi'$) is an edge containing x_1. Let y_1 (resp. y_1') be the other endpoint of this edge. By Lemma 6.1 (1), y_1 and y_1' are not of type III, hence they are of type II.

Suppose Π_1 has $2n$ edges. Since s is an acute corner, $n \geq 3$. Since $\Pi \cap \Pi_1$ (resp. $\Pi \cap \Pi_1'$) has only one edge, by Lemma 4.11 and Lemma 4.6, there is a 2–cell $C_1 \subset \mathrm{St}(y_1, Q_R)$ (resp. $C_1' \subset \mathrm{St}(y_1', Q_R)$) such that $C_1 \cap \Pi_1$ (resp. $C_1' \cap \Pi_1$) is a path with $n-1$ edges, moreover, C_1, C_1' and Π_1 are in the same block. Let x_2 be the vertex in $\partial \Pi_1$ that is opposite to x_1. Then $x_2 \in (C_1 \cap C_1' \cap \Pi_1)$. Note that $x_2 \in r_K \subset Q_R$. Thus by Lemma 4.11 and Lemma 4.5, x_2 is of type I. Let Π_2 be the other 2–cell in $\mathrm{St}(x_2, Q_R)$ besides C_1, C_1' and Π_1. Either x_2 is the tip of both Π_1 and Π_2, or x_2 is the tip of both C_1 and C_1'. We deduce from Lemma 4.5 and $n \geq 3$ that the latter case is impossible. Also we deduce from Lemma 4.5 that $C_1 \cap \Pi_2$ is an edge. Let y_2 be the endpoint of this edge that is not x_2. We claim y_2 is of type I.

Since y_2 is an interior vertex of Q_R, Lemma 4.11 describes all possibilities of $\mathrm{St}(y_2, Q_R)$. Lemma 4.11 (1) can be ruled out immediately. Note that y_2

is not of type III since C_1 and Π_2 are in the same block. If y_2 is of type II, then Lemma 4.6 (2) and (3) imply $C_1 \cap \Pi_2$ has $n-1$ edges, which is contradictory to $n \geq 3$. Thus the above claim follows.

Note that y_2 is not a tip of Π_2 (since x_2 is a tip of Π_2). Hence y_2 is a tip of C_1. By Lemma 4.5, there is a 2–cell $C_2 \subset \mathrm{St}(y_2, Q_R)$ such that $C_2 \cap \Pi_2$ is a path with $n-1$ edges. We define C_2' similarly. Let x_3 be the vertex in $\partial\Pi_2$ opposite to x_2. Again we know $x_3 \in r_K \subset Q_R$ and x_3 is contained in three cells C_2, C_2' and Π_2 which are in the same block. Hence x_3 is of type I and the pair (Π_2, Π_3) plays the role of (Π_1, Π_2) in Figure 9. By repeating this argument, we can inductively produce a sequence of 2–cells $\{\Pi_i\}_{i=1}^{\infty}$ such that $\Pi_i \cap \Pi_{i+1}$ is a vertex of type I contained in r_K and (Π_i, Π_{i+1}) plays the role of (Π_1, Π_2) in Figure 9, which finishes the proof. $\square$

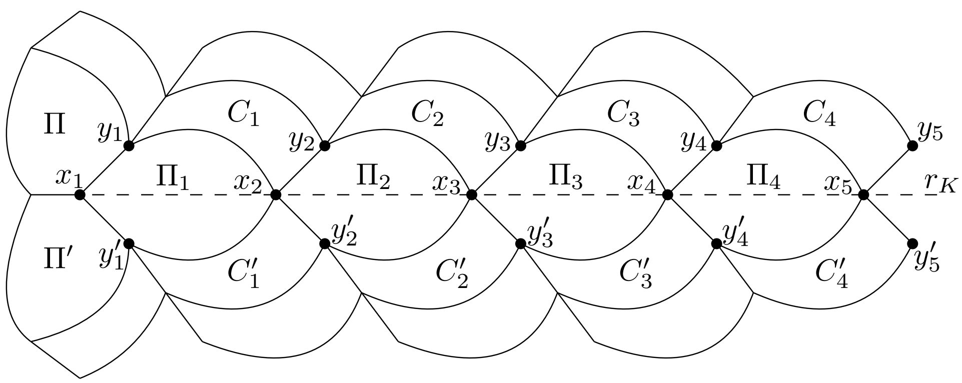


FIGURE 35.

Corollary 8.2. *Suppose that outside any compact subset of Q_R there is a $\triangle$–vertex (i.e. type III vertex whose support is a triangle). Then there is either a diamond ray, or a Coxeter ray in Q_R.*

Proof. Consider an irreducible (x, ℓ)–half Coxeter region $\mathcal{H}'$. If $\mathcal{H}'$ is unbounded, then we obtain a Coxeter ray by Proposition 7.5. If $\mathcal{H}'$ contains an acute corner, then we obtain a diamond ray by Lemma 8.1. Now we assume $\mathcal{H}'$ is bounded and does not contain any acute corner. By Lemma 7.6, $\mathcal{H}'$ contains a unique right-angled corner s.

Let K be the unique chamber in $\mathcal{H}'$ containing s. We extend the two borders of $\mathcal{H}'$ meeting at s to two CAT(0) geodesic lines ℓ_1, ℓ_2 in Q'. Since $\mathcal{H}'$ is bounded and s is the only corner of $\mathcal{H}'$, we know that ℓ, ℓ_1 and ℓ_2 bound a triangle which contains K. Note that the angle between ℓ_1 (or ℓ_2) and ℓ is $\geq 30°$. Thus by choosing s sufficiently far away from the base point x_0 of Q', we know at least one of ℓ_1 and ℓ_2, say ℓ_1, satisfies that $\ell_1 \subset Q_R$ and that K and x_0 are on the same side of ℓ_1.

Let $\mathcal{C}_K'$ be the Coxeter region containing K. Let $\ell_1' \subset \ell_1$ be the connected component of $\mathcal{C}_K' \cap \ell_1$ containing s. By local convexity of $\mathcal{C}_K'$ along ℓ_1', we know the intersection of $\mathcal{C}'_K$ and an ϵ–neighborhood of ℓ_1 (for ϵ small enough) is on one side of ℓ_1. If ℓ'_1 is unbounded, i.e. ℓ'_1 is a ray based at s, then by Corollary 6.2 and the argument in Lemma 7.3, we know a sub-ray of ℓ'_1 gives rise to a Coxeter ray. If ℓ'_1 is bounded, let t be the other endpoint of ℓ'_1. Then t satisfies Corollary 6.2 (1). The proof of Lemma 7.6 implies that the angle at t is acute, then we can produce a diamond ray by repeating the argument in Lemma 8.1 (we define a ray r based at t as in the first paragraph of the proof of Lemma 8.1, then r and K are in different sides of ℓ_1, hence, r and the base point x_0 are in different sides of ℓ_1, thus $r \subset Q_R$, the rest of the proof is identical). $\square$

Proposition 8.3. *Q_R contains a singular ray.*

Proof. By Corollary 8.2, it suffices to consider the case that all the $\triangle$–vertices of type III are contained in a bounded subset of Q'. We assume without loss of generality that Q_R does not contain any $\triangle$–vertex.

Suppose there is an interior vertex $y_0 \in Q_R$ such that y_0 is of type II or y_0 is a $\square$–vertex. Pick edges e_0, e_1 of Q_R containing y_0 such that

(1) for $i = 0, 1$, e_i is the intersection of two 2–cells of $\mathrm{St}(y_0, Q_R)$ that are in different blocks;
(2) e_0 and e_1 are not contained in the same 2–cell.

By Lemma 4.6 and Lemma 4.8, e_0 and e_1 always exist (such pair is unique when y_0 is of type II). By Lemma 4.2 (2), e_0 and e_1 are also edges in Q', and the concatenation of e_1 and e_0 is a CAT(0) geodesic segment in Q'.

Let r^+ and r^- be two CAT(0) geodesic rays based at y_0 containing e_1 and e_0 respectively. Then the angle between $\overline{x_0 y_0}$ (recall that x_0 is the base point of Q') and at least one of r^+ and r^-, say r^+, is $\geq 90°$. Then $d(r^+, x_0) \geq d(y_0, x_0)$ by CAT(0) geometry. Hence $r^+ \subset Q_R$. We claim r^+ is a plain ray. Let y_1 be the other endpoint of e_1. Since e_1 is contained in two 2–cells that are in different blocks, y_1 is not of type I. Hence y_1 is of type II or y_1 is a $\square$–vertex. By Lemma 4.6 and Lemma 4.8, there is another edge e_2 containing y_1 such that the above two conditions hold with e_0, e_1, y_0 replaced by e_1, e_2, y_1. Note that $e_2 \subset r^+$. By repeating such argument, we know r^+ is a union of edges $\{e_i\}_{i=1}^{\infty}$ such that $e_i \cap e_{i+1}$ is a vertex of type II for each i. Hence r^+ is a plain ray.

It remains to consider the case when every interior vertex of Q_R is of type I. One readily verifies that there is a diamond ray in such case. $\square$

9. Development of singular rays

A *flat sector (with angle α)* in Q' is a subset of Q' isometric to a sector in the Euclidean plane $\mathbb{E}^2$ bounded by two rays meeting at an angle α with $0 \leq \alpha \leq \pi$. Since Q' is homeomorphic to $\mathbb{R}^2$, we specify an orientation of Q'. Let $\gamma \subset Q'$ be a CAT(0) geodesic ray. We think of γ as being oriented from its base point to infinity (if γ happens to be mapped to the 1–skeleton of X^*_Γ, then such orientation of γ may be different from the induced orientation from X^*_Γ). Recall that we use d_H to denote the Hausdorff distance. A flat sector S with angle $0 \le \alpha \le \pi$ is *on the right side* of γ if

(1) one boundary ray γ_1 of S satisfies that $d_H(\gamma_1, \gamma) < \infty$;
(2) the interior of S has empty intersection with γ;
(3) S is to the right of γ with respect to the orientation of γ and Q'.

Since a flat sector with angle 0 is just a geodesic ray of Q', we know from the above definition the meaning of a geodesic ray γ' on the right side of γ.

The goal of this section is the following theorem. Recall the Definition 5.17 of atomic sectors and Table 2 on page 41.

Theorem 9.1. *Let $r \subset Q_R$ be a singular ray. Then there is a CAT(0) geodesic ray γ in the 1–skeleton of Q' with $\gamma \subset r$. And there exists a singular ray $r_1 \subset Q_R$ such that*

(1) the CAT(0) geodesic ray $\gamma_1 \subset r_1$ satisfies that $d_H(\gamma_1, \gamma) < \infty$;
(2) r_1 is a boundary ray of an atomic sector S_1 in Q_R;
(3) there is a flat sector S'_1 in Q' such that $S'_1 \subset S_1$ and $d_H(S_1, S'_1) < \infty$;
(4) S'_1 is on the right side of γ;
(5) the angle of S'_1 is $\ge \pi/6$.

Since Q' is flat outside a compact subset, (1) actually means something stronger: a sub-ray of γ and a sub-ray of γ_1 bound an isometrically embedded flat strip $[0, \infty) \times [0, a]$ in Q'.

Assuming Theorem 9.1, we can deduce the following, being one of our main results (see Theorem 1.1 in Introduction).

Theorem 9.2. *There are finitely many mutually disjoint atomic sectors $\{S_i\}_{i=1}^n$ in Q_R such that $d_H(\cup_{i=1}^n S_i, Q_R) < \infty$. Hence each 2–quasiflat in X^*_Γ is at bounded distance from a union of finitely many atomic sectors.*

Proof. By Proposition 8.3, there exists at least one singular ray r_1 in Q_R. Now we apply Theorem 9.1 to r_1 to obtain an atomic sector $S_1 \subset Q_R$ and a CAT(0) sector $S'_1 \subset Q'$. Let r_2 be a boundary ray of S_1 such that $d_H(r_2, r_1) = \infty$. We apply Theorem 9.1 to r_2 to produce S_2 and S'_2. Let S_i and S'_i be the sectors produced by repeating this process for i times. Consider $\{\partial_T S'_i\}$ in the Tits boundary $\partial_T Q'$ of Q'. Since Q' comes from a quasiflat, $\partial_T Q'$ is a circle whose length is $\ge 2\pi$ and $< \infty$. By Theorem 9.1, the length of each $\partial_T S'_i$ is between $\pi/6$ and π, and $\partial_T Q'$ inherits an orientation from Q' such that the terminal point of $\partial_T S'_i$ is the starting point of $\partial_T S'_{i+1}$. So there exists n such that $\cup_{i=1}^n S'_i$ covers $\partial_T Q'$. We take n to be the smallest possible. It is clear that $d_H(\cup_{i=1}^n S'_i, \partial_T Q') < \infty$, hence $d_H(\cup_{i=1}^n S_i, Q_R) < \infty$.

Now we show that it is possible to arrange $\{S_i\}_{i=1}^n$ such that they are mutually disjoint in Q_R (however, we are not claiming their images in X^*_Γ are also mutually disjoint). It suffices to show the terminal point of $\partial_T S'_n$ is the starting point η of $\partial_T S'_1$, then we can pass to suitable sub-sectors of S_i such that they are mutually disjoint. Suppose by contradiction that η

is in the interior of $\partial S'_n$. Then there is a boundary ray r of S_1 such that r is contained in S_n (modulo a compact subset of r), but r is not at finite Hausdorff distance away from any of the boundary rays of S_n. However, this is impossible by definitions of atomic sectors. Now the theorem follows. $\square$

In the rest of this section, we prove Theorem 9.1. The proof is divided into three cases when r is a plain ray (Section 9.1), a Coxeter ray (Section 9.2), or a diamond ray (Section 9.3). We conclude the proof at the end of the section.

9.1. **Development of plain rays.** We start with several preparatory observations. An interior vertex $v \in Q_R$ is *good* if $\mathrm{St}(v, Q_R)$ has four 2–cells $\{\Pi_i\}_{i=1}^4$ as in Figure 11 such that

(1) $\partial\Pi_1$ and $\partial\Pi_2$ have $2n_1$ edges, $\partial\Pi_3$ and $\partial\Pi_4$ have $2n_2$ edges;
(2) $\Pi_3 \cap \Pi_2 = \Pi_1 \cap \Pi_4 = \{v\}$;
(3) $\Pi_1 \cap \Pi_2$ has $n_1 - 1$ edges, $\Pi_3 \cap \Pi_4$ has $n_2 - 1$ edges, $\Pi_1 \cap \Pi_3$ is an edge e_1 and $\Pi_2 \cap \Pi_4$ is an edge e_2.

Suppose v is good. If $\Pi_1, \Pi_2, \Pi_3, \Pi_4$ are contained in the same block, then v is of type I (see Table 1 on page 33). If $\{\Pi_i\}_{i=1}^4$ are contained in two different blocks, then v is of type II. If $\{\Pi_i\}_{i=1}^4$ are contained in more than two different blocks, then v is a $\square$–vertex. We record the following observations, which follow from the argument in the second paragraph of the proof of Lemma 4.6 (see also Figure 12).

Lemma 9.3. *Suppose v is a good vertex and let e_1 and e_2 be as above. Let u be the endpoint of $\Pi_1 \cap \Pi_2$ with $u \neq v$. Then*

(1) e_1 and e_2 are also edges in Q' and their concatenation forms a CAT(0) geodesic segment in Q';
(2) the convex hull of e_1 (or e_2) and u in Q' is a flat triangle with a right angle at v and an angle of $\pi/2n$ at u, where $2n$ is the number of edges in $\partial\Pi_1$.

Let Z be $\mathbb{E}^2$ tiled by unit squares. Suppose the tiling is compatible with the coordinate system on $\mathbb{E}^2$. So it makes sense to talk about vertical edges, or horizontal edges of Z. A subcomplex of $P \subset Q_R$ is *quasi-rectangular*, or *Z'–quasi-rectangular*, if there exists a homeomorphism h from a convex subcomplex $Z' \subset Z$ to P such that for each square $\sigma \subset Z'$, $h(\sigma)$ is a 2–cell C of P, $h(\partial\sigma) = \partial C$ and h maps the two vertical edges of σ to a pair of opposite edges on ∂C. We refer to Figure 36 right for an example of a $[0,4] \times [0,5]$–quasi-rectangular subcomplex of Q_R. Each interior vertex of the quasi-rectangular region P is either a vertex of type O, or a good vertex.

