Dismantlability of weakly systolic complexes and applications

Victor Chepoi[1] and Damian Osajda[2]

[1]Laboratoire d'Informatique Fondamentale,
Aix-Marseille Université,
Faculté des Sciences de Luminy,
F-13288 Marseille Cedex 9, France
chepoi@lif.univ-mrs.fr

[2]Universität Wien, Fakultät für Mathematik
Nordbergstraße 15, 1090 Wien, Austria
and
Instytut Matematyczny, Uniwersytet Wrocławski (on leave)
pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland
dosaj@math.uni.wroc.pl

Abstract. The main goal of this paper is proving the fixed point theorem for finite groups acting on weakly systolic complexes. As corollaries we obtain results concerning classifying spaces for the family of finite subgroups of weakly systolic groups and conjugacy classes of finite subgroups. As immediate consequences we get new results on systolic complexes and groups.
The fixed point theorem is proved by using a graph-theoretical tool — dismantlability. In particular we show that 1–skeleta of weakly systolic complexes, i.e. weakly bridged graphs, are dismantlable. On the way we show numerous characterizations of weakly bridged graphs and weakly systolic complexes.

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1. Introduction

In his seminal paper [Gro87], among many other results, Gromov gave a pretty combinatorial characterization of CAT(0) cubical complexes as simply connected cubical complexes in which the links of vertices are simplicial flag complexes. Based on this result, [Che00, Rol98] established a bijection between the 1–skeleta of CAT(0) cubical complexes and the median graphs, well-known in metric graph theory; cf. [BC08]. A similar combinatorial characterization of CAT(0) simplicial complexes having regular Euclidean simplices as cells seems to be out of reach. Nevertheless, Chepoi [Che00] characterized the bridged complexes (i.e., the simplicial complexes having bridged graphs as 1–skeleta) as the simply connected simplicial complexes in which the links of vertices are flag complexes without embedded 4– and 5–cycles; the bridged graphs are exactly the graphs which satisfy one of the basic features of

CAT(0) spaces: the balls around convex sets are convex. Bridged graphs have been introduced in [FJ87, SC83] as graphs without embedded isometric cycles of length greater than 3 and have been further investigated in several graph-theoretical and algebraic papers; cf. [AF88, BC96, Che97, Pol02, Pol00] and the survey [BC08]. Januszkiewicz-Świątkowski [JŚ06] and Haglund [Hag03] rediscovered this class of simplicial complexes (they call them *systolic complexes*) using them (and groups acting on them geometrically — *systolic groups*) fruitfully in the context of geometric group theory. Systolic complexes and groups turned out to be good combinatorial analogs of CAT(0) (nonpositively curved) metric spaces and groups; cf. [Hag03, JŚ06, Osa07, OP09, Prz08, Prz09].

One of the characteristic features of systolic complexes, related to the convexity of balls around convex sets, is the following $SD_n(\sigma^*)$ property introduced in [Osa10]: *if a simplex σ of a simplicial complex $\mathbf{X}$ is located in the sphere of radius $n+1$ centered at some simplex σ^* of $\mathbf{X}$, then the set of all vertices x such that $\sigma \cup \{x\}$ is a simplex and x has distance n to σ^* is a nonempty simplex σ_0 of $\mathbf{X}$.* Relaxing this condition, Osajda [Osa10] called a connected simplicial complex $\mathbf{X}$ *weakly systolic* if the property $SD_n(\sigma^*)$ holds whenever σ^* is a vertex (i.e., a 0–dimensional simplex) of $\mathbf{X}$. He further showed that this SD_n property is equivalent with the $SD_n(\sigma^*)$ property in which σ^* is a vertex and σ is a vertex or an edge (i.e., a 1–dimensional simplex) of $\mathbf{X}$. Finally it is shown in [Osa10] that weakly systolic complexes can be characterized as simply connected simplicial complexes satisfying some local combinatorial conditions, cf. also Theorem A below. This is analogous to the cases of $CAT(0)$ cubical complexes and systolic complexes. In graph-theoretical terms, the 1–skeleta of weakly systolic complexes (which we call *weakly bridged graphs*) satisfy the so-called triangle and quadrangle conditions [BC96], i.e., weakly bridged graphs are weakly modular. Median graphs and bridged graphs (i.e., the 1–skeleta of respectively CAT(0) cubical complexes and systolic complexes) are two other subclasses of weakly modular graphs. From the results of [Osa10] and of the present paper it follows that the properties of weakly systolic complexes resemble very much the properties of spaces of non-positive curvature.