Let $r \subset Q_R$ be a plain ray based at a vertex y. By Definition 5.4 and Table 1, each vertex in r that is not y and is not adjacent to y is good. Thus by passing to a sub-ray, we assume each vertex of r is good. By Lemma 9.3 (1), r is a CAT(0) geodesic ray in Q'. Let ℓ_r be a CAT(0) geodesic line orthogonal to r at y. Let H_r be the halfspace which is bounded by ℓ_r and contains r. Up to passing to a sub-ray of r, we assume H_r is sufficiently far away from the base point x_0 of Q' so that H_r is in the interior of Q_R.

The thickening T_r of r is defined to be the union of 2–cells of Q_R which contains an edge of r. We label these 2–cells and vertices of r as in Figure 36 left, namely, the 2–cells that contain the edge $\overline{y_{0,j-1}y_{0j}}$ are denoted by C_{0j} and C_{1j}. Suppose ∂C_{01} (resp. ∂C_{11}) is made of $2n_0$ (resp. $2n_1$) edges. Since each y_{0j} is a good vertex, ∂C_{ij} is made of $2n_i$ edges for $i = 0, 1$ and $j \geq 1$ and $C_{ij} \cap C_{i,j+1}$ is a path with $n_i - 1$ edges.

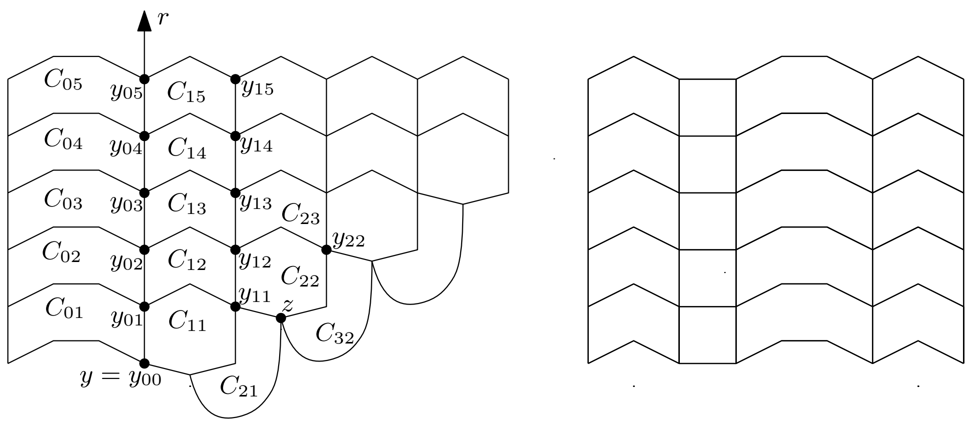


FIGURE 36.

Now we look at the vertex y_{11} in Figure 36, i.e. $y_{11} = \partial T_r \cap C_{12} \cap C_{11}$. Since y_{01} is good, by Lemma 9.3 (2), y_{01}, y_{11} and y_{00} span a triangle in Q' with a right-angle at y_{01}. Hence $y_{11} \in H_r$ and y_{11} satisfies Lemma 4.11 by our choice of H_r.

Lemma 9.4.

(1) If y_{11} is not good, then r bounds a diamond-plain sector (see Table 2 on page 41) on the right side of r.

(2) If y_{11} is good, then y_{1i} is good for all $i \geq 1$.

The reader might find Table 1 helpful for the following proof.

Proof. We prove (1). Since $C_{11} \cap C_{12}$ has $n_1 - 1$ edges, y_{11} is not a $\triangle$–vertex. Since y_{11} is not good, it is of type I and $n_1 \geq 3$. Let C_{22} and C_{21} be other two 2–cells containing y_{11}. For $j \geq 2$, let C_{2j} be the 2–cell containing $\overline{y_{1j}y_{1,j-1}}$ such that $C_{2j} \neq C_{1j}$. Note that $C_{12} \cap C_{22} = \overline{y_{11}y_{12}}$ (since other edges of ∂C_{12} are also contained in either C_{13}, or C_{11}, or C_{02}). By Lemma 4.5 (2) and (3), either y_{11} is a tip of both C_{11} and C_{22}, or y_{11} is a tip of both C_{12} and C_{21}. If the latter case is true, then by Lemma 4.5 (4), $C_{11} \cap C_{21}$ has one edge (since $C_{11} \cap C_{12}$ has $n_1 - 1$ edges) and $C_{22} \cap C_{21}$ has $n_1 - 1$ edges (since $C_{12} \cap C_{22}$ has one edge), which implies y_{11} is good and leads to a contradiction. Thus the former case holds.

Since C_{12}, C_{22} and C_{13} are in the same block, y_{12} is of type I. We deduce from $C_{12} \cap C_{22}$ is an edge and Lemma 4.5 (4) that y_{12} is good. Moreover, since y_{12} is not a tip of C_{22}, y_{12} is a tip of both C_{12} and C_{23}. By repeating this argument (note that $y_{1j} \in H_r \subset Q_R$), we deduce that for each $j \geq 2$, y_{1j} is a good vertex of type I and y_{1j} is a tip of C_{1j} and $C_{2,i+1}$.

Suppose $z \in \partial C_{22}$ is the endpoint of $C_{21} \cap C_{22}$ such that $z \neq y_{11}$. It is clear that $z \in H_r$, hence z satisfies Lemma 4.11. We claim z is of type I. Since C_{21} and C_{22} are in the same block, z is either type I or type II (note that $z \in Q_R$). However, since y_{11} is not good, $C_{22} \cap C_{21}$ has $< n_1 - 1$ edges, which contradicts Lemma 4.6 (3). Thus z is of type I.

Let y_{22} be the end point of $C_{22} \cap C_{23}$ with $y_{22} \neq y_{12}$. Since both y_{01} and y_{02} are good, we deduce from the relation between Q' and Q_R that the convex hull of y_{01}, y_{02}, y_{11} and y_{22} in Q' is isometric to a flat rectangle. Since y_{12} is good, by Lemma 9.3 (2), y_{12}, y_{11} and y_{22} span a flat triangle in Q' with a right angle at y_{12}. Hence $y_{22} \in H_r$ and y_{22} satisfies Lemma 4.11.

Since y_{11} is a tip of C_{22}, z is not a tip of C_{22}. Thus z is a tip of both C_{21} and C_{32}. By Lemma 4.5 (4), $C_{22} \cap (C_{21} \cup C_{32})$ is a half of C_{22}. Thus $C_{22} \cap C_{32}$ is a path from z to y_{22}. We deduce that y_{22} is not good and is of type I, and we can apply the previous discussion to y_{22}.

The above argument enable us to build inductively a diamond ray $L = \cup_{i=1}^{\infty} C_{ii}$ and a sector between r and L. Moreover, for each $i \geq 1$, $\cup_{j=1}^{\infty} C_{j,j+i}$ is a diamond ray. Since $y_{i,i+k}$ is good for each $i \geq 0$ and $k \geq 1$, one readily verify that this sector is a diamond-plain sector.

(2) already follows from the proof of (1). □

By Lemma 9.3, the vertices $\{y_{ii}\}_{i=0}^{\infty}$ are contained in a CAT(0) geodesic ray L' emanating from y_{00}, and the convex hull of r and L' in Q' is a flat sector with angle $\frac{n_1-1}{2n_1}\pi \geq \pi/3$. This flat sector is at finite Hausdorff distance from the diamond-plain sector of Q_R produced above in Lemma 9.4.

It follows from Lemma 9.4 that exactly one of the following holds:

(a) r bounds a $[0,\infty) \times [0,\infty)$–quasi-rectangular region P on its right side with $r = \{0\} \times [0,\infty)$;

(b) there are a $[0,n] \times [0,\infty)$–quasi-rectangular region P_1 and a diamond-plain sector P_2 such that r bounds P_1 on its right side with $r = \{0\} \times [0,\infty)$ and $P_1 \cap P_2 = \{n\} \times [0,\infty)$.

Now we look at case (a) in more detail. We identify P with $[0,\infty) \times [0,\infty)$ via $h\colon [0,\infty) \times [0,\infty) \to P$ (note that not all vertices of P have integer x–coordinates). Let ℓ' be a CAT(0) geodesic line in Q' that contains r, and let H_P be the half-space bounded by ℓ' with $P \subset H_P$. By possibly replacing $[0,\infty) \times [0,\infty)$ by $[m,\infty) \times [0,\infty)$, we assume H_P is sufficiently far away from the base point x_0 of Q' such that H_P is contained in the interior of Q_R. There are three cases to consider.

Case 1: Suppose each vertex of P in $[0,\infty) \times \{0\}$ is either of type O, or a good vertex. If P is made of squares, then it is a plain sector. Now we assume at least one 2–cell of P is not a square. For each integer $k \geq 0$, let

$w_k = \{k\} \times \{0\}$. Let Π_k and Π'_k be the two 2–cells in Q_R that contain w_k and are not contained in P. There exists an integer k_0 such that the vertex w_{k_0} is contained in a cell Π of P with ≥ 6 edges. Since w_{k_0} is of type II, $\Pi_{k_0} \cap \Pi'_{k_0}$ is one edge. Now we can prove inductively that $\Pi_k \cap \Pi'_k$ is one edge for all $k \geq 0$. Thus we can enlarge P to be a $[0,\infty) \times [-1,\infty)$–quasi-rectangular region. If we can repeat this process for infinitely many times to obtain a $[0,\infty) \times (-\infty,\infty)$–quasi-rectangular region P', then at least one of the following cases hold:

- a sub-region of P' of form $[m,\infty) \times (-\infty,\infty)$ is made of squares, in which case P' contains a plain sector on the right side of r;
- P' is a PCH as in Definition 5.16 (see Table 2 on page 41), in which case P' is Hausdorff close to a half-flat in Q' by Lemma 9.3;
- there exists $m \geq 0$ such that the subcomplex $P'' \subset P'$ corresponding to $[m,\infty) \times (-\infty,\infty)$ is contained in a block. Since P'' is quasi-rectangular, it is actually contained in a diamond-plain flat. Consequently P'' contains diamond-plain sector on the right side of r.

Case 2: Suppose there is a vertex of P in $[0,\infty)\times\{0\}$, say $y_{00} = \{0\}\times\{0\}$, of type I but not good. Then $n_1 \geq 3$ (recall that C_{11} has $2n_1$ edges). Let C_{00}, C_{10} be 2–cells containing y_{00} other than C_{01} and C_{11}.

If y_{00} is a tip of both C_{00} and C_{11} (we refer to Figure 37 right) then, by Lemma 4.5, $C_{01} \cap (C_{00} \cup C_{11})$ is a half of ∂C_{01}. Thus $C_{01} \cap C_{00}$ has $n_1 - 1$ edges. Since y_{00} is not good, $C_{10} \cap C_{11}$ has $< n_1 - 1$ edges. Let z be the end point of $C_{10} \cap C_{11}$ other than y_{00}. Then z is an interior vertex of the path from y_{00} to y_{10}. Since C_{10} and C_{11} are in the same block, z is either of type I or of type II. In the former case, since z is not a tip of C_{11}, z is a tip of C_{10} and C_{20}. Then $C_{11} \cap (C_{10} \cup C_{20})$ is a half of ∂C_{11}. Hence y_{10} is an interior point of the interval $C_{11} \cap C_{20}$, which contradicts that Q_R is a manifold around y_{10}. In the later case, we still have $C_{11} \cap (C_{10} \cup C_{20})$ is a half of ∂C_{11}, which can be deduced from Lemma 4.6 (2) and (3). Thus we reach a contradiction as before.

Thus y_{00} is a tip of both C_{01} and C_{10} (we refer to Figure 37 left). In this case, we argue as in Lemma 9.4 to produce a diamond ray $\cup_{i=0}^{\infty} C_{i,1-i}$ such that r and this diamond ray bound a diamond-plain sector S_1. Moreover, S_1 contains a flat sector $S'_1 \subset Q'$ with angle $= \frac{n_1+1}{2n_1}\pi$ (recall that $2n_1$ is the number of edges in ∂C_{11}).

Case 3: Suppose there is a vertex v of P in $[0,\infty) \times \{0\}$ such that v is a $\triangle$–vertex. In the light of Case 1 and Case 2, this is the only possibility left for discussion. Let x_v be the x–coordinate of v and let $\{v_i\}_{i=0}^{\infty}$ be consecutive vertices of P on $[x_v,\infty) \times \{0\}$ with $v_0 = v$.

If each v_i is of type III, then $[x_v,\infty) \times \{0\}$ or its appropriate sub-ray can be realized as part of the outer border of some half Coxeter region of Q_R. Then by Lemma 7.3, there is a sub-ray r' of $[x_v,\infty) \times \{0\}$ such that r' is the boundary of a Coxeter ray. We deduce from P being quasi-rectangular

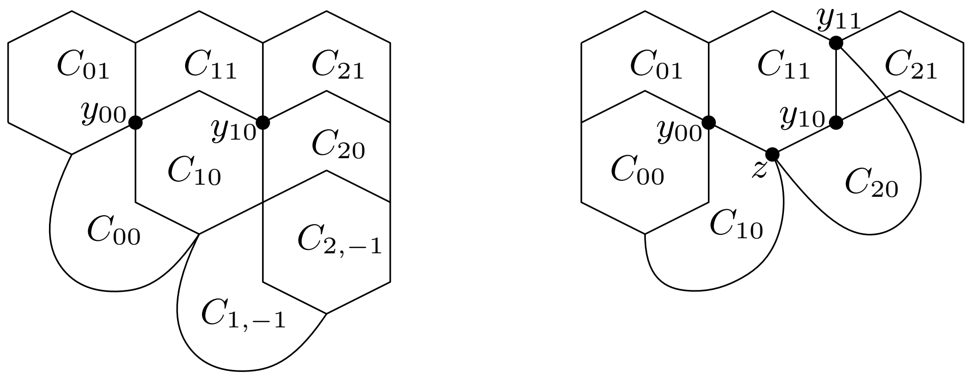


FIGURE 37.

that $r' \times [0, \infty)$ is a Coxeter-plain sector, moreover, by Lemma 9.3, P is Hausdorff close to a flat sector of angle $\pi/2$ in Q'.

Now we assume at least one of $\{v_i\}$ is not of type III and assume without loss of generality that v_1 is the first vertex which is not of type III. Let Π_1, Π_2 be the two 2–cells containing $\overline{v_0 v_1}$ with $\Pi_1 \subset P$. Since v_0 is of type III, Π_1 and Π_2 are in different blocks. Thus v_1 is either a $\triangle$–vertex with the same support as v_0 (Lemma 4.13), which can be ruled out by our assumption, or v_1 is of type II. Let Π_3, Π_4 be another two 2–cells such that $\Pi_1 \cap \Pi_3$ contains at least one edge. Suppose Π_i has $2m_i$ edges. Since $\Pi_1 \cap \Pi_2 = \overline{v_0 v_1}$ is one edge, by Lemma 4.6, $\Pi_3 \cap \Pi_1$ has $m_1 - 1$ edges. We deduce that $\Pi_1 \cap (\Pi_2 \cup \Pi_3)$ has m_1 edges. Since $\Pi_1 \cap ([0, \infty) \times \{0\})$ has $m_1 - 1$ edges, $\Pi_1 \cap \Pi_3$ has at least one edge which is not contained in $[0, \infty) \times \{0\}$. Hence $\Pi_3 \subset P$. Moreover, $\Pi_1 \cap \Pi_3$ is an edge since P is quasi-rectangular. Thus $m_i = 2$ and both Π_1 and Π_3 are squares.

We claim either all v_i with $i \geq 2$ are of type II, or there exists v_{i_0} with $i_0 \geq 2$ such that v_{i_0} is a $\triangle$–vertex with the same support as v_0 and v_i is of type II for each $1 \leq i < i_0$. To see this, note that v_2 is contained in Π_3 and Π_4 which are in different blocks, so v_2 is either of type II or type III. If v_2 is of type II, let Π_5 and Π_6 be another two 2–cells of Q_R containing v_2. Then Π_5 is a square in P and $|\partial \Pi_6| = |\partial \Pi_4| = |\partial \Pi_2|$ ($|\partial \Pi_i|$ denotes the number of edges in $\partial \Pi_i$). If v_2 is of type III, since Π_4 and Π_2 (and Π_3 and Π_1) are in the same block, v_2 is a $\triangle$–vertex with the same support as v_0. We now deduce the claim by repeating this argument.

If v_i is of type II for each $i \geq 2$, then there is a sub-region $P_1 \subset P$ of form $[m, \infty) \times [0, \infty)$ for some m such that each cell in P_1 is a square. Thus P_1 is a plain sector and it is isometric to a right-angled flat sector.

It remains to consider the case where v_{i_0} in the above claim exists. We claim for all $i \geq i_0$, v_i is a $\triangle$–vertex with the same support as v_{i_0}. Given this claim, one can find as before a sub-ray r'' of $[x_v, \infty) \times \{0\}$ such that $r'' \times [0, \infty)$ is a Coxeter-plain sector. We argue by contradiction and assume there is v_{i_1} such that v_{i_1} is not of type III and v_i is of type III for $i_0 \leq i < i_1$.

We deduce as before that there exist two squares in P containing v_{i_1}. The same holds for v_{i_0-1}. Then v_{i_0-1} and v_{i_1} give rise to two right-angled corners of a Coxeter region, and by the proof of Lemma 7.6, this is impossible.

9.2. **Development of Coxeter rays.** Let r be a Coxeter ray. Then r is quasi-rectangular. Hence we identify r with $[0,\infty)\times[0,1]$. Let v be the vertex of r with coordinates $(1,0)$ and let $\{v_i\}_{i=0}^{\infty}$ be consecutive vertices of r on $[1,\infty)\times\{0\}$ with $v_0=v$.

Lemma 9.5.

1. *If v is of type III, then each vertex of r on $[1,\infty)\times\{0\}$ is of type III.*
2. *If v is of type II, then each vertex of r on $[1,\infty)\times\{0\}$ is either of type O or of type II.*

Proof. For (1) we use induction. Suppose v_i is of type III. Let Π_1 and Π_2 be the two 2–cells of Q_R containing $\overline{v_iv_{i+1}}$ with $\Pi_1\subset r$. Since Π_1 and Π_2 are in different blocks, v_{i+1} is of type II or type III. If v_{i+1} is of type II, let Π_3 be the cell containing v_{i+1} such that $\Pi_3\cap\Pi_1$ has at least one edge and $\Pi_3\neq\Pi_2$. We argue as in Section 9.1 to deduce that $\Pi_3\subset r$ and that Π_1 and Π_3 are squares which intersect along an edge. However, this is impossible by the definition of Coxeter ray. Thus v_{i+1} is of type III. (2) can be proved by using Lemma 4.6 □

Note that the two 2–cells in r containing v_0 are in different blocks, thus v_0 is either of type II or type III. By Lemma 9.5, we can assume (up to passing to a sub-ray of r) that either all vertices of r on $[0,\infty)\times\{0\}$ are of type III, or all vertices with coordinate $(n,0)$ with $n\geq 0$ are of type II. We deduce from Lemma 4.6 and Lemma 4.8 that in either cases, each vertical edge of r (i.e. edge of r of form $\{n\}\times[0,1]$ with $n\geq 0$) is also an edge of Q' and different vertical edges are the opposite sides of a flat rectangle in Q'. Thus there is a CAT(0) geodesic ray $\gamma\subset Q'$ such that it is based at $(0,1/2)$ and intersects each vertical edge of r orthogonally at its midpoint.

We think of γ as being oriented from its base point to infinity and suppose $[0,\infty)\times\{0\}$ (resp. $[0,\infty)\times\{1\}$) is on the right (resp. left) side of γ with respect to the orientation of γ and Q'. $[0,\infty)\times\{0\}$ (resp. $[0,\infty)\times\{0\}$) is called the *right border* (resp. *left border*) of r.

Let $\ell_r\subset Q'$ be a CAT(0) geodesic line containing γ and let $H'_r\subset Q'$ be the half-space bounded by ℓ_r on the right side of γ. By repeatedly applying Lemma 9.6 below, we can replace r by an appropriate Coxeter ray such that H'_r is sufficient far away from the base point x_0 of Q' so that H'_r and its carrier in Q_R are contained in the interior of Q_R.

Lemma 9.6. *Suppose r is a Coxeter ray in Q_R with its associated CAT(0) geodesic ray γ. There there is a Coxeter ray $r_1\subset Q_R$ on the right side of r such that its associated CAT(0) geodesic γ_1 satisfies that a sub-ray of γ_1 and a sub-ray of γ bound a flat strip in Q' isometric to $[0,\infty)\times[0,1]$.*

Proof. If Lemma 9.5 (2) holds, then there is another Coxeter ray r_1 such that $r_1 \cap r = [0, \infty) \times \{0\}$. We define γ_1 to be the CAT(0) geodesic ray associated with r_1. If Lemma 9.5 (1) holds, then there is a half Coxeter region $\mathcal{H}' \subset Q'$ such that a sub-ray of γ' is a real border of $\mathcal{H}'$. As in the first paragraph of the proof of Lemma 7.3, there exists a wall $\mathcal{W} \subset \mathcal{H}'$ such that $\mathcal{W} \cap \gamma' = \emptyset$ and $\mathcal{W}$ is an infinite ray parallel to γ'. We define r_1 to be the carrier (in Q_R) of an appropriate sub-ray of $\mathcal{W}$. Then r_1 is a Coxeter ray by Lemma 7.2. □

Let $v = (1, 0) \in r = [0, \infty) \times [0, 1]$ be as before. We now consider the case when v is of type III in more detail. Let $\mathcal{H}'_r \subset Q'$ be the (v, ℓ_r)–half Coxeter region and let $\mathcal{H}_r \subset Q_R$ be the associated half Coxeter region (see Section 6). Note that $\mathcal{H}'_r$ contains a sub-ray of γ by Lemma 9.5 (2).

Let K be a chamber of $\mathcal{H}'_r$. The angle between a real border and a fake border of $\mathcal{H}'_r$ must be a multiple of an (internal) angle of K, and the angle between two real borders of $\mathcal{H}'_r$ is equal to an angle of K (Corollary 6.2 (1)). Since a sub-ray of γ is contained in the real border of $\mathcal{H}'_r$, a computation of angles yields that exactly one of the following possibilities holds for $\mathcal{H}'_r$:

(a) $\mathcal{H}'_r = H'_r$;
(b) $\mathcal{H}'_r$ has two parallel unbounded borders, one real and one fake, and one bounded border, in this case $d_H(\mathcal{H}'_r, \gamma) < \infty$;
(c) $\mathcal{H}'_r$ is a flat strip isometric to $(-\infty, \infty) \times [0, s]$;
(d) $\mathcal{H}'_r$ has two unbounded borders, one real and one fake, and they are not parallel, in this case $\mathcal{H}'_r$ has at most one bounded border and has finite Hausdorff distance from a flat sector in $\mathcal{H}'_r$.

Lemma 9.7.

(1) If case (d) holds, then Theorem 9.1 holds with S_1 being a Coxeter sector.
(2) If case (a) holds, then Theorem 9.1 holds with S_1 being a Coxeter sector or a CCH of type II.

Proof. We first verify assertion (1). Recall that each wall of $\mathcal{H}'_r$ has an orientation by Lemma 7.2. A wall $\mathcal{W}$ of $\mathcal{H}'_r$ with $\mathcal{W} \nsubseteq \ell_r$ is *positively oriented* if the half-space $H_{\mathcal{W}}$ induced by the orientation of $\mathcal{W}$ satisfies that $H_{\mathcal{W}}$ contains γ up to a finite segment. Recall that $\mathcal{H}'_r$ has a simplicial structure where each 2–simplex is a chamber of $\mathcal{H}'_r$. Let α be the value of the smallest angle of a chamber of $\mathcal{H}'_r$. A vertex u of $\mathcal{H}'_r$ is *special* if u is contained in a chamber whose angle at u is α.

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two borders of $\mathcal{H}'_r$ with $\mathcal{B}_1$ fake (hence $\mathcal{B}_1 \subset \ell_r$). Let $\{\mathcal{W}_i\}_{i=1}^{\infty}$ be parallel walls in $\mathcal{H}'_r$ such that $\mathcal{W}_1 = \mathcal{B}_1$ and $\mathcal{W}_i$ and $\mathcal{W}_{i+1}$ are not separated by other walls of $\mathcal{H}'_r$ for each $i \geq 1$. Note that for each i, $\mathcal{W}_i$ has a sub-ray $\mathcal{W}'_i$ such that the carrier of $\mathcal{W}'_i$ (in Q_R) is a Coxeter ray. This is clear when $i = 1$ since r is a Coxeter ray. The $i > 1$ case follows from Lemma 7.2. We assume without loss of generality that the starting point w_i of each $\mathcal{W}'_i$ is a special vertex of $\mathcal{H}'_r$ and that each wall $\mathcal{W}$ of $\mathcal{H}'_r$ that intersects $\mathcal{W}_i$ transversely is positively oriented. By Lemma 7.3, the carrier (in Q_R) of a sub-ray of $\mathcal{B}_2$ is a Coxeter ray. Hence there exists $i_0 \geq 1$ such that $\mathcal{W}_i$ and $\mathcal{W}_j$ are oriented towards the same direction for any $i, j \geq i_0$.

Suppose $\mathcal{W}_i$ is negatively oriented for $i \geq i_0$. Let $\mathcal{S}'_i$ be the flat sector with angle α in $\mathcal{H}'_r$ based at w_i with one side of $\mathcal{S}'_i$ being $\mathcal{W}'_i$ (such $\mathcal{S}'_i$ always exists and is a convex subcomplex of $\mathcal{H}'_r$). One readily verifies that for each $i \geq i_0$, the carrier of $\mathcal{S}'_i$ in Q_R is a Coxeter sector. We define S'_1 in Theorem 9.1 to be any $\mathcal{S}'_i$ with $i \geq i_0$.

It remains to consider the case where each $\mathcal{W}_i$ for $i \geq i_0$ is positively oriented. Though we can still consider the carrier of the sector $\mathcal{S}'_i$ as before, one boundary ray of this carrier does not have the correct orientation in order for it to be a Coxeter ray. So we will consider a different sector as below.

We first claim the Tits angle (cf. [BH99, Definition II.9.4]) between $\mathcal{B}_2$ and $\mathcal{B}_1$ is $\pi - \alpha$. Suppose the claim is not true (i.e. the Tits angle is $< \pi - \alpha$). Then there exists a special vertex $u \in \mathcal{B}_2$ of form $u = \mathcal{W}_{i_1} \cap \mathcal{B}_2$ for some $i \geq i_0$ such that $\mathcal{W}_u \cap \mathcal{W}'_1 \neq \emptyset$ where $\mathcal{W}_u$ is the wall of $\mathcal{H}'_r$ containing u and intersecting $\mathcal{B}_1$ in an angle $= \alpha$ (this can always be arranged if one chooses i_1 sufficiently large). We refer to Figure 38. Our choice implies that both

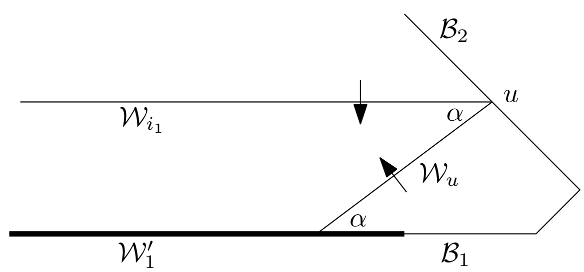


FIGURE 38.

$\mathcal{W}_u$ and $\mathcal{W}_{i_1}$ are positively oriented. Let C_u be the 2–cell in $\mathcal{H}_r$ dual to u. Then there is an orientation reserving vertex $y \in \partial C_u$ contained in the sector of $\mathcal{H}'_r$ bounded by $\mathcal{W}_{i_1}$ and $\mathcal{W}_u$. By Lemma 7.1, y is contained in an edge of C_u that is dual to $\mathcal{B}_2$. This yields a contradiction since $\mathcal{W}_{i_1}, \mathcal{W}_u$ and $\mathcal{B}_2$ are pairwise distinct walls.

Let $\mathcal{S}'_i$ be the flat sector with angle $\pi - \alpha$ in $\mathcal{H}'_r$ based at w_i such that one boundary ray of $\mathcal{S}'_i$ is $\mathcal{W}'_i$. The existence of $\mathcal{S}'_i$ follows from the previous claim. By the argument in the previous paragraph, we can find orientation reversing vertex on the boundary of each 2–cell C of $\mathcal{H}_r$ dual to a special vertex in $\mathcal{S}'_i$ ($i \geq i_0$), hence the orientation of each edge of C can be determined. Now one readily verify that the carrier of $\mathcal{S}'_i$ in $\mathcal{H}_r$ is a Coxeter sector and conclude the proof as before.

It remains to prove assertion (2). Let $\mathfrak{W}$ be the collection of walls in $\mathcal{H}'_r$ intersecting $\mathcal{W}_1$ transversally. Let $\mathfrak{W} = \sqcup_{k=1}^m \mathfrak{W}_k$ be the decomposition into parallel classes.

Suppose there exists k such that there is a pair of walls $\mathcal{W}, \mathcal{W}'$ in $\mathfrak{W}_k$ with opposite orientation. By the argument in Proposition 7.5, we know that $\mathcal{W}$ has a sub-ray whose carrier in Q_R is a Coxeter ray. Thus there exists $i_0 \geq 0$ such that all $\mathcal{W}_i$ are oriented towards the same direction for $i \geq i_0$, then we argue as before to produce a Coxeter sector.

Suppose for each k, each pair of walls in $\mathfrak{W}_k$ are oriented towards the same direction. Recall that $\mathcal{W}_1$ contains a ray $\mathcal{W}'_1$ whose carrier in Q_R is a Coxeter ray. Since each $\mathfrak{W}_k$ contains at least one wall that intersects $\mathcal{W}'_1$, either all walls in $\mathfrak{W}$ are positively oriented, or they are all negatively oriented. Hence $\mathcal{W}_1$ is a Coxeter line. If there exists $i_0 \geq 0$ such that all $\mathcal{W}_i$ are oriented towards the same direction for $i \geq i_0$, then we argue as before. If such i_0 does not exist, then $\mathcal{H}_r$ is a CCH of type II. □

The angle α defined in the proof of Lemma 9.7 satisfies $\alpha \geq \pi/6$. Hence the associated CAT(0) sector has angle $\geq \pi/6$.

Now we are ready to prove Theorem 9.1 in the case when r is a Coxeter ray. Starting from $r \cong [0,\infty) \times [0,1]$. If Lemma 9.5 (2) holds, then there is another Coxeter ray r_1 such that $r_1 \cup r$ is a quasi-rectangular region of Q_R, which we identify with $[0,\infty) \times [-1,1]$. By Lemma 9.3 (2), r_1 is contained in the half-space H'_r defined before Lemma 9.6. Hence Lemma 4.11 holds for vertices of r_1 and we can apply Lemma 9.5 to $[0,\infty) \times \{-1\}$. Then either we are able to repeat this argument for infinitely many times to produce a Coxeter-plain sector with one of its boundary ray being r, or after finitely many steps we find a vertex of type III on the boundary $[0,\infty) \times \{-m\}$ of the quasi-rectangular region we produced. In the latter case we assume without loss of generality that $m = 0$, which leads to (a), (b), (c) and (d) discussed before. By Lemma 9.7, it remains to consider cases (b) and (c).

Suppose (b) or (c) holds for $\mathcal{H}'_r$. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two parallel unbounded borders of $\mathcal{H}'_r$ such that $\mathcal{B}_2$ is not real. By Lemma 7.3, there exists a sub-ray of $\mathcal{B}_2$ such that its carrier r_2 in Q_R is a Coxeter ray and $d_H(r, r_2) < \infty$. By the previous discussion, either r_2 is the boundary ray of a Coxeter-plain sector, or we find a Coxeter ray r_3 on the right side of r_2 such that r_3 has a vertex of type III on its right border. Note that $r_2 \neq r_3$ by our choice of r_2. Moreover, the left border of r_3 satisfies Lemma 9.5 (2). Now we define $\mathcal{H}'_{r_3}$, H'_{r_3}, γ_3 and ℓ_{r_3} as before. Let $\mathcal{C}'_{r_3}$ be the Coxeter region containing $\mathcal{H}'_{r_3}$. Since the left border of r_3 satisfies Lemma 9.5 (2), the intersection of a sufficiently small ball around each interior point of γ_3 with $\mathcal{C}'_{r_3}$ satisfies Corollary 6.2 (2). Now we consider the connected component P of $\partial\mathcal{C}'_{r_3}$ that contains γ_3. By Corollary 6.2 and an induction argument, we know either $P = \ell_{r_3}$, or P travels into the interior of H'_{r_3} at some point and remains there. Thus $\mathcal{C}'_{r_3} \subset H'_{r_3}$. It follows that $\mathcal{C}'_{r_3} = \mathcal{H}'_{r_3}$. We claim $\mathcal{H}'_{r_3}$ can not satisfy (b). By $\mathcal{C}'_{r_3} = \mathcal{H}'_{r_3}$ and Corollary 6.2 (1), the only way $\mathcal{H}'_{r_3}$ can satisfy (b) is that the bounded border of $\mathcal{H}'_{r_3}$ is orthogonal to both of its unbounded borders, however, this can be ruled out by Lemma 7.6.