The initial motivation of [Osa10] for introducing weakly systolic complexes was to exhibit a class of simplicial complexes with some kind of simplicial nonpositive curvature that will include the systolic complexes and some other classes of complexes appearing in the context of geometric group theory. As we noticed already, systolic complexes are weakly systolic. Moreover, for every simply connected locally 5–large cubical complex (i.e. CAT(-1) cubical complex [Gro87]) there exists a canonically associated simplicial complex, which is weakly systolic [Osa10]. In particular, the class of *weakly systolic groups*, i.e., groups acting geometrically by automorphisms on weakly systolic complexes, contains the class of CAT(-1) cubical groups and is therefore essentially bigger than the class of systolic groups; cf. [Osa07]. Other classes of weakly systolic groups are presented in [Osa10]. The ideas and results from [Osa10] permit the construction in [Osa13] of new examples of Gromov hyperbolic groups of arbitrarily large (virtual) cohomological dimension. Furthermore, Osajda [Osa10] and

Osajda-Świątkowski [OŚ10] provide new examples of high dimensional groups with interesting asphericity properties. On the other hand, as we will show below, the class of weakly systolic complexes seems also to appear naturally in the context of graph theory and has not been studied before from this point of view.

In this paper, we present further characterizations and properties of weakly systolic complexes and their 1–skeleta, weakly bridged graphs. Relying on techniques from graph theory we establish dismantlability of locally-finite weakly bridged graphs. This result is used to show some interesting nonpositive-curvature-like properties of weakly systolic complexes and groups (see [Osa10] for other properties of this kind). As corollaries, we also get new results about systolic complexes and groups. We conclude this introductory section with the formulation of our main results (see respective sections for all missing definitions and notations as well as for other related results).

We start with a characterization of weakly systolic complexes proved in Section 3:

Theorem A. *For a connected flag simplicial complex* $\mathbf{X}$ *the following conditions are equivalent:*

(a) $\mathbf{X}$ *is weakly systolic;*
(b) *the* 1*–skeleton of* $\mathbf{X}$ *is a weakly modular graph without induced* C_4*;*
(c) *the* 1*–skeleton of* $\mathbf{X}$ *is a weakly modular graph with convex balls;*
(d) *the* 1*–skeleton of* $\mathbf{X}$ *is a graph with convex balls in which any* C_5 *is included in a* 5*–wheel* W_5*;*
(e) $\mathbf{X}$ *is simply connected, satisfies the* $\widehat{W}_5$*–condition, and does not contain induced* C_4.

In Section 4 we prove the following result:

Theorem B. *Any LexBFS ordering of vertices of a locally-finite weakly systolic complex* $\mathbf{X}$ *is a dismantling ordering of its* 1*–skeleton.*

This result allows us to prove in Section 5 the following fixed point theorem concerning group actions:

Theorem C. *Let* G *be a finite group acting by simplicial automorphisms on a locally-finite weakly systolic complex* $\mathbf{X}$*. Then there exists a simplex* $\sigma \in \mathbf{X}$ *which is invariant under the action of* G*.*

The barycenter of an invariant simplex is a point fixed by G. An analogous theorem holds in the case of $CAT(0)$ spaces; cf. [BH99, Corollary 2.8]. As a direct corollary of Theorem C, we get the fixed point theorem for systolic complexes. This was conjectured by Januszkiewicz-Świątkowski (personal communication) and Wise [Wis03], and later was formulated in the collection of open questions [08, Conjecture 40.1 on page 115]. A partial result in the systolic case was proved by Przytycki [Prz08]. In fact, in Section 7, based on a result of Polat [Pol02] for bridged graphs, we prove an even stronger version of the fixed point theorem in this case.

The use of dismantlability of the underlying graph to prove the fixed point theorem for finite group actions is, due to our knowledge, a novelty brought by the current paper. It should be noticed, that there are well known examples of contractible, or even collapsible simplicial complexes admitting finite group actions without fixed points. Thus it seems that dismantlability is a right strengthening of those properties in the context of fixed point results. Subsequently, many other complexes studied in connection with group actions have dismantling properties. There, our approach gives new results concerning sets of fixed points; cf. e.g. [PS12].