From the above discussion, we deduce that at least one of following holds:

(1) there is a Coxeter sector or a Coxeter-plain sector S on the right side of r such that one boundary ray r' of S satisfies $d_H(r,r')<\infty$;
(2) there is a Coxeter line L on the right side of r such that L coarsely contains r and L bounds a CCH of type II;
(3) there are subcomplexes $\{U_i\}_{i=1}^{\infty}$ of Q_R such that
    (a) each U_i coarsely contains r and is on the right side of r;
    (b) each U_i is either a $(-\infty,\infty)\times[0,m_i]$–quasi-rectangular region ($m_i>0$) or a Coxeter region such that the associated Coxeter region U'_i in Q' is isometric to a flat strip $(-\infty,\infty)\times[0,m_i]$; moreover, there does not exist i such that both U_i and U_{i+1} are quasi-rectangular;
    (c) $U_i\cap U_{i+1}$ is a boundary line of both U_i and U_{i+1}, and each vertex in $U_i\cap U_{i+1}$ is either of type O or of type II;
    (d) there does not exist i_0 such that all U_i are quasi-rectangular regions for $i\geq i_0$ (otherwise we will be in case (1) with S being a Coxeter-plain region).

To finish the proof of Theorem 9.1 in the Coxeter ray case, it remains to show $\cup_{i=1}^{\infty}U_i$ is a CCH of type I in case (3). Assume without loss of generality that U_1 is a Coxeter region. It suffices to prove there are no orientation reversing vertices on $U_1\cap U_2$, since one can deduce from this, Lemma 7.2 and Remark 4.7 that there are no orientation reversing vertices on $U_i\cap U_{i+1}$ for each i, which implies the claim. By (3b) and (3c), $U_i\cap U_{i+1}$ is the outer border of a Coxeter region for each i. Hence by Lemma 7.4, there is at most one orientation reversing vertex on $U_i\cap U_{i+1}$. Suppose there is an orientation reversing vertex $v\in U_1\cap U_2$. We first look at the case U_2 is quasi-rectangular. Then U_3 is a Coxeter region and there is exactly one orientation reversing vertex in $U_2\cap U_3$ by Remark 4.7. Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be two parallel boundary walls of the Coxeter region $U'_3\subset Q'$ associated with U_3. Suppose $2n$ is the largest possible number of edges on the boundary of a 2–cell in U_3. Choose a vertex $v\in\mathcal{W}_1$ such that it is dual to a 2–cell with $2n$ edges in U_3. Consider the zig-zag line ℓ_v containing v as in Figure 39 such that ℓ_v is a union of straight line segments from a point in $\mathcal{W}_1$ to a point in $\mathcal{W}_2$ such that each segment has angle π/n with $\mathcal{W}_1$. We deduce that each segment of ℓ_v is contained in a wall of U'_3. Let $\mathfrak{W}$ be the collection of walls which contain a segment of ℓ_v. We assign a total order on $\mathfrak{W}$ by looking at

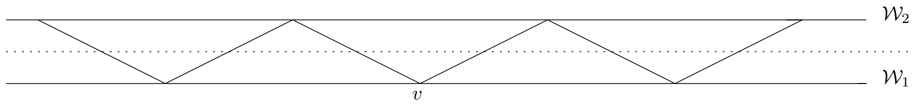


FIGURE 39.

intersection points of elements in $\mathfrak{W}$ along a straight line ℓ (see the dotted line in Figure 39) between $\mathcal{W}_1$ and $\mathcal{W}_2$. Since there is exactly one orientation reversing vertex in $U_2\cap U_3$, there is exactly one pair of walls $\mathcal{W},\mathcal{W}'$ in $\mathfrak{W}$

which are adjacent with respect to the total order such that one is oriented towards left and one is oriented towards right. Note that $u = \mathcal{W} \cap \mathcal{W}'$ is either in $\mathcal{W}_1$ or $\mathcal{W}_2$. Let C_u be the 2–cell in U_3 dual to u. By Lemma 7.1, there are no orientation reversing vertices in $\partial C_u \cap \partial U_3$. Thus either both $\mathcal{W}$ and $\mathcal{W}'$ are oriented towards left, or they are oriented towards right. This leads to a contradiction. The case U_2 is a Coxeter region is similar.

Each CCH in Q_R contains a CAT(0) half-flat of Q' which is Hausdorff close this CCH. To see this, note that a CCH has infinitely many Coxeter lines, each of them contains a CAT(0) geodesic line of Q' by a similar argument as before. A pair of such CAT(0) geodesic lines bounds a flat strip, and the union of these flat strips gives the half-flat as required.

9.3. **Development of diamond rays.** Let $r = \cup_{i=0}^{\infty} C_i$ be a diamond ray in Q_R with C_i satisfying Definition 5.1. There is a CAT(0) geodesic ray $\gamma \subset r$ defined as follows. For each C_i, let h_i be the CAT(0) geodesic segment in Q' connecting the two tips of C_i. By Lemma 4.5, Lemma 3.9 and Lemma 3.12, $\gamma = \cup_{i=1}^{\infty}$ is a CAT(0) geodesic ray in Q'. As before we think of γ as being oriented from its base vertex to infinity, and we talk about right side or left side of r and γ with respect to the orientation of γ and Q'.

Lemma 9.8. *Suppose r is contained in the interior of Q_R. Then there is a diamond ray r' such that $r \cap r'$ is contained in the right boundary of r.*

Proof. Let $v_i = C_i \cap C_{i+1}$ for $i \geq 1$. Since C_i and C_{i+1} are in the same block, but they only intersect along one vertex, we know from Lemma 4.5 and Lemma 4.6 that v_i is of type I (see Table 1 on page 33) for each i. Let C'_i be the 2–cell containing v_i on the right side of r. We claim $C'_i \cap C'_{i+1}$ is a tip of both C'_i and C'_{i+1} for $i \geq 0$. Then $r' = \cup_{i=0}^{\infty} C'_i$ is a diamond ray by this claim.

Now we prove the claim. By Lemma 4.5, $C'_i \cap (C_i \cup C_{i+1})$ is a half of $\partial C'_i$. Let v'_i be the endpoint of the path $C'_i \cap C_{i+1}$ such that $v'_i \neq v_i$. Then v'_i is a tip of C'_i by Lemma 3.1 (2). Since C'_i and C_{i+1} are in the same block, v'_i is either of type I or type II. Let C'''_{i+1} be the 2–cell different from C_{i+1}, C'_i such that $v'_i \in C'''_{i+1}$ and $C'''_{i+1} \cap C_{i+1}$ contains at least an edge. By Lemma 4.5 and Lemma 4.6, $C'''_{i+1} \cup C'_i$ is a point and $C_{i+1} \cap (C'''_{i+1} \cup C'_i)$ is a half of ∂C_{i+1}. Thus $v_{i+1} \in C'''_{i+1}$. It follows that $C'''_{i+1} = C'_{i+1}$ since v_{i+1} is of type I. Now we know C'''_{i+1}, C'_i and C_{i+1} are in the same block. Thus v'_i is of type I, and C'_{i+1} and C'_i both contain v'_i as a tip. Hence the claim follows. $\square$

In the above proof, by reading off the label of edges in the path $C_{i+1} \cap C'_i$ traveling from v_i to v'_i, we obtain a word, and this word does not depend on i. We denote this word and its length by $\delta_{r,r'}$ and $|\delta_{r,r'}|$. Let $2n$ be the number of edges in the boundary of C_0. Then $1 \leq |\delta_{r,r'}| \leq n-1$. Suppose $|\delta_{r,r'}| > 1$. Let v'_{-1} be the tip of C'_0 other than v'_0. Since C_0 and C'_0 are in the same block and $C_0 \cap C'_0$ has $n - |\delta_{r,r'}| < n-1$ edges, v'_{-1} can not be of type II by Lemma 4.6 (2) and (3). Thus v'_{-1} is of type I. Let C'_{-1} be the 2–cell such that $C'_{-1} \neq C'_0$ and C'_{-1} contains v'_{-1} as a tip. Then $\cup_{i=-1}^{\infty} C'_i$ is a diamond ray, which is defined to be the *right shift* of r. If $|\delta_{r,r'}| = 1$, then the *right shift* of r is defined to be $\cup_{i=0}^{\infty} C'_i$.

Let $\ell_r \subset Q'$ be a CAT(0) geodesic line that contains the geodesic ray γ associated with r and let H'_r be the half-space bounded by ℓ_r on the right side of ℓ_r. By applying finitely many right shifts to r, we assume without loss of generality that H'_r is sufficiently far away from the base point x_0 of Q' such that H'_r and its carrier in Q_R are contained in the interior of Q_R.

Lemma 9.9. *Let r' be a right shift of r. If $1 < |\delta_{r,r'}| < n$, then we can extend r and r' to diamond lines L and L' in Q_R respectively such that $L \cap L'$ is a boundary line of both L and L'.*

Proof. Let $r = r_0 = \cup_{i=0}^{\infty} C_i$, $r' = r'_{-1} = \cup_{i=-1}^{\infty} C'_i$. Let v_{-1} be the tip of C_0 with $v_{-1} \neq v_0$. Then v_{-1} is an endpoint of $C_0 \cap C'_{-1}$, and the number of edges in $C_0 \cap C'_{-1} = |\delta_{r,r'}|$ is $< n$. Thus v_{-1} is type I. Let $C_{-1} \subset Q_R$ be the 2–cell containing v_{-1} as a tip such that $C_{-1} \neq C_0$. Now we can enlarge r to a new diamond ray $\cup_{i=-1}^{\infty} C_i$. Note that $r_{-1} = \cup_{i=-1}^{\infty} C_i$ is contained in the interior of Q_R by our choice of H'_r. Note that $\delta_{r_{-1},r'_{-1}} = \delta_{r,r'} > 1$. Thus we can define the right shift $r'_{-2} = \cup_{i=-2}^{\infty} C'_i$ of r_{-1}, then enlarge r_{-1} to r_{-2} as before. The lemma follows by repeating this braiding process. □

Now we define a sequence of diamond rays $\{r_i\}_{i=0}^{\infty}$ such that $r_0 = r$, and r_i is the right shift of r_{i-1}. By our choice of H'_r, it is possible to define these diamond rays.

Lemma 9.10. *If there exists i_0 such that $|\delta_{r_{i_0-1},r_{i_0}}| = 1$ and $|\delta_{r_{i_0},r_{i_0+1}}| = n-1$, then there exists a diamond line L containing r_{i_0}.*

Proof. We assume without loss of generality $i_0 = 1$. Let $r_0 = \cup_{i=0}^{\infty} C_i$, $r_1 = \cup_{i=0}^{\infty} C'_i$ and $r_2 = \cup_{i=-1}^{\infty} C''_i$. Let $v'_0 \in C'_0$. Then v'_0 is an endpoint of both $C'_0 \cap C_0$ and $C'_0 \cap C''_{-1}$ by Lemma 3.1 (2). By $|\delta_{r_1,r_2}| = n-1$, we know $C'_0 \cap C''_{-1}$ has $n-1$ edges. By $|\delta_{r_0,r_1}| = 1$, we know $C_1 \cap C'_0$ has one edge, hence $C_0 \cap C'_0$ has $n-1$ edges. Now the lemma follows from the same argument in Lemma 8.1 (note that C_0, C'_0 and C''_{-1} play the role of C_1, Π_1 and C'_1 in Figure 35 respectively). □

Now we are ready to prove Theorem 9.1 in the case r is a diamond line. Note that this is the only case left to be verified.

If there exists i_0 such that $|\delta_{r_{i_0},r_{i_0+1}}| = n-1$ for all $i \geq i_0$, or $|\delta_{r_{i_0},r_{i_0+1}}| = 1$ for all $i \geq i_0$, then $\cup_{i=i_0}^{\infty} r_i$ is a diamond-plain sector.

We assume such i_0 does not exist. If there exists i_1 such that $1 < |\delta_{r_{i_1},r_{i_1+1}}| < n$, then we can extend r_{i_1} to a diamond line L_{i_1} by Lemma 9.9. Hence each r_i for $i \geq i_1$ can be extended to a diamond line L_i by the argument in Lemma 9.8. We deduce from the non-existence of i_0 that $\cup_{i=i_1}^{\infty} L_i$ is a DCH. If for all i, $|\delta_{r_{i_1},r_{i_1+1}}|$ is either 1 or $n-1$, then there exists i_2 such that the assumption of Lemma 9.10 is satisfied. So we can extend r_{i_2} to a diamond line L_{i_2} and deduce as before that L_{i_2} bounds a DCH. Note that each DCH in Q_R contains a CAT(0) half-flat of bounded Hausdorff distance from this DCH. Indeed, each L_i contains a CAT(0) geodesic line ℓ_i, and ℓ_i and ℓ_{i+1} bound a flat strip isometric to $[0, m_i] \times \mathbb{E}^1$, then the union of these flat strips gives the half-flat.

In Sections 9.1, 9.2, and 9.3 we treated all the cases for the singular ray r. Therefore, Theorem 9.1 follows.

10. Quasi-isometry invariants for 2–dimensional Artin groups

10.1. Preservation of completions of atomic sectors.

Definition 10.1. Let $\{S_j\}_{j=1}^n$ be a collection of subsets of X^*_Γ. We say $\{S_j\}_{j=1}^n$ *touch along discs* if there exist L, A and a sequence of (L, A)–quasi-isometric embeddings $f_i\colon D_i \to X^*_\Gamma$ such that each D_i is a disc in the Euclidean plane with radius $m_i \to \infty$ and $f(D_i) \subset \cap_{j=1}^n N_A(S_j)$.

Here X^*_Γ has a piecewise Euclidean structure inherited from X_Γ, which gives rise to a length metric on X^*_Γ that is quasi-isometric to A_Γ. We use $N_A(X)$ to denote the A neighborhood of X.

Lemma 10.2. *Let S_1 and S_2 be atomic sectors. If they touch along discs, then $d_H(S_1 \cap S_2, S_1) < \infty$ and $d_H(S_1 \cap S_2, S_2) < \infty$.*

Proof. By modifying the map f_i in Definition 10.1, there are L' and A' independent of i such that for each D_i, there are continuous (L', A')–quasi-isometric embeddings $g_i\colon D_i \to S_1$ and $h_i\colon D_i \to S_2$ such that for any $x \in D_i$, $d_{X^*_\Gamma}(g_i(x), h_i(x)) < A'$. By Lemma 5.21, we can assume g_i and h_i are also (L', A')–quasi-isometric embeddings with respect to the intrinsic metric on S_1 and S_2. For the convenience of later computation, we assume S_1 and S_2 are tilings of flat Euclidean sectors (this can always be arranged up to quasi-isometries). Let σ be a singular chain representing the fundamental class in $H_2(D_i, \partial D_i)$. Let $\alpha_i = (g_i)_*\sigma$ and $\beta_i = (h_i)_*\sigma$.

We claim there exists $M > 1$ independent of i such that $[\partial\alpha_i]$ is nontrivial in $H_1(S_1 \setminus B_{S_1}(g_i(c_i), \frac{n_i}{M}))$, where $B_{S_1}(g_i(c_i), \frac{n_i}{M})$ denotes the ball in S_1 centered at $g_i(c_i)$ and has radius $\frac{n_i}{M}$ with respect to the flat metric on S_1. To see the claim, we construct a continuous map $g_i^{-1}\colon \operatorname{Im} g_i \to D_i$ such that $g^{-1} \circ g$ and $g \circ g^{-1}$ are identity maps up to a bounded error which depends only on L' and A'. Hence $g^{-1} \circ g$ and $g \circ g^{-1}$ are homotopic to identity where the homotopy moves points by a distance $\le N = N(L', A')$. Now we assume without loss of generality that $n_i \gg L'$, $n_i \gg A'$ and $n_i \gg N$ for all i. Then the claim follows. It follows from the claim that the image of $[\alpha_i]$ is non-trivial under $H_2(S_1, S_1 \setminus B_{S_1}(g_i(c_i), \frac{n_i}{M})) \to H_2(S_1, S_1 \setminus \{p\}) \to H_2(X^*_\Gamma, X^*_\Gamma \setminus \{p\})$ (recall that S_1 is embedded in X^*_Γ by Lemma 5.19, hence the two maps here are both injective). Hence the support set $S_{[\alpha]}$ of the homology class $[\alpha] \in H_2(X^*_\Gamma, g_i(\partial D_i))$ satisfies $B_{S_1}(g_i(c_i), \frac{n_i}{M}) \subset S_{[\alpha]}$.

Since $d_{X^*_\Gamma}(g_i(x), h_i(x)) < A'$ for each $x \in \partial D_i$ and X^*_Γ is uniformly contractible, there exists a continuous map $F\colon \partial D_i \times [0, 1] \to X^*_\Gamma$ such that $F|_{\partial D_i\times\{0\}} = g_i|_{\partial D}$, $F|_{\partial D_i\times\{1\}} = h_i|_{\partial D}$ and $d_H(\operatorname{Im} F, g_i(\partial D_i)) < A_1$ for A_1 independent of i. Then F gives rise to a 2–chain γ_i such that $\partial\gamma_i = \partial\alpha_i - \partial\beta_i$ and $d_H(\operatorname{Im}\gamma_i, \operatorname{Im}\partial\alpha_i) < A_2$ for A_2 independent of i. Then $\partial(\gamma_i+\beta_i) = \partial\alpha_i$. Since $H_2(X^*_\Gamma)$ is trivial, there exists a 3–chain ρ such that $\partial\rho = \gamma_i + \beta_i - \alpha_i$. Thus $[\gamma_i+\beta_i] = [\alpha_i]$ in $H_2(X^*_\Gamma, g_i(\partial D_i))$ and $S_{[\gamma_i+\beta_i]} = S_{[\alpha_i]} \supset B_{S_1}(g_i(c_i), \frac{n_i}{M})$. By assuming $n_i \gg A_2$ and possibly enlarging M, we have $B_{S_1}(g_i(c_i), \frac{n_i}{M}) \cap \operatorname{Im}\gamma_i = \emptyset$. Thus $[\beta_i]$ is nontrivial in $H_2(X^*_\Gamma, X^*_\Gamma \setminus \{p\})$ for any $p \in B_{S_1}(g_i(c_i), \frac{n_i}{M})$. Hence $B_{S_1}(g_i(c_i), \frac{n_i}{M}) \subset \operatorname{Im}\beta_i \subset S_2$. Thus $S_1 \cap S_2$ contains larger and larger discs and the lemma follows from Lemma 5.20. □

Lemma 10.3. *Let S be an atomic sector and let $Q \subset X^*_\Gamma$ be a 2–dimensional quasiflat. Let $\{S_i\}_{i=1}^n$ and Q_R be as in Theorem 9.2. If S and Q touch along discs, then there exists an S_i such that $d_H(S\cap S_i, S) < \infty$ and $d_H(S\cap S_i, S_i) < \infty$. In particular, $S \subset N_A(Q)$ for some A.*

Proof. Let $q'\colon Q' \to X_\Gamma$ be the CAT(0) approximation of Q as in Theorem 2.7. Let D be the image of an (L,A)–quasi-isometric embedding from a disc of radius m to Q. Since q' is Lipschitz, $(q')^{-1}(D)$ contains a ball of radius $\frac{m}{M}$ in Q' with M depending on L, A. By Theorem 9.1 and Theorem 9.2, there are disjoint flat sectors $\{S'_i\}_{i=1}^n$ in Q' such that $d_H(Q', \cup_{i=1}^n S'_i) < \infty$ and $d_H(S_i, S'_i) < \infty$. Since Q' is a CAT(0) plane, for sufficiently large m, any ball of radius $\frac{m}{M}$ in Q' (with respect to the CAT(0) metric on Q) must contain a ball of radius $\frac{m}{M'}$ in one of the S'_i for some $M' > M$ (with respect to the CAT(0) metric on S'_i). By Lemma 5.21, the CAT(0) metric on S'_i and the metric on S_i induced from X^*_Γ are quasi-isometric. Let $m \to \infty$. By passing to a subsequence we know S and at least one of the S_i touch along discs. Then the lemma follows from Lemma 10.2. □

Lemma 10.4. *Let $\{Q_i\}_{i=1}^n$ be a finite collection of 2–dimensional quasiflats. If they touch along discs, then there exists an atomic sector S and $A > 0$ such that $S \subset \cap_{i=1}^n N_A(Q_i)$.*

Proof. By Theorem 9.2, we can assume each Q_i is a union of finitely many atomic sectors. By the proof of Lemma 10.3, we know that for each (L,A)–quasi-isometric embedding $D_m \to \cap_{i=1}^n N_A(Q_i)$ (where D_m denotes a disc of radius m in the Euclidean plane), there exist constants $L', A', M > 1$ independent of m, atomic sector S_i of Q_i for each i and an (L',A')–quasi-isometric embedding $D_{\frac{m}{M}} \to \cap_{i=1}^n N_{A'}(S_i)$. Since each Q_i has only finitely many atomic sectors, we can assume S_i does not change as $m \to \infty$ by passing to a subsequence. Thus the collection $\{S_i\}_{i=1}^n$ touch along discs and we are done by Lemma 10.2. □

Lemma 10.5. *Let S be an atomic sector and let Q be a 2–dimensional quasiflat. Let Q_R and $\{S_i\}_{i=1}^n$ be as in Theorem 9.2. If there exists $A > 0$ such that $S \subset N_A(Q)$, then the completion $\bar{S}$ of S satisfies that $d_H(\bar{S} \cap Q_R, \bar{S}) < \infty$. In particular, $\bar{S} \subset N_{A'}(Q)$ for some $A' > 0$.*

Proof. By Lemma 5.22, it suffices to consider the case when S is a diamond-plain sector or a plain sector. By Lemma 10.2, we can assume $S = S_1$.

First we assume S_1 is a diamond-plain sector. Suppose S_1 is bounded by a plain ray r_1 and a diamond ray r_1'. Then the discussion in Section 9.1 implies that we can assume S_2 has a boundary ray r_2 such that

(1) r_2 is a plain ray;
(2) r_1 and r_2 bound a (possibly degenerate) quasi-rectangular region P, in particular, $d_H(r_1, r_2) < \infty$.

Also we know from Section 9.1 that the possibilities for S_2 are diamond-plain sector, Coxeter-plain sector, plain sector and PCH. Let $a \in \Gamma$ be the label of edges in r_1 and $\overline{ab}$ be the defining edge of the block B that contains S. The interesting case is that every vertex in the odd component (cf. Section 5.4) containing a is a leaf. Thus either a is a leaf and $\overline{ab}$ is labeled by an even number > 2, or $\overline{ab}$ has odd label and $\overline{ab}$ is a connected component of Γ. In both cases we can deduce the following

(1) $P \subset B$;
(2) r_2 is single labeled by either a or b;
(3) S_2 is a diamond-plain sector in B.

Thus a sub-half-flat of $\bar{S}$ is contained in Q_R.

Now assume S_1 is a plain sector. Write $S_1 = r_1 \times r_1'$, where r_1 and r_1' are boundary rays of S_1. Let V_1 and V_1' be the collection of labels of edges in r_1 and r_1' respectively. First we consider the case when V_1 is good (in the sense of Section 5.4) and V_2 is not good. Since we always have $V_1 \subset V_2^{\perp}$, V_1 has to be a singleton. V_2 either satisfies $|V_2| \geq 2$ and $V_2^{\perp} = V_1$, or $|V_2| = 1$ and V_2 is a leaf. In the latter case we also have $V_2^{\perp} = V_1$.

Let S_2, r_2 and P be as before. We claim that if P is non-degenerate, then P is made of squares. Now we prove the claim. In the case $|V_2| \geq 2$, there exists a vertex $v \in r_1$ such that the two edges in r_1 containing v have different labels. Since the two squares of S_1 containing v are in different blocks, v is either of type II, or is a $\square$–vertex. By Lemma 4.6, the other two cells containing v are also squares. Now we walk along r_1 from v to other vertices to prove inductively that each vertex in r_1 which is not the endpoint of r_1 is contained in four squares. This shows that the first layer of P is made of squares and we can repeat this argument to show P is made of squares. In the case $|V_2| = 1$, if a vertex $v \in r_1$ is contained in a non-square cell, then V_2 is adjacent to another vertex in Γ via an edge whose label is not 2. This contradicts that $V_2^{\perp} = V_1$ and V_2 is a leaf, hence the first layer of P is made of squares. Now we conclude the proof as before.

The above argument also shows that S_2 is made of squares, hence S_2 is a plain sector. And $V_2^{\perp} = V_1$ determines what this plain sector is. Thus a sub-half-flat of $\bar{S}$ is contained in Q_R. The remaining case that both V_1 and V_1' are not good is similar. $\square$

Corollary 10.6. *Suppose that A_{Γ_1} and A_{Γ_2} are two-dimensional Artin groups. Let $q\colon X^*_{\Gamma_1} \to X^*_{\Gamma_2}$ be a quasi-isometry. Then for any atomic sector*

S_1 in $X^*_{\Gamma_1}$, there exists an atomic sector S_2 in $X^*_{\Gamma_2}$ such that $d_H(q(\bar{S}_1), \bar{S}_2) < \infty$, where $\bar{S}_i$ denotes the completion of S_i.

Proof. First we show for each S_1, there exists S_2 such that $\bar{S}_2 \subset N_{A_1}(q(\bar{S}_1))$ for some $A_1 > 0$. By Corollary 5.23, $\bar{S}_1$ is the coarse intersection of finitely many quasiflats $\{Q_i\}_{i=1}^n$. Thus $q(\bar{S}_1)$ is the coarse intersection of $\{q(Q_i)\}_{i=1}^n$. In particular, $\{q(Q_i)\}_{i=1}^n$ touch along discs. Lemma 10.4 implies there is an atomic sector $S_2 \subset \cap_{i=1}^n N_A(q(Q_i))$ for some A. By Lemma 10.5, $\bar{S}_2 \subset \cap_{i=1}^n N_{A'}(q(Q_i))$ for some A'. Thus $\bar{S}_2 \subset N_{A_1}(q(\bar{S}_1))$ for some A_1. Let q^{-1} be a quasi-isometric inverse of q. As before, we know there exists an atomic sector S_3 of $X^*_{\Gamma_1}$ such that $\bar{S}_3 \subset N_{A_2}(q^{-1}(\bar{S}_2))$. Thus $\bar{S}_3 \subset N_{A_3}(\bar{S}_1)$ for some $A_3 > 0$. Note that $\bar{S}_1$ is a union of finitely many atomic sectors $\{T_i\}$ such that the completion of any T_i is Hausdorff close to $\bar{S}_1$. Since $S_3 \subset \bar{S}_3 \subset N_{A_3}(\bar{S}_1)$, we know from the argument in Lemma 10.3 that S_3 and one of $\{T_i\}$ (say T_1) touch along discs. Thus we can assume $S_3 = T_1$ by Lemma 10.2. Hence $\bar{S}_3 = \bar{T}_1$ has finite Hausdorff distance from $\bar{S}_1$. It follows that $d_H(q(\bar{S}_1), \bar{S}_2) < \infty$. □

10.2. **Preservation of stable lines.**

Definition 10.7. A *stable line* in X^*_Γ is one of the following objects:

(1) a diamond line;
(2) a Coxeter line;
(3) a single labeled plain line such that its label a satisfies that all edges of Γ containing a are labeled by 2, $|a^\perp| \geq 2$ and $(a^\perp)^\perp = a$;
(4) a single labeled plain line such that its label a is not a leaf and there is an edge containing a with label ≥ 3;
(5) a single labeled plain line such that its label a is a leaf, a is connected to a vertex b by an odd-labeled edge and b is not a leaf.

Recall that $a^\perp$ is the collection of vertices in Γ that are adjacent to a along an edge labeled by 2.

Definition 10.7 is motivated by looking for a $\mathbb{Z}$–subgroup which is the $\mathbb{Z}$ factor of some $F_k \times \mathbb{Z}$ subgroup ($k \geq 2$), moreover, we do not want this $F_k \times \mathbb{Z}$ subgroup to be further contained in an $F_k \times F_{k'}$ subgroup ($k' \geq 2$). See the conjecture below for a precise formulation, which is motivated by properties of Dehn twists in mapping class groups of surfaces.

Conjecture 10.8. *Suppose A_Γ has dimension ≤ 2. A $\mathbb{Z}$–subgroup A acts on a stable line of X^*_Γ in the sense of Definition 10.7 if and only if*

(1) the centralizer $Z_{A_\Gamma}(A)$ of A has a finite index subgroup isomorphic to $F_k \times A$, where F_k is a free group with k generators for $k \geq 2$;
(2) there does not exist another cyclic subgroup $B \leq A_\Gamma$ such that $B \cap A$ is the trivial subgroup and the projection of $Z_{A_\Gamma}(B) \cap (F_k \times A)$ to the F_k factor has finite index in F_k.

We will not need this conjecture in the later part of the paper.

It follows from the construction of X_Γ and the structure of its vertex links that for each atomic sector $S \subset X^*_\Gamma$, there exists a unique subcomplex $S' \subset X_\Gamma$ such that

(1) S' with the induced piecewise Euclidean structure is isometric to a flat sector in the Euclidean plane;
(2) the center of any 2–cell of S is contained in S', and S' is the convex hull of such centers, where the convex hull is taken inside S' with respect to the flat metric on S'.

The subcomplex S' is called the *shadow* of S. Similarly, the completion of each atomic sector also has a *shadow* which is either a CAT(0) sector or a flat plane. Note that if we view S' and S as subsets of X_Γ, then they are at finite Hausdorff distance. Moreover, if two atomic sectors satisfy $S_1 \subset S_2$, then the shadow of S_1 is contained in the shadow of S_2.

Lemma 10.9. *A singular line L is stable if and only if there exists an atomic sector S such that L bounds $\bar{S}$ and the shadow of $\bar{S}$ has angle $= \pi$.*

Proof. We prove the "if" direction. Suppose that $\bar{S}$ satisfies the assumption of the lemma. If S is a DCH or S is a diamond-plain sector with $\bar{S}$ being a half-flat, then Definition 10.7 (1) holds. If S is a CCH, then Definition 10.7 (2) holds. If S is a plain sector with $\bar{S}$ being a half-flat, then Definition 10.7 (3) holds. If S is a PCH, then Definition 10.7 (4) or (5) holds. These are all the possibilities.

Now we prove the converse. Let L be a Coxeter line. Let F_1 (resp. F_2) be the Coxeter-plain flat (resp. Coxeter flat) containing L. There are two basic moves to extend L to a CCH. The first move is to enlarge L inside F_1 to a finite width infinite strip bounded by L, the second move is to enlarge L inside F_2 to obtain a thickened Coxeter line with one side bounded by L. By alternating between these two moves for infinitely many times, we can construct a CCH bounded by L. The other cases are similar. Note that in Definition 10.7 (3), we need to use all the vertices in $a^\perp$ to obtain a plain sector whose completion satisfies all the requirements. Definition 10.7 (5) can be reduced to (4) by first constructing a thickened plain line between an a–labeled plain line and a b–labeled plain line. □

Lemma 10.10. *There exist a constant A and a function $k\colon (0,\infty) \to (0,\infty)$ such that the following hold for any stable line L:*

(1) there are three (A, A)–quasiflats Q_1, Q_2, Q_3 such that for any $C \geq A$, we have $d_H(L, \cap_{i=1}^3 N_C(Q_i)) < k(C)$;
(2) each Q_i is at finite Hausdorff distance from the disjoint union of the completion of two atomic sectors such that their shadows have angle $= \pi$.

Proof. Since there are finitely many A_Γ–orbits of stable lines in X^*_Γ, it suffices to prove the lemma for a single stable line. Consider a case of a Coxeter line L with its boundaries denoted by ℓ_1 and ℓ_2. We extend L along ℓ_1 to a CCH U'_1 (resp. U'_2) bounded L by first applying the first move (resp. the second move) in Lemma 10.9 and then alternating between first move and second move. For $i = 1, 2$, let U_i be the sub-half-flat of U'_i bounded by ℓ_1. Similarly, we extend L along ℓ_2 to a CCH U_3 bounded by L. By the argument in the proof of Lemma 5.19 and Lemma 5.21, we know $U_i \cup U_j$ is embedded and quasi-isometrically embedded for any $1 \leq i \neq j \leq 3$. Hence L is the coarse intersection of these three quasiflats. Other cases of L are similar. □

Theorem 10.11. *Suppose that A_{Γ_1} and A_{Γ_2} are two-dimensional Artin groups. Let $q\colon X^*_{\Gamma_1} \to X^*_{\Gamma_2}$ be an (L, A)–quasi-isometry. Then there exists a constant D such that for any stable line $L_1 \subset X^*_{\Gamma_1}$, there is a stable line $L_2 \subset X^*_{\Gamma_2}$ such that $d_H(q(L_1), L_2) < D$.*

The weaker statement $d_H(q(L_1), L_2) < \infty$ follows from Corollary 10.6 and Lemma 10.9. We need more work to make D independent of L_1.