There are several important group theoretical consequences of Theorem C. The first one follows directly from this theorem and [Prz08, Remarks 7.7&7.8].

Theorem D. *Let $k \geq 6$. Free products of k–systolic groups amalgamated over finite subgroups are k–systolic. HNN extensions of k–systolic groups over finite subgroups are k–systolic.*

The following result (Corollary 5.4 below) also has its $CAT(0)$ counterpart; cf. [BH99, Corollary 2.8]:

Corollary. *Let G be a weakly systolic group. Then G contains only finitely many conjugacy classes of finite subgroups.*

The next important consequence of the fixed point theorem concerns classifying spaces for proper group actions. Recall that if a group G acts properly on a space $\mathbf{X}$ such that the fixed point set for any finite subgroup of G is contractible (and therefore non-empty), then we say that $\mathbf{X}$ is a *model for $\underline{E}G$* — the classifying space for finite subgroups. If additionally the action is cocompact, then $\mathbf{X}$ is a *finite model for $\underline{E}G$*. A (finite) model for $\underline{E}G$ is in a sense a "universal" G–space (see [Lüc05] for details). The following theorem is a direct consequence of Theorem C and Proposition 6.6 below.

Theorem E. *Let G act properly by simplicial automorphisms on a finite dimensional weakly systolic complex $\mathbf{X}$. Then $\mathbf{X}$ is a finite dimensional model for $\underline{E}G$. If, moreover, the action of G on $\mathbf{X}$ is cocompact, then $\mathbf{X}$ is a finite model for $\underline{E}G$.*

As an immediate consequence we get an analogous result about $\underline{E}G$ for systolic groups. This was conjectured in [08, Chapter 40]. Przytycki [Prz09] showed that the Rips complex (with the constant at least 5) of a systolic complex is an $\underline{E}G$ space. Our result gives a systolic — and thus much nicer — model of $\underline{E}G$ in that case.

In the final Section 7 we present some further results about systolic complexes and groups. Besides a stronger version of Theorem C, we remark on another approach to this theorem initiated by Zawiślak [Zaw04] and Przytycki [Prz08]. In particular, our Proposition 7.5 proves their conjecture about round complexes; cf. [Zaw04, Conjecture 3.3.1] and [Prz08, Remark 8.1]. Finally, we show (cf. the end of Section 7) how our results about $\underline{E}G$ apply to the questions of existence of particular boundaries of systolic groups (and thus to the Novikov conjecture for systolic groups with torsion). This relies on earlier results of Osajda-Przytycki [OP09].

2. Preliminaries

2.1. **Graphs and simplicial complexes.** We continue with basic definitions used in this paper concerning graphs and simplicial complexes (see [Die10] for graph theoretical notions used in this paper). All graphs $G = (V, E)$ occurring here are undirected, connected, and without loops or multiple edges. A graph G is *complete* if any two of its vertices are connected by an edge. A graph $H = (V', E')$ is an *induced subgraph* of the graph G if $V' \subseteq V$, and $uv \in E'$ iff $uv \in E$. The *distance* $d(u, v)$ between two vertices u and v is the length of a shortest (u, v)–path, and the *interval* $I(u, v)$ between u and v consists of all vertices on shortest (u, v)–paths, that is, of all vertices (metrically) *between* u and v:

$$I(u, v) = \{x \in V : d(u, x) + d(x, v) = d(u, v)\}.$$

An induced subgraph of G (or the corresponding vertex set A) is called *convex* if it includes the interval of G between any of its vertices. By the *convex hull* $\mathrm{conv}(W)$ of $W \subseteq V$ in G we mean the smallest convex subset of V (or induced subgraph of G) that contains W. An *isometric subgraph* of G is an induced subgraph in which the distances between any two vertices are the same as in G. In particular, convex subgraphs are isometric. The *neighborhood* $N(x)$ of a vertex x consists of all vertices $y \neq x$ adjacent to x in G. The *ball* $B_r(x)$ of center x and radius $r \geq 0$ consists of all vertices of G at distance at most r from x. In particular, the unit ball $B_1(x)$ comprises x and the neighborhood $N(x)$ of x. The *sphere* $S_r(x)$ of center x and radius $r \geq 0$ consists of all vertices of G at distance exactly r from x. The ball $B_r(S)$ centered at a convex set S is the union of all balls $B_r(x)$ with centers x from S. The *sphere* $S_r(S)$ of center S and radius $r \geq 0$ consists of all vertices of G at distance exactly r from S.