Proof. Suppose L_1 is the coarse intersection of three quasiflats O_1, P_1, Q_1 as in Lemma 10.10. Then there are atomic sectors S_1, T_1 such that $\bar{S}_1 \cap \bar{T}_1 = \emptyset$ and $d_H(\bar{S}_1 \cup \bar{T}_1, O_1) < \infty$. By Corollary 10.6, there are atomic sectors S_2 and T_2 in X^*_Γ such that their completions satisfy $d_H(q(\bar{S}_1), \bar{S}_2) < \infty$ and $d_H(q(\bar{T}_1), \bar{T}_2) < \infty$. Note that the shadows of $\bar{S}_2$ and $\bar{T}_2$ have angle $\leq \pi$.

Let $O_2 = q(O_1)$. Let $f'\colon O'_2 \to X_{\Gamma_2}$ be a simplicial map approximating the quasiflat O_2 satisfying all the conditions (1)–(5) in Theorem 2.7, with O'_2 being a CAT(0) plane. Lemma 10.10 and Theorem 2.7 imply that $d_H(\operatorname{Im} f', O_2) < D_0$ for D_0 independent of L_1.

We claim O'_2 is flat. Let β be the length of the Tits boundary $\partial_T O'_2$. It suffices to prove $\beta = 2\pi$. We have $\beta \geq 2\pi$ by CAT(0) geometry. It remains to show $\beta \leq 2\pi$. By Theorem 9.2, O_2 is at bounded Hausdorff distance away from a union of finitely many atomic sectors $\{\widehat{S}_i\}_{i=1}^k$. Since $d_H(O_2, \bar{S}_2 \cup \bar{T}_2) < \infty$, we know from Lemma 10.2 that up to passing to a sub-sector, each $\widehat{S}_i$ is contained in at least one of $\bar{S}_2$ or $\bar{T}_2$. Let α_i be the angle of the CAT(0) sector $\widehat{S}'_i \subset O'_2$ associated with $\widehat{S}_i$ as in Theorem 9.1. Then $\beta = \sum_{i=1}^k \alpha_i$. We can assume $f'(\widehat{S}'_i)$ is the shadow of $\widehat{S}_i$. Each $f'(\widehat{S}'_i)$ is contained in at least one of $\bar{S}'_2$ or $\bar{T}'_2$, which are shadows of $\bar{S}_2$ or $\bar{T}_2$ respectively. Moreover, if both $f'(\widehat{S}'_i)$ and $f'(\widehat{S}'_j)$ $(i \neq j)$ are contained in $\bar{S}'_2$, then $f'(\widehat{S}'_i) \cap f'(\widehat{S}'_j)$ can not contain a flat sector with angle > 0 (otherwise $\widehat{S}_i$ and $\widehat{S}_j$ touch along discs, which implies $d_H(\widehat{S}_i, \widehat{S}_j) < \infty$). Thus $\sum_{i=1}^k \alpha_i$ is bounded above by the sum of the angle of $\bar{S}'_2$ and $\bar{T}'_2$, which is 2π. This finishes the proof of the claim.

By Lemma 4.3, the map ρ in Section 4.1 is well-defined on O'_2. Thus we can give a new cell structure on O'_2 and a new locally injective cellular map $f\colon \widetilde{O}_2 \to X^*_\Gamma$ ($\widetilde{O}_2$ is O'_2 with the new cell structure) as in Section 4.3 such that each vertex of $\widetilde{O}_2$ satisfies Lemma 4.11. Note that $d_H(\operatorname{Im} f, O_2) < D_1$ for D_1 independent of L_1. By Lemma 10.3 and Lemma 10.5, we can assume $\bar{S}_2 \subset \widetilde{O}_2$. By using Lemma 4.11 and the argument in Section 9, we deduce that the topological closure of $\widetilde{O}_2 \setminus \bar{S}_2$ in $\widetilde{O}_2$ is one of the following forms:

(1) when $\bar{S}_2$ is bounded by a diamond line, the topological closure is a union of diamond lines;
(2) when $\bar{S}_2$ is bounded by a Coxeter line, it is a CCH;
(3) when $\bar{S}_2$ is bounded by a plain line, it is a union of thickened plain lines.

Moreover, by the proof of Lemma 5.19, $\widetilde{O}_2$ is embedded.

Now we define $\widetilde{P}_2$ and $\widetilde{Q}_2$ in a similar way. Assume without loss of generality that $\bar{S}_2$ is the coarse intersection of $\widetilde{O}_2$ and $\widetilde{P}_2$. Then by Lemma 10.3 and Lemma 10.5, we can further assume $\bar{S}_2 \subset \widetilde{O}_2 \cap \widetilde{P}_2$. Then the topological closure of $\widetilde{P}_2 \setminus \bar{S}_2$ in $\widetilde{P}_2$ is one of the forms listed from (1)-(3) in the previous paragraph. By mapping $\widetilde{O}_2$ and $\widetilde{P}_2$ to a suitable tree as in Lemma 5.19 and Lemma 5.7, we know $\widetilde{O}_2 \cup \widetilde{P}_2$ gives an embedded triplane in $X^*_{\Gamma_2}$, moreover, $\partial(\widetilde{O}_2 \cap \widetilde{P}_2)$ is either a plain line, or a boundary line of a diamond line, or a Coxeter line. Let $\widehat{Q}_2$ be the topological closure of the symmetric difference of $\widetilde{O}_2$ and $\widetilde{P}_2$. Then $\widehat{Q}_2$ is homeomorphic to $\mathbb{R}^2$ and $d_H(\widehat{Q}_2, \widetilde{Q}_2) < \infty$. Since both $\widehat{Q}_2$ and $\widetilde{Q}_2$ are homeomorphic to $\mathbb{R}^2$ and are quasi-isometrically embedded, we use the argument from the proof of Lemma 10.2 to show larger and larger discs of $\widehat{Q}_2$ centered at a fixed vertex are contained in $\widetilde{Q}_2$, hence $\widehat{Q}_2 \subset \widetilde{Q}_2$. Thus $\widehat{Q}_2 = \widetilde{Q}_2$. Then there is a singular line L_2 such that $\widetilde{O}_2 \cap \widetilde{P}_2 \cap \widetilde{Q}_2$ is either contained in L_2, or a boundary of L_2. The line L_2 is actually a stable line by Lemma 10.9. One now readily verifies that $d_H(q(L_1), L_2) < D$ for some D independent of L_1. □

Remark 10.12. Note that for each stable line, there is a $\mathbb{Z}$–subgroup of A_Γ acting cocompactly on the line. Thus Theorem 10.11 also implies that the collection of $\mathbb{Z}$–subgroups corresponding to stable lines is preserved by quasi-isometries up to finite Hausdorff distance.

10.3. **Preservation of intersection graphs.** Two singular lines are *parallel* if their Hausdorff distance is finite. The *parallel set* P_L of a stable line L is defined to be the union of all stable lines that are parallel to L.

The following is motivated by the fixed set graph defined by Crisp [Cri05] and the extension graph for right-angled Artin groups defined by Kim and Koberda [KK13].

Definition 10.13 (Intersection graph)**.** Let A_Γ be a 2–dimensional Artin group. The *intersection graph* $\mathcal{I}_\Gamma$ of A_Γ is defined as follows. There is a one to one correspondence between vertices of $\mathcal{I}_\Gamma$ and parallelism classes of diamond lines, Coxeter lines and single labeled plain lines in the universal cover of the presentation complex of A_Γ. Let v_1, v_2 be two vertices representing two such parallelism classes $\mathcal{P}_1$ and $\mathcal{P}_2$. Then v_1 and v_2 are joined by an edge if and only if for $i = 1, 2$, there exists a $\mathbb{Z}$ subgroup $Z_i \leq A_\Gamma$ such that Z_i stabilizes a line in $\mathcal{P}_i$, and Z_1, Z_2 generate a free abelian subgroup of rank 2 in A_Γ. The *stable subgraph* of $\mathcal{I}_\Gamma$ is the full subgraph of $\mathcal{I}_\Gamma$ spanned by vertices corresponding to stable lines.

It could happen that the stable subgraph of $\mathcal{I}_\Gamma$ is empty. For example, this is the case when Γ is a 4–gon with each edge labeled by 2. Of course, in such case, one should not expect a quasi-isometry to map any $\mathbb{Z}$ subgroups to other $\mathbb{Z}$ subgroups up to finite Hausdorff distance.

In the rest of this subsection, we prove Theorem 10.16.

Lemma 10.14. *Suppose A_Γ is 2–dimensional. Suppose that*

(1) Γ is connected and does not contain valence one vertex;
(2) for any vertex $u \in \Gamma$ which commutes with all of its adjacent vertices, we have $(u^\perp)^\perp = u$.

Then the stable subgraph of $\mathcal{I}_\Gamma$ is all of $\mathcal{I}_\Gamma$. In particular, if the outer automorphism group $\mathrm{Out}(A_\Gamma)$ is finite and Γ has more than two vertices, then the stable subgraph of $\mathcal{I}_\Gamma$ is all of $\mathcal{I}_\Gamma$.

Proof. The first statement follows from Definition 10.7. It remains to prove the "in particular" statement. Suppose $\mathrm{Out}(A_\Gamma)$ is finite. If Γ has a valence one vertex v, then any vertex $w \in \Gamma$ adjacent to v is a separating vertex (since Γ has more than two vertices). This leads to a Dehn twist automorphism defined in [Cri05, pp. 1383] and hence $\mathrm{Out}(A_\Gamma)$ is infinite. Thus Γ does not have valence one vertices. Similarly, we know Γ is connected. Let $u \in \Gamma$ be vertex such that all edges of Γ containing u are labeled by 2. If there is a vertex $v \neq u$ such that $v \in (u^\perp)^\perp$, then sending v to vu and fixing all other generators yields an infinite order element in $\mathrm{Out}(A_\Gamma)$. Thus we must have $(u^\perp)^\perp = u$ for all such u. $\square$

Lemma 10.15. *Let $L \subset X^*_\Gamma$ be a diamond line, or a Coxeter line, or a single-labeled plain line. Then any singular line L' parallel to L is of the same type. Moreover, there exists a $\mathbb{Z}$–subgroup $Z \leq A_\Gamma$ such that*

(1) Z stabilizes any singular line that is parallel to L;
(2) for any $g \in Z$, there is a constant $C > 0$ only depending on g such that $d(x, gx) < C$ for any vertex x in the parallel set of L.

Proof. Suppose that L' and L are parallel with their stabilizers denoted by Z and Z'. Then by [MSW11, Corollary 2.4], $Z_0 = Z \cap Z'$ is of finite index in both Z and Z'. Then L' and L are of the same type by the following:

- if L is a diamond line, then the fixed point set $\mathrm{Fix}(Z)$ of the Z–action on the Deligne complex D_Γ is a vertex of rank 2, and $\mathrm{Fix}(Z)$=$\mathrm{Fix}(Z_0)$ by [Cri05, Lemma 8 (ii)];
- if L is a single-labeled plain line, then $\mathrm{Fix}(Z)$ is a tree containing a vertex of rank 1, and $\mathrm{Fix}(Z)$=$\mathrm{Fix}(Z_0)$ by [Cri05, Lemma 8 (i)];
- if L is a Coxeter line, then $\mathrm{Fix}(Z)$ is empty and Z acts by translations on a geodesic line in D_Γ.

Now we prove the "moreover" statement. By Section 5.1, each such L is quasi-isometrically embedded with constants independent of L. Thus (2) follows from (1). It suffices to prove (1). Let L, L', Z and Z' be as before. If L is a single-labeled plain line, then $\mathrm{Fix}(Z)$ =$\mathrm{Fix}(Z_0)$ =$\mathrm{Fix}(Z')$. By the definition of D_Γ, we know L (resp. L') corresponds to a vertex in $\mathrm{Fix}(Z)$ (resp. $\mathrm{Fix}(Z')$), thus Z also stabilizes L' and (1) holds for single-labeled plain lines. If L is a diamond line, then the above discussion implies that L and L' are in the same block B. Thus the centralizer of the stabilizer of B will stabilize all the diamond lines in B.

It remains to consider the case L is a Coxeter line. We claim there is a finite chain $L_0 = L, L_1, \ldots, L_{n-1}, L_n = L'$ such that each L_i is a Coxeter line and for $0 \leq i \leq n-1$, either L_i and L_{i+1} are contained in the same Coxeter-plain flat, or they are contained in the same Coxeter flat. The claim will imply that the stabilizer of L also stabilizes L', which finishes the proof.

Now we prove the above claim. Let $\mathcal{W}$ and $\mathcal{W}'$ be the walls of L and L' respectively. Let $\pi\colon X^b_\Gamma \to D_\Gamma$ be the A_Γ–equivariant simplicial map defined at the beginning of Section 5.3. Let $\ell = \pi(\mathcal{W})$ and $\ell' = \pi(\mathcal{W}')$. Then ℓ and ℓ' are parallel geodesic lines in a $CAT(0)$ space D_Γ. If $\ell = \ell'$, then by the definition of D_Γ, we know L and L' are contained in the same Coxeter-plain flat, hence the claim follows. Now we assume $\ell \neq \ell'$, then they bound a flat strip $E \subset D_\Gamma$. Since D_Γ is 2–dimensional, E is a subcomplex of D_Γ. Let F be the Coxeter-plain flat containing L. Let $L^\perp$ be a thickened plain line in F intersecting L in a 2–cell C. Let $v \in C$ be the center of C and let u, w be two vertices in $\mathcal{W}$ adjacent to v (with respect to the simplicial structure on X^b_Γ). Then u and w are in two different boundary lines of $L^\perp$. Let $\Delta \subset E$ be the triangle containing $\bar{v} = \phi(v)$ and $\bar{u} = \phi(u)$. Then Δ has a rank 0 vertex $\bar{x}$. Let $x = \rho^{-1}(\bar{x})$. Then by the definition of D_Γ, we know x and u are contained in the same boundary line of $L^\perp$. Let C_x be a 2–cell in $L^\perp$ containing x. Then ℓ cuts $\rho(C_x)$ into two halves such that one of them (denoted by K) contains both Δ and $\rho(w)$. Since $K \cap E$ is a convex subcomplex of K, we have $K \subset E$. Let L_x be the Coxeter line in F intersecting $L^\perp$ in C_x. Similarly, we know that ℓ cuts $\rho(L_x)$ into two halves with one of them contained in E. Let F_1 be the Coxeter flat containing L_x, and let $\mathcal{W}_x$ and $\mathcal{W}'_x$ be two distinct parallel walls in F_1 such that $\mathcal{W}_x \subset L_x$, x is between $\mathcal{W}_x$ and $\mathcal{W}'_x$, and the distance between $\mathcal{W}_x$ and $\mathcal{W}'_x$ is as small as possible. Then ρ maps the region of F_1 between $\mathcal{W}_x$ and $\mathcal{W}'_x$ to a flat strip U in D_Γ. Since a half of $\rho(L_x)$ is contained in $U \cap E$, by convexity, we know $U \subset E$. Now we repeat the previous argument with L replaced by the carrier of $\mathcal{W}'_x$. This process terminates after finitely many steps and gives rise to the finite chain as required. □

Theorem 10.16. *Let A_Γ and $A_{\Gamma'}$ be two 2–dimensional Artin groups and let $q\colon A_\Gamma \to A_{\Gamma'}$ be a quasi-isometry. Then q induces an isomorphism from the stable subgraph of $\mathcal{I}_\Gamma$ onto the stable subgraph of $\mathcal{I}_{\Gamma'}$. If both A_Γ and $A_{\Gamma'}$ satisfy conditions (1) and (2) in Lemma 10.14, then q induces an isomorphism between their intersection graphs.*

Thus if both A_Γ and $A_{\Gamma'}$ have finite outer automorphism group and Γ has more than two vertices, then q induces an isomorphism between their intersection graphs.

Proof. Let Θ_Γ and $\Theta_{\Gamma'}$ be the stable subgraphs of $\mathcal{I}_\Gamma$ and $\mathcal{I}_{\Gamma'}$ respectively. It follows from Theorem 10.11 that q_* induces a well-defined injective map on the vertex set of Θ_Γ. Now we show if two vertices v_1, v_2 are adjacent in Θ_Γ, then $q_*(v_1)$ and $q_*(v_2)$ are adjacent in $\Theta_{\Gamma'}$.