A graph G is called *thin* if for any two nonadjacent vertices u, v of G any two neighbors of v in the interval $I(u, v)$ are adjacent. A graph G is *weakly modular* [BC96, BC08] if its distance function d satisfies the following conditions:

Triangle condition (T): for any three vertices u, v, w with $1 = d(v, w) < d(u, v) = d(u, w)$ there exists a common neighbor x of v and w such that $d(u, x) = d(u, v) - 1$.

Quadrangle condition (Q): for any four vertices u, v, w, z with $d(v, z) = d(w, z) = 1$ and $2 = d(v, w) \leq d(u, v) = d(u, w) = d(u, z) - 1$, there exists a common neighbor x of v and w such that $d(u, x) = d(u, v) - 1$.

An abstract *simplicial complex* $\mathbf{X}$ is a collection of sets (called *simplices*) such that $\sigma \in \mathbf{X}$ and $\sigma' \subseteq \sigma$ implies $\sigma' \in \mathbf{X}$. The *geometric realization* $|\mathbf{X}|$ of a simplicial complex is the polyhedral complex obtained by replacing every face σ of $\mathbf{X}$ by a "solid" regular simplex $|\sigma|$ such that realization commutes with intersection, that is, $|\sigma'| \cap |\sigma''| = |\sigma' \cap \sigma''|$ for any two simplices σ' and σ''. Then $|\mathbf{X}| = \bigcup\{|\sigma| : \sigma \in \mathbf{X}\}$. $\mathbf{X}$ is called *simply connected* if $|\mathbf{X}|$ is connected and if every continuous mapping of the 1–dimensional sphere S^1 into $|\mathbf{X}|$ can be extended to a continuous mapping of the disk D^2 with boundary S^1 into $|\mathbf{X}|$.

For a simplicial complex $\mathbf{X}$, denote by $V(\mathbf{X})$ and $E(\mathbf{X})$ the *vertex set* and the *edge set* of $\mathbf{X}$, namely, the set of all 0–dimensional and 1–dimensional simplices of $\mathbf{X}$. The pair $(V(\mathbf{X}), E(\mathbf{X}))$ is called the *(underlying) graph* or the 1*–skeleton* of $\mathbf{X}$ and is denoted by $G(\mathbf{X})$. Conversely, for a graph G one can derive a simplicial complex $\mathbf{X}(G)$ (the *clique complex* of G) by taking all complete subgraphs (cliques) as simplices of the complex. A simplicial complex $\mathbf{X}$ is a *flag complex* (or a *clique complex*) if any set of vertices is included in a face of $\mathbf{X}$ whenever each pair of its vertices is contained in a face of $\mathbf{X}$ (in the theory of hypergraphs this condition is called conformality). A flag complex can therefore be recovered by its underlying graph $G(\mathbf{X})$: the complete subgraphs of $G(\mathbf{X})$ are exactly the simplices of $\mathbf{X}$. The *link* of a simplex σ in $\mathbf{X}$, denoted $\mathrm{lk}(\sigma, \mathbf{X})$ is the simplicial complex consisting of all simplices σ' such that $\sigma \cap \sigma' = \emptyset$ and $\sigma \cup \sigma' \in \mathbf{X}$. For a simplicial complex $\mathbf{X}$ and a vertex v not belonging to $\mathbf{X}$, the *cone* with apex v and base $\mathbf{X}$ is the simplicial complex $v * \mathbf{X} = \mathbf{X} \cup \{\sigma \cup \{v\} : \sigma \in \mathbf{X}\}$.

For a simplicial complex $\mathbf{X}$ and any $k \geq 1$, the *Rips complex* $\mathbf{X}_k$ is a simplicial complex with the same set of vertices as $\mathbf{X}$ and with a simplex spanned by any subset $S \subseteq V(\mathbf{X})$ such that $d(u, v) \leq k$ in $G(\mathbf{X})$ for each pair of vertices $u, v \in S$ (i.e., S has diameter $\leq k$ in the graph $G(\mathbf{X})$); cf. e.g. [Gro87]. From the definition it immediately follows that the Rips complex of any complex is a flag complex. Alternatively, the Rips complex $\mathbf{X}_k$ can be viewed as the clique complex $\mathbf{X}(G^k(\mathbf{X}))$ of the kth power of the graph of $\mathbf{X}$ (the *kth power* G^k of a graph G has the same set of vertices as G and two vertices u, v are adjacent in G^k if and only if $d(u, v) \leq k$ in G).

All simplicial complexes occurring in this paper are flag complexes not containing infinite simplices. Analogously, we will consider only graphs not containing infinite complete subgraphs.

2.2. SD_n **property and weakly systolic complexes.** The following generalization of systolic complexes has been presented by Osajda [Osa10]. Let $\mathbf{X}$ be a flag simplicial complex and σ^* be a simplex of $\mathbf{X}$. Then $\mathbf{X}$ satisfies the $SD_n(\sigma^*)$ *property* if for each $i \leq n$ and each simplex σ located in the sphere $S_{i+1}(\sigma^*)$ the set $\sigma_0 := V(\mathrm{lk}(\sigma, \mathbf{X})) \cap B_i(\sigma^*)$ spans a non-empty simplex of $\mathbf{X}$ (SD stands for *simple descent* on balls). Systolic complexes are exactly the flag complexes which satisfy the $SD_n(\sigma^*)$ property for all simplices σ^* and all natural numbers n. On the other hand, the 5–wheel W_5 (see the definition at the beginning of Section 3) is an example of a (2–dimensional) simplicial complex which satisfies the $SD_1(\sigma^*)$ property for σ^* being any vertex or triangle but not for σ^* being a boundary edge. In view of this analogy and of subsequent results, we define a *weakly systolic* complex to be a connected flag simplicial complex $\mathbf{X}$ which satisfies the $SD_n(v)$ property for all vertices $v \in V(\mathbf{X})$ and for all natural numbers n. We will also define a *weakly bridged* graph to be the underlying graph of a weakly systolic complex. It can be shown (cf. Theorem 3.1) that $\mathbf{X}$ is a weakly systolic complex if for each vertex v and every i it satisfies the following two conditions:

Vertex condition (V): for every vertex $w \in S_{i+1}(v)$, the intersection $V(\mathrm{lk}(w, \mathbf{X})) \cap B_i(v)$ is a single simplex;

Edge condition (E): for every edge $e \in S_{i+1}(v)$, the intersection $V(\mathrm{lk}(e, \mathbf{X})) \cap B_i(v)$ is nonempty.

In fact, this is the original definition of a weakly systolic complex given in [Osa10]. Notice that these two conditions imply that weakly systolic complexes are exactly the flag complexes whose underlying graphs are thin and satisfy the triangle condition.

2.3. **Dismantlability of graphs and LC-contractibility of complexes.** Let $G = (V, E)$ be a graph and u, v two vertices of G such that any neighbor of v (including v itself) is also a neighbor of u, i.e. $B_1(v) \subseteq B_1(u)$. Then there is a retraction of G to $G - v$ taking v to u. Following [HN04], we call this retraction a *fold* and we say that v is *dominated* by u (if $B_1(v) \subsetneq B_1(u)$, then we say that v is *strictly dominated* by u). A finite graph G is *dismantlable* if it can be reduced, by a sequence of folds, to a single vertex. In other words, an n–vertex graph $G = (V, E)$ is dismantlable if its vertices can be ordered $v_1, \ldots, v_n$ so that for each vertex $v_i, 1 \leq i < n$, there exists another vertex v_j with $j > i$, such that $B_1(v_i) \cap V_i \subseteq B_1(v_j) \cap V_i$, where $V_i := \{v_i, v_{i+1}, \ldots, v_n\}$. This order is called a *dismantling order*. We now consider the analogue of dismantlability for a simplicial complex $\mathbf{X}$ investigated in the papers [CY07, Mat08]. A vertex v of $\mathbf{X}$ is *LC-removable* if $\mathrm{lk}(v, \mathbf{X})$ is a cone. If v is an LC-removable vertex of $\mathbf{X}$, then $\mathbf{X} - v := \{\sigma \in \mathbf{X} : v \notin \sigma\}$ is obtained from $\mathbf{X}$ by an *elementary LC-reduction* (link-cone reduction) [Mat08]. Then $\mathbf{X}$ is called *LC-contractible* [CY07] if there is a sequence of elementary LC-reductions transforming $\mathbf{X}$ to one vertex. For flag simplicial complexes, the LC-contractibility of $\mathbf{X}$ is equivalent to dismantlability of its graph $G(\mathbf{X})$ because an LC-removable vertex v is dominated by the apex of the cone $\mathrm{lk}(v, \mathbf{X})$ and vice versa the link of any dominated vertex v is a cone having the vertex dominating v as its apex. LC-contractible simplicial complexes are collapsible (see [CY07, Corollary 6.5]).