Let L_1 and L_2 be two stable lines representing v_1 and v_2. For $i = 1, 2$, let P_i be the parallel set of L_i and suppose $\mathbb{Z}_i$ is a $\mathbb{Z}$–subgroup stabilizing L_i. We assume $\mathbb{Z}_1$ and $\mathbb{Z}_2$ generate a free abelian subgroup of rank 2. Let $Q = \mathbb{Z}_1 \oplus \mathbb{Z}_2$ and we view Q as a subset of X^*_Γ. Then Q is at finite Hausdorff distance away from a union of stable lines parallel to L_1. By Theorem 10.11, there exists a stable line L'_i such that $Q' = q(Q)$ is at finite Hausdorff distance away from a union of stable lines parallel to L'_1. Let g'_1 be a generator of a $\mathbb{Z}$–subgroup $\mathbb{Z}'_1$ that stabilizes L'_1. Then Lemma 10.15 (2) implies that $d_H(Q', g'_1Q') < \infty$. Similarly, we define L'_2, g'_2 and $\mathbb{Z}'_2$ and we have $d_H(Q', g'_2Q') < \infty$.

Note that Q' is a quasiflat. Now we assume there is a continuous quasi-isometric embedding $f\colon \mathbb{E}^2 \to X^*_{\Gamma'}$ such that $d_H(\operatorname{Im} f, Q') < \infty$. Let $[\mathbb{E}^2] \in H^{\mathrm{lf}}_2(\mathbb{E}^2)$ be the fundamental class and let $[\alpha] = f_*([\mathbb{E}^2]) \in H^{\mathrm{lf}}_2(X^*_{\Gamma'})$. By [BKS16, Lemma 4.3], the support set $S_{[\alpha]}$ of $[\alpha]$ is non-empty and satisfies $d_H(\operatorname{Im} f, S_{[\alpha]}) < \infty$. Let f_i be f post-composed with the action of g'_i. Lemma 10.15 (2) implies that there is a uniform bound on $d(f(x), f_i(x))$ for any $x \in \mathbb{E}^2$. Since $X^*_{\Gamma'}$ is uniformly contractible, there is a proper homotopy between f and f_i. Thus $[\alpha] = (f_i)_*[\mathbb{E}^2] = (g'_i)_*[\alpha]$. Thus the support set $S_{[\alpha]}$ is invariant under the subgroup $H = \langle \mathbb{Z}'_1, \mathbb{Z}'_2 \rangle$ of $A_{\Gamma'}$. Then H is quasi-isometric to $\mathbb{E}^2$, hence is virtually $\mathbb{Z} \oplus \mathbb{Z}$. It follows that a finite index subgroup of $\mathbb{Z}'_1$ and a finite index subgroup of $\mathbb{Z}'_2$ generate a free abelian subgroup. Hence $q_*(v_1)$ and $q_*(v_2)$ are adjacent.

In summary, we now have a graph embedding $q_*\colon \Theta_\Gamma \to \Theta_{\Gamma'}$. By considering the quasi-isometry inverse of q and repeating the above argument, we know q_* is actually an isomorphism. The "in particular" statement of the theorem follows from Lemma 10.14. □

Corollary 10.17. *Let A_Γ be a 2–dimensional Artin group such that Γ satisfies (1) and (2) in Lemma 10.14. If $q\colon A_\Gamma \to A_\Gamma$ is an (L, A)–quasi-isometry inducing the identity map on $\mathcal{I}_\Gamma$, then there exists $C = C(L, A, \Gamma)$ such that $d(q(x), x) \le C$ for any $x \in A_\Gamma$. Thus the map $\mathrm{QI}(A_\Gamma) \to \mathrm{Aut}(\mathcal{I}_\Gamma)$ described in Theorem 10.16 is an injective homomorphism.*

Thus if we know in addition that the homomorphism $A_\Gamma \to \mathrm{Aut}(\mathcal{I}_\Gamma)$ induced by the action $A_\Gamma \curvearrowright \mathrm{Aut}(\mathcal{I}_\Gamma)$ has finite index image, then any finitely generated groups quasi-isometric to A_Γ is virtually A_Γ.

Proof. First, observe that, by virtue of Definition 10.13 and Theorem 10.11 the map $\mathrm{QI}(A_\Gamma) \to \mathrm{Aut}(\mathcal{I}_\Gamma)$ is a homomorphism.

Let $x \in A_\Gamma$. It follows from the assumption on Γ that there exist $F_1, F_2, F_3 \subset X^*_\Gamma$ containing x such that

(1) each F_i is either a diamond-plain flat (cf. Definition 5.6) or a cubical flat spanned by two commuting elements of Γ;

(2) $x = F_1 \cap F_2 \cap F_3$, moreover, there exists $f\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\cap_{i=1}^{3} N_r(F_i)$ is contained in the $f(r)$–ball around x.

Note that each F_i corresponds to an abelian subgroup of $A_i \leq A_\Gamma$ such that $\cap_{i=1}^{3} A_i$ is the trivial subgroup, thus (2) follows from [MSW11, Corollary 2.4]. We can assume that the function f depends only on A_Γ (but not on x) by applying a left translation. As q induces the identity map on $\mathcal{I}_\Gamma$, it follows from the third paragraph of the proof of Theorem 10.16 that if we consider the support set S_i associated with $q(F_i)$ as there, then S_i is invariant under a finite index subgroup of A_i. Thus $d_H(q(F_i), F_i) < \infty$. Actually this can be improved to $d_H(q(F_i), F_i) < C$ for C depending only on L, A, A_Γ, as $d_H(S_i, q(F_i)) < C = C(L, A, A_\Gamma)$ and $S_i = F_i$. This and the property (2) above imply the corollary. $\square$

11. Artin groups of type CLTTF

In this section, we restrict ourselves to groups A_Γ belonging to the class of CLTTF Artin groups, as in Definition 1.5. Note that there are no Coxeter lines in X^*_Γ. Let $\mathcal{I}_\Gamma$ be the intersection graph of A_Γ and let Θ_Γ be the stable subgraph (cf. Definition 10.13).

11.1. **The fixed set graphs and chunks.** We recall several notions from [Cri05]. Crisp defined a notion of *fixed set graph* for A_Γ in [Cri05, Section 4]. It follows from [Cri05, Lemma 9] that a single-labeled line is stable if and only if the centralizer of the stabilizer of this line is not virtually abelian (recall that we require A_Γ to be of large type). Now by [Cri05, Lemma 12] and the sentence before this lemma, we know that the there is a natural isomorphism between the fixed set graph and Θ_Γ.

Crisp divides the vertex set of Θ_Γ into two disjoint subsets, one being $\mathcal{V}_\Gamma$, which consists of vertices coming from diamond lines, and one being $\mathcal{F}_\Gamma$, which consists of vertices coming from stable plain lines. Note that there is a one to one correspondence between elements in $\mathcal{V}_\Gamma$ and blocks in X^*_Γ. It is clear that there is an action $A_\Gamma \curvearrowright \Theta_\Gamma$ which preserves $\mathcal{V}_\Gamma$ and $\mathcal{F}_\Gamma$.

We choose an identification of A_Γ with the 0–skeleton of X^*_Γ. Let Γ_2 be the first subdivision of Γ. Each vertex of Γ_2 corresponds to a unique single-labeled plain line or a diamond line that contains the identity element $*$ of A_Γ. Let $\widehat{\Gamma}$ be the full subgraph of Γ_2 spanned by the non-terminal vertices. Then vertices of $\widehat{\Gamma}$ give stable lines containing $*$ (however, it is possible that a terminal vertex of Γ_2 also gives a stable line). This induces a graph embedding $\widehat{\Gamma} \to \Theta_\Gamma$. From now on, we shall identify $\widehat{\Gamma}$ as a subgraph of Θ_Γ via such embedding. Note that Θ_Γ is the union of A_Γ translates of $\widehat{\Gamma}$.

Similarly, the vertices of D_Γ that are associated with left cosets of A_Γ that contain $*$ span a convex subcomplex K of D_Γ, which is called the *fundamental chamber* of D_Γ. As a simplicial complex, K is isomorphic to the cone over Γ_2. D_Γ is the union of A_Γ translates of K.

There is another cell structure on D_Γ, where one ignores all the edges between a rank 0 vertex and a rank 2 vertex and combines triangles of D_Γ into squares. We denote D_Γ with this new cell structure by $\Box_\Gamma$. When Γ is CLTTF, we know that $\Box_\Gamma$ is a CAT(0) cube complex (actually, a more general version of this fact was proved in [CD95b]).

Lemma 11.1. [Cri05, Lemma 39] *Let $v \in \Box_\Gamma$ be a rank 2 vertex and let m be the label of the defining edge of the block that is associated with v. Every simple cycle in $\mathrm{lk}(v, \Box_\Gamma)$ has edge length at least $2m$, and there exists at least one simple cycle of $2m$ edges in $\mathrm{lk}(v, \Box_\Gamma)$.*

Proposition 11.2. [Cri05, Proposition 40] *Let v be a rank 2 vertex in the fundamental chamber of $\Box_\Gamma$ and let $A_{s,t}$ be the standard subgroup of A_Γ associated with v such that its defining edge is $\overline{st} \subset \Gamma$. Let E be the edge in $\mathrm{lk}(v, \Box_\Gamma)$ spanned by vertices corresponding to A_s and A_t. If τ is a graph automorphism of $\mathrm{lk}(v, \Box_\Gamma)$ that fixes E, then either τ is the identity on $\mathrm{lk}(v, \Box_\Gamma)$, or τ is induced by the group inversion such that $s \to s^{-1}$ and $t \to t^{-1}$.*

The following notion of chunks was introduced in [Cri05, Section 7]

Definition 11.3. Let A be a connected full subgraph of Γ. A is *indecomposable* if, for every decomposition $\Gamma = \Gamma_1 \cup_T \Gamma_2$ of Γ over a separating edge or vertex T, either $A \subset \Gamma_1$ or $A \subset \Gamma_2$. A *chunk* of Γ is a maximal indecomposable (connected and full) subgraph of Γ. Clearly, any two chunks of Γ intersect, if at all, along a single separating edge or vertex. A chunk of Δ shall be said to be *solid* if it contains a simple closed circuit of Δ. Note that any chunk is either solid or consists of just a single edge of the graph.

$\widehat{\Gamma}$ can be viewed both as a subset of Γ and as a subgraph of Θ_Γ. A *(solid) chunk* of $\widehat{\Gamma}$ is the intersection of $\widehat{\Gamma}$ with a (solid) chunk of Γ. This defines a subgraph of $\widehat{\Gamma}$ which shall be thought of as lying inside Θ_Γ. A *(solid) chunk* of Θ_Γ is any translate of a (solid) chunk of $\widehat{\Gamma}$ by an element of A_Γ.

An isomorphism ϕ from Θ_Γ to $\Theta_{\Gamma'}$ is a *$\mathcal{VF}$–isomorphism* if $\phi(\mathcal{V}_\Gamma) = \mathcal{V}_{\Gamma'}$ and $\phi(\mathcal{F}_\Gamma) = \mathcal{F}_{\Gamma'}$.

Proposition 11.4. [Cri05, Proposition 41] *Suppose A_Γ and $A_{\Gamma'}$ are CLTTF. If Γ is not a tree, then any isomorphism $\Theta_\Gamma \to \Theta_{\Gamma'}$ is a $\mathcal{VF}$–isomorphism.*

Proposition 11.5. [Cri05, Proposition 23] *Suppose A_Γ and $A_{\Gamma'}$ are CLTTF. Any $\mathcal{VF}$–isomorphism $\Theta_\Gamma \to \Theta_{\Gamma'}$ maps each solid chunk of Θ_Γ onto a solid chunk of $\Theta_{\Gamma'}$.*

11.2. **Quasi-isometries of CLTTF Artin groups.**

Lemma 11.6. *Suppose that A_Γ and $A_{\Gamma'}$ are CLTTF Artin groups. Let $q\colon X^*_\Gamma \to X^*_{\Gamma'}$ be an (L, A)–quasi-isometry. Then*

(1) q induces a graph isomorphism $q_*\colon \Theta_\Gamma \to \Theta_{\Gamma'}$;

(2) *if in addition* Γ *is not a tree, then there exists a constant* $D = D(L, A, \Gamma, \Gamma')$ *such that for any block* $B \subset X^*_\Gamma$*, there exists a block* $B' \subset X^*_{\Gamma'}$ *such that* $d_H(q(B), B') < D$*; moreover,* q_* *sends the vertex in* Θ_Γ *associated with* B *to the vertex in* $\Theta_{\Gamma'}$ *associated with* B'*.*

Proof. Theorem 10.16 implies (1). Now we prove (2). By Proposition 11.4, $q_*\colon \Theta_\Gamma \to \Theta_{\Gamma'}$ is a $\mathcal{VF}$–isomorphism. Now suppose B is the parallel set of a diamond line L. By Theorem 10.11, there exists a stable line $L' \subset X^*_{\Gamma'}$ such that $d_H(P_{L'}, q(P_L)) < D$ for $D = D(L, A, \Gamma_1, \Gamma_2)$. Note that L' has to be a diamond line. Thus $P_{L'}$ is a block B' and $d_H(B', q(B)) < D$. □

Theorem 11.7. *Let* A_Γ *and* $A_{\Gamma'}$ *be CLTTF Artin groups. Suppose* Γ *does not have separating vertices and edges. Let* $q\colon A_\Gamma \to A_{\Gamma'}$ *be an* (L, A)*–quasi-isometry. Then there exist a constant* $D = D(L, A, \Gamma)$ *and a bijection* $\tilde{q}\colon A_\Gamma \to A_{\Gamma'}$ *such that*

(1) $d(q(x), \tilde{q}(x)) < D$ *for any* $x \in A_\Gamma$*;*

(2) *if* $\Gamma_1 \subset \Gamma$ *is an edge (resp. vertex), then for any* $g \in A_\Gamma$*, there exists an edge (resp. vertex)* $\Gamma'_1 \subset \Gamma'$ *and* $g' \in A_{\Gamma'}$ *such that* $\tilde{q}$ *maps* gA_{Γ_1} *bijectively onto* $g'A_{\Gamma'_1}$*; and similar properties hold for* $\tilde{q}^{-1}$*.*

Moreover, such $\tilde{q}$ *is unique.*

Proof. Pick a vertex $x \in X^*_\Gamma$ and let $\{B_i\}_{i=1}^n$ be the collection of blocks containing x. Note that $x = \cap_{i=1}^n B_i$. Let $\{v_i\}_{i=1}^n$ be the collection of vertices in $\mathcal{V}_\Gamma$ corresponding to $\{B_i\}_{i=1}^n$. Let $g \in A_\Gamma$ be the element such that it sends the identity element $*$ to x. Then $\{v_i\}_{i=1}^n \subset g\widehat{\Gamma} \subset \Theta_\Gamma$. Since Γ has no separating vertices and edges, $g\widehat{\Gamma}$ is a chunk. Let $q_*\colon \Theta_\Gamma \to \Theta_{\Gamma'}$ be the graph isomorphism in Lemma 11.6, which is a $\mathcal{VF}$–isomorphism by Proposition 11.4. Let B'_i be the block such that $d_H(q(B_i), B'_i) < D$ as in Lemma 11.6 (2). Proposition 11.5 implies that $q_*(g\widehat{\Gamma})$ is a chunk of $\Theta_{\Gamma'}$, thus the vertices of $\Theta_{\Gamma'}$ associated with $\{B'_i\}_{i=1}^n$ are contained in a chunk. Hence $\cap_{i=1}^n B'_i \neq \emptyset$. Since $\cap_{i=1}^n B'_i$ is bounded, it is a vertex x' of $X^*_{\Gamma'}$. We define $\tilde{q}(x) = x'$. It is clear that for any vertex $x \in X^*_\Gamma$, we have $d(q(x), \tilde{q}(x)) < D$ for $D = D(L, A, \Gamma, \Gamma')$. By construction, $\tilde{q}$ satisfies all the requirements except we still need to verify the bijectivity and uniqueness of $\tilde{q}$.