The simplest algorithmic way to order the vertices of a locally-finite graph is to apply the *Breadth-First Search* (BFS) starting from the root vertex (base point) u. We number with 1 the vertex u and put it on the initially empty queue. We repeatedly remove the vertex v at the head of the queue and consequently number (in an arbitrary order) and place onto the queue all still unnumbered neighbors of v. BFS constructs a spanning tree T_u of G with the vertex u as a root. Then a vertex v is the *father* in T_u of any of its neighbors w in G included in the queue when v is removed (notation $f(w) = v$). Notice that the distance from any vertex v to the root u is the same in G and in T_u. Another method to order the vertices of a graph is the *Lexicographic Breadth-First Search* (LexBFS) proposed by Rose-Tarjan-Lueker [RTL76]. According to LexBFS, the vertices of a graph G are numbered in decreasing order. The *label* $L(w)$ of an unnumbered vertex w is the list of its numbered neighbors. As the next vertex to be numbered, select the vertex with the lexicographic largest label, breaking ties arbitrarily. As in case of BFS, we number and remove the vertex v at the head of the queue and consequently label according to the lexicographic order and place onto the queue all still unnumbered neighbors of v. LexBFS is a particular instance of BFS, i.e., every ordering produced by LexBFS can also be generated by BFS.

Anstee-Farber [AF88] established that bridged graphs are dismantlable. Chepoi [Che97] noticed that any order of a bridged graph returned by BFS is a dismantling order. Namely, he showed a stronger result: *for any two adjacent vertices v_i, v_j with $i < j$, their fathers $f(v_i), f(v_j)$ either coincide or are adjacent and moreover $f(v_j)$ is adjacent to v_i.* Polat [Pol02, Pol00] defined dismantlability and BFS for arbitrary (not necessarily locally-finite) graphs and extended the results of [AF88, Che97] to all bridged graphs.

2.4. **Group actions on simplicial complexes.** Let G be a group acting by automorphisms on a simplicial complex $\mathbf{X}$. By $\mathrm{Fix}_G\mathbf{X}$ we denote the *fixed point set* of the action of G on $\mathbf{X}$, i.e. $\mathrm{Fix}_G\mathbf{X} = \{x \in \mathbf{X} |\, Gx = \{x\}\}$. Recall that the action is *cocompact* if the orbit space $G\backslash\mathbf{X}$ is compact. The action of G on a locally-finite simplicial complex $\mathbf{X}$ is *properly discontinuous* if stabilizers of simplices are finite. Finally, the action is *geometric* (or *G acts geometrically* on $\mathbf{X}$) if it is cocompact and properly discontinuous.

3. Characterizations of weakly systolic complexes

We continue with the characterizations of weakly systolic complexes and their underlying graphs; some of those characterizations have been presented also in [Osa10]. We denote by C_k a k–cycle and by W_k a k–wheel, i.e., a k–cycle $x_1, \ldots, x_k$ plus a central vertex c adjacent to all vertices of C_k. W_k can also be viewed as a 2–dimensional simplicial complex consisting of k triangles $\sigma_1, \ldots, \sigma_k$ sharing a common vertex c and such that σ_i and σ_j intersect in an edge $x_i c$ exactly when $|j - i| = 1 \pmod k$. In other words, $\mathrm{lk}(c, \mathbf{W}_k) = C_k$, i.e. W_k is a cone over C_k. By $\widehat{W}_k$ we denote a k–wheel W_k plus a triangle ax_ix_{i+1} for some $i < k$ (we suppose that $a \neq c$ and that a is not adjacent to any other vertex of W_k). We continue with a condition which basically characterizes weakly systolic complexes among simply connected flag simplicial complexes:

$\widehat{W}_5$–condition: *for any $\widehat{W}_5$, there exists a vertex $v \notin \widehat{W}_5$ such that $\widehat{W}_5$ is included in* $\mathrm{lk}(\mathrm{v}, \mathbf{X})$, *i.e., v is adjacent in $G(\mathbf{X})$ to all vertices of $\widehat{W}_5$* (see Fig. 1).