We define a map F from A_Γ to chunks in $\widehat{\Gamma}'$ as follows. Let x, x' and g be as before and let $g' \in A_{\Gamma'}$ be the element sending x' to the identity element $*' \in A_{\Gamma'}$. Then $F(x)$ is defined to be the chunk $g'(q_*(g\widehat{\Gamma}))$. We claim F is a constant function. It suffices to show if x_1 and x_2 are two adjacent vertices in X^*_Γ, then $F(x_1) = F(x_2)$. Suppose $F(x_i) = g'_i(q_*(g_i\widehat{\Gamma}))$ for $i = 1, 2$. Then $g_1\widehat{\Gamma} \cap g_2\widehat{\Gamma}$ contains the closed star (in $g_1\widehat{\Gamma}$) of some $\mathcal{F}$–type vertex w, where w corresponds to the single-labeled plain line ℓ passing through x_1 and x_2. Our assumption on Γ implies $g_1\widehat{\Gamma} \cap g_2\widehat{\Gamma}$ has at least two edges. Let ℓ' be the stable line associated with $q_*(w)$ that passes through $\tilde{q}(x_1)$. Then $\tilde{q}(x_2) \subset \ell'$, hence g'_1 and g'_2 differ by an element in the stabilizer of ℓ'. Now one readily deduces that $F(x_1) \cap F(x_2)$ contains at least two edges that intersect along an $\mathcal{F}$–type vertex. Thus $F(x_1) = F(x_2)$ by the definition of chunks.

Note that the image of F gives a subgraph $\Gamma'_1 \subset \Gamma'$. Note that if x_1 and x_2 are adjacent in X^*_Γ, then $\tilde{q}(x_1)$ and $\tilde{q}(x_2)$ are in the same left coset of $A_{\Gamma'_1}$ in $A_{\Gamma'}$. Thus $\tilde{q}(x)$ is contained in a left coset of $A_{\Gamma'_1}$. Since $\operatorname{Im}\tilde{q}$ is A–dense in $A_{\Gamma'}$, we have $\Gamma'_1 = \Gamma'$. Now we repeat the previous discussion for a quasi-isometry inverse of q to deduce the bijectivity statement in the theorem. To see the uniqueness, note that two blocks have finite Hausdorff distance if and only if they are equal. Thus for any block $B \subset X^*_\Gamma$, there is a unique block $B' \subset X^*_\Gamma$ such that $d_H(q(B), B') < \infty$. By using (2) and repeating the opening paragraph of the proof, we know $\tilde{q}$ is unique. □

Corollary 11.8. *Let A_Γ and $A_{\Gamma'}$ be CLTTF Artin groups. Suppose Γ does not have separating vertices and edges. Then A_Γ and $A_{\Gamma'}$ are quasi-isometric if and only if Γ and Γ' are isomorphic as labeled graphs.*

Proof. Let $\tilde{q}$ be as in Theorem 11.7. Then by Theorem 11.7 (2), $\tilde{q}$ induces an isomorphism $\phi\colon D_\Gamma \to D_{\Gamma'}$. By restricting ϕ to the fundamental chamber of D_Γ, we obtain a graph isomorphism $\Gamma \to \Gamma'$. By Lemma 11.1, this isomorphism preserves labels of edges. □

Theorem 11.9. *Let A_Γ be a CLTTF Artin group such that Γ does not have separating vertices and edges. Let $\mathrm{QI}(A_\Gamma)$ be the quasi-isometry group of A_Γ. Let $\mathrm{Isom}(A_\Gamma)$ be the isometry group of A_Γ with respect to the word distance with respect to the standard generating set. Then the following hold.*

(1) Any quasi-isometry from A_Γ to itself is uniformly close to an element in $\mathrm{Isom}(A_\Gamma)$.
(2) There are isomorphisms $\mathrm{QI}(A_\Gamma) \cong \mathrm{Isom}(A_\Gamma) \cong \mathrm{Aut}(D_\Gamma)$, where $\mathrm{Aut}(D_\Gamma)$ is the simplicial automorphism group of D_Γ.

Proof. We first define a homomorphism $h_1\colon \mathrm{Aut}(D_\Gamma) \to \mathrm{Isom}(A_\Gamma)$ as follows. Let $\phi \in \mathrm{Aut}(D_\Gamma)$. By looking at the restriction of ϕ to rank 0 vertices, we obtain a bijection $\varphi\colon A_\Gamma \to A_\Gamma$ satisfying Lemma 11.6 (2). We claim $\varphi \in \mathrm{Isom}(A_\Gamma)$. First we show if $x_1, x_2 \in A_\Gamma$ are adjacent vertices in X^*_Γ, then so is $\varphi(x_1)$ and $\varphi(x_2)$. Up to pre-composing and post-composing φ with suitable left translations, we assume x_1 is the identity element $*$, and $\varphi(x_1) = *$. Then the corresponding ϕ sends the fundamental chamber to itself. By the proof of Corollary 11.8, ϕ induces a label-preserving automorphism α of Γ. By modifying φ by an automorphism of A_Γ induced by α, we assume ϕ is the identity on the fundamental chamber. Thus there exists a rank 2 vertex v in the fundamental chamber such that the standard subgroup $A_{s,t}$ associated with v contains $x_1 = \varphi(x_1)$ and $x_2 = \varphi(x_2)$. Note that ϕ induces a graph automorphism $\tau\colon \mathrm{lk}(v, \square_\Gamma) \to \mathrm{lk}(v, \square_\Gamma)$ which is compatible with $\varphi|_{A_{s,t}}\colon A_{s,t} \to A_{s,t}$. Since ϕ is the identity on the fundamental chamber, τ fixes the edge in $\mathrm{lk}(v, \square_\Gamma)$ that are spanned by vertices associated with A_s and A_t. By Proposition 11.2, $\varphi|_{A_{s,t}}$ is either the identity or a group inversion. Thus $\varphi(x_1)$ and $\varphi(x_2)$ are adjacent in X^*_Γ. We deduce that φ is 1–Lipschitz with respect to the word metric. Similarly, we know

φ^{-1} is 1–Lipschitz. Thus $\varphi \in \mathrm{Isom}(A_\Gamma)$. We define $h_1(\phi) = \varphi$ and one readily verifies that $h_1 \colon \mathrm{Aut}(D_\Gamma) \to \mathrm{Isom}(A_\Gamma)$ is a homomorphism.

It is clear there is a homomorphism $h_2 \colon \mathrm{Isom}(A_\Gamma) \to \mathrm{QI}(A_\Gamma)$. By Theorem 11.7, there is a homomorphism $h_3 \colon \mathrm{QI}(A_\Gamma) \to \mathrm{Aut}(D_\Gamma)$. It is clear that $h_3 \circ h_2 \circ h_1$ is the identity map. Thus the theorem follows. □

A group G is *strongly rigid* if any element in $\mathrm{QI}(G)$ is uniformly close to an automorphism of G.

Theorem 11.10. *Let A_Γ be a large-type and triangle-free Artin group. Then A_Γ is strongly rigid if and only if Γ satisfies all of the following conditions:*

(1) Γ is connected and has ≥ 3 vertices;
(2) Γ does not have separating vertices and edges;
(3) any label preserving automorphism of Γ which fixes the neighborhood of a vertex is the identity.

Moreover, if a large-type and triangle-free Artin group A_Γ satisfies all the above conditions and H is a finitely generated group quasi-isometric to A_Γ, then there exists a finite index subgroup $H' \leq H$ and a homomorphism $\phi \colon H' \to A_\Gamma$ with finite kernel and finite index image.

Recall that a *global inversion* of A_Γ is the automorphism of A_Γ sending each generator to its inverse. This will be used in the following proof.

Proof. First we prove the "if" direction. Let $q \colon A_\Gamma \to A_\Gamma$ be a quasi-isometry. Suppose $\tilde{q}$ is the map in Theorem 11.7. Let $\phi \colon D_\Gamma \to D_\Gamma$ be the isomorphism induced by $\tilde{q}$. As in Theorem 11.9, by modifying $\tilde{q}$ by a left translation and an automorphism of A_Γ induced by label-preserving graph automorphism of Γ, we assume ϕ is the identity on the fundamental chamber K. Also by Proposition 11.2, ϕ is either the identity or induced by the group inversion on the link of each rank 2 vertex of K. By connectedness of Γ, we know if $\tilde{q}$ is an inversion on A_s for some vertex $s \in \Gamma$, then $\tilde{q}$ is an inversion on A_t for any vertex $t \in \Gamma$. Thus by possibly modifying $\tilde{q}$ by a global inversion, we can assume ϕ is the identity on both K and the closed star of any rank 2 vertex in K. Now the argument in the last paragraph of [Cri05, pp. 1436] implies that ϕ is the identity on D_Γ (assumption (3) is used in this step). Hence $\tilde{q}$ is the identity and the if direction follows.

Before turning to the "only if" direction we prove the last statement in the theorem. Since H is quasi-isometric A_Γ, there is a discrete and cobounded quasi-action of H on A_Γ (see [KL01, Section 2] for setting up on quasi-actions). By Theorem 11.9, we can replace each quasi-isometry in the quasi-action by a unique element in $\mathrm{Isom}(A_\Gamma)$, which gives a homomorphism $h \colon H \to \mathrm{Isom}(A_\Gamma)$. Since the quasi-action is discrete, h has finite kernel. Let N be the finite subgroup of $\mathrm{Aut}(A_\Gamma)$ generated by global inversion and automorphisms arise from label-preserving graph automorphisms of Γ. Then $\mathrm{Isom}(A_\Gamma) \cong A_\Gamma \rtimes N$ by the previous paragraph (the copy of A_Γ in $\mathrm{Isom}(A_\Gamma)$ corresponds to left translations). Now the coboundedness of the quasi-action implies $\operatorname{Im} h$ has finite index in $\operatorname{Isom}(A_\Gamma)$. Since A_Γ is finite index in $\operatorname{Isom}(A_\Gamma)$, the moreover statement follows.

It remains to prove the "only if" direction. If Γ is not connected, then A_Γ is a free product and clearly there are quasi-isometries which are not uniformly close to any automorphisms. Thus Γ is connected. If Γ has only two vertices, then A_Γ is quasi-isometric to either $\mathbb{Z}\oplus\mathbb{Z}$ or $F_2\oplus\mathbb{Z}$ (F_2 denotes the free group of two generators). Again A_Γ is not strongly rigid. Thus (1) holds.

Now we assume Γ has a separating vertex or edge. Let $\Gamma=\Gamma_1\cup_T\Gamma_2$, where Γ_1 and Γ_2 are full subgraphs of Γ such that T is an edge or a vertex, $\Gamma_1\cup\Gamma_2=\Gamma$ and $\Gamma_1\cap\Gamma_2=T$. We also assume $T\neq\Gamma_1$ and $T\neq\Gamma_2$. Let g be an element in the centralizer of A_T. Then there is an automorphism ϕ of A_Γ defined by $\phi(v)=gvg^{-1}$ for any vertex $v\in\Gamma_2$, and $\phi(v)=v$ for any vertex in Γ_1. This is called a *Dehn twist automorphism* by Crisp. We denote the standard presentation complex of A_Γ by P_Γ. Let $f\colon P_\Gamma\to P_\Gamma$ be a homotopy equivalence such that f induces ϕ on the fundamental groups. We can assume $f|_{P_{\Gamma_1}}$ is the identity map and $f(P_{\Gamma_2})\subset P_{\Gamma_2}$. We pull apart P_{Γ_1} and P_{Γ_2} in P_Γ to form a graph of spaces $X=P_{\Gamma_1}\cup P_T\times[0,1]\cup P_{\Gamma_2}$ with the attaching maps defined in the natural way. Then f induces a homotopy equivalence $g\colon X\to X$. Let T be the associated Bass-Serre tree. There is a projection map $p\colon\widetilde{X}\to T$ where $\widetilde{X}$ is the universal cover of X. Let $\tilde{g}\colon\widetilde{X}\to\widetilde{X}$ be a lift of g. We can assume $\tilde{g}$ is the identity on a lift $\widetilde{P}_{\Gamma_1}\cup\widetilde{P}_T\times[0,1]$ of $P_{\Gamma_1}\cup P_T\times[0,1]$ in $\widetilde{X}$. Let $E\subset T$ be the edge which is the p–image of such lift and let V_1 be the vertex of E associated with $\widetilde{P}_{\Gamma_1}$. The midpoint of E divides T into two halfspace. Let H_1 be the halfspace containing V_1 and H_2 be the other halfspace. Now we define a new map $\tilde{h}\colon\widetilde{X}\to\widetilde{X}$ such that $\tilde{h}$ is the identity on $p^{-1}(H_1)$ and $\tilde{h}=\tilde{g}$ on $p^{-1}(H_2)$. One readily verifies that $\tilde{h}$ is a quasi-isometry, and $\tilde{h}$ is not uniformly close to an automorphism.

Now we assume (1) and (2) hold, but (3) fails. Let $v\in\Gamma$ be a vertex and let α be a non-trivial label-preserving automorphism of Γ which fixes the closed star of v pointwise. Let f be the automorphism of A_Γ induced by α. Then f induces an automorphism $\phi\colon\square_\Gamma\to\square_\Gamma$. Let K be the fundamental chamber in $\square_\Gamma$ and let E be the edge in K between the rank 0 vertex in K and the rank 1 vertex in K corresponding to the subgroup generated by v. Let H_E be the hyperplane in $\square_\Gamma$ dual to E. Suppose Γ_1 is the full subgraph of Γ spanned by the closed star of v. Then there is a natural isometric embedding $\square_{\Gamma_1}\to\square_{\Gamma_2}$. Note that $H_E\subset\square_{\Gamma_1}$ and ϕ fixes H_E pointwise. Let H_1 and H_2 be two halfspaces bounded by H_E such that $v\in H_1$. Now we define an automorphism ϕ' of $\square_\Gamma$ such that ϕ' is the identity on H_1 and $\phi'=\phi$ on H_2. Then ϕ' induces an isometry $f'\colon A_\Gamma\to A_\Gamma$ as in the proof of Theorem 11.9. Now we show f' is not uniformly close to an automorphism. Since we already assume (1) and (2) hold, by [Cri05, Theorem 1], $\operatorname{Aut}(A_\Gamma)$ is generated by automorphisms induced by label-preserving automorphisms of Γ, global inversions and inner automorphisms. On the other hand, $f'|_K$ is induced by α and f' is the identity on some translate of K. No elements in $\mathrm{Aut}(A_\Gamma)$ can induce such automorphism of $\square_\Gamma$. Thus f' is not uniformly close to any automorphism by Theorem 11.9 (2). $\square$

12. Ending remarks and open questions

Here we leave some remarks concerning perspectives of extending the results of Section 10 and Section 11 to more general 2–dimensional Artin groups and ask several related questions.

Question 12.1. *Let A_Γ and $A_{\Gamma'}$ be two 2–dimensional Artin groups with finite outer automorphism groups. Suppose Γ has more than two vertices. Suppose A_Γ and $A_{\Gamma'}$ are quasi-isometric. Are they isomorphic?*

By Theorem 10.16, Question 12.1 reduces to the following question:

Question 12.2. *Let A_Γ and $A_{\Gamma'}$ be two 2–dimensional Artin groups with finite outer automorphism groups. Suppose that Γ has more than two vertices and suppose that $\mathcal{I}_\Gamma$ and $\mathcal{I}_{\Gamma'}$ are isomorphic as graphs. Are A_Γ and $A_{\Gamma'}$ isomorphic as groups?*

To answer this question, one need to generalize [Cri05, Proposition 23]. This question has a positive answer in the right-angled case [BKS08,Hua17a]. Even in the case of large-type Artin groups answering Question 12.1 and Question 12.2 will be interesting.

In view of [Cri05, Proposition 23], it is also natural to ask when an analogue of "Ivanov's theorem" holds for 2–dimensional Artin groups.

Question 12.3. *Let A_Γ be a 2–dimensional Artin group. Find conditions on Γ such that each automorphism of A_Γ induces a graph automorphism of $\mathcal{I}_\Gamma$ and vice versa.*

Of course, one can ask more questions of this kind by relaxing the assumptions of the results in Section 11.2. A question of different flavor, motivated by the work of Kim and Koberda [KK14], and Behrstock, Hagen and Sisto [BHS17a], is the following.

Question 12.4. *Suppose that A_Γ is 2–dimensional and that Γ is connected. Is $\mathcal{I}_\Gamma$ Gromov-hyperbolic? Does A_Γ have a Mazur-Minsky hierarchy structure with $\mathcal{I}_\Gamma$ being the "largest Gromov-hyperbolic space"?*

It is known that in the right-angled case, $\mathcal{I}_\Gamma$ is always a quasi-tree [KK14]. We suspect this is not true outside the right-angled world, since the intersection pattern of flats will be substantially more complicated.

Question 12.5. *Under what condition, the graph $\mathcal{I}_\Gamma$ is not a quasi-tree?*

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