Theorem 3.1 (Characterizations)**.** *For a connected flag simplicial complex $\mathbf{X}$ the following conditions are equivalent:*

- (i) $\mathbf{X}$ *is weakly systolic;*
- (ii) $\mathbf{X}$ *satisfies the vertex condition (V) and the edge condition (E);*
- (iii) $G(\mathbf{X})$ *is a weakly modular thin graph;*
- (iv) $G(\mathbf{X})$ *is a weakly modular graph without induced C_4;*
- (v) $G(\mathbf{X})$ *is a weakly modular graph with convex balls;*
- (vi) $G(\mathbf{X})$ *is a graph with convex balls in which any C_5 is included in a 5–wheel W_5;*
- (vii) $\mathbf{X}$ *is simply connected, satisfies the $\widehat{W}_5$–condition, and does not contain induced C_4.*

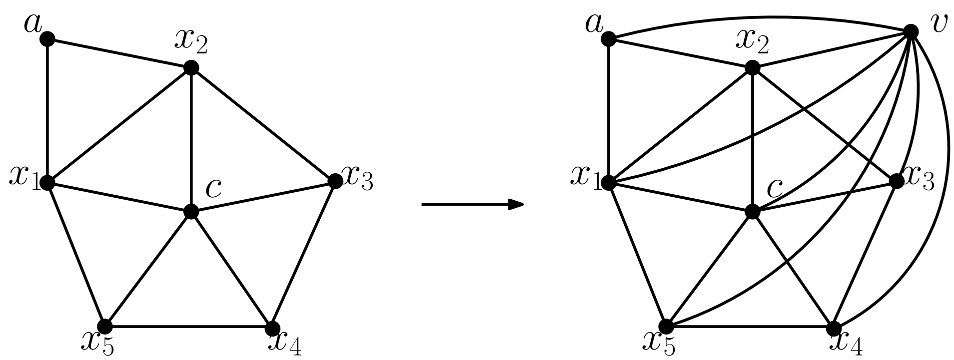

FIGURE 1. The $\widehat{W}_5$–condition

Proof. First we show that the conditions (i) through (v) are equivalent and then we show that these conditions are equivalent to (vi) and to (vii). The implications (i)⇒(ii) and (iii)⇒(iv) are obvious.

(ii)⇒(iii): The condition (V) implies that all vertices of $I(u,v)$ adjacent to v are pairwise adjacent, i.e., that $G(\mathbf{X})$ is thin. On the other hand, from the condition (E) we conclude that if $1 = d(v,w) < d(u,v) = d(u,w) = i+1$, then v and w have a common neighbor x in the sphere $S_i(u)$, implying the triangle condition. Finally, in thin graphs the quadrangle condition is automatically satisfied if the triangle condition is satisfied. This shows that $G(\mathbf{X})$ is a weakly modular thin graph.

(iv)⇒(v): Let $B_i(u)$ be any ball in $G(\mathbf{X})$. Since $G(\mathbf{X})$ is weakly modular and $B_i(u)$ is a connected subgraph, to show that $B_i(u)$ is convex it suffices to show that $B_i(u)$ is locally-convex, i.e., *if* $x,y \in B_i(u)$ *and* $d(x,y) = 2$, *then* $I(x,y) \subseteq B_k(u)$; cf. [Che89, Theorem 7(a)] (compare also [BC00, Lemma 1]). Suppose by way of contradiction that $z \in I(x,y) \setminus B_i(u)$. Then necessarily $d(x,u) = d(y,u) = i$ and $d(z,u) = i+1$. Applying the quadrangle condition, we infer that there exists a vertex z' adjacent to x and y at distance $i-1$ from u. As a result, the vertices x,z,y,z' induce a forbidden 4–cycle, a contradiction.

(v)⇒(i): Pick a simplex σ in the sphere $S_{i+1}(u)$. Denote by σ_0 the set of all vertices $x \in S_i(u)$ such that $\sigma \cup \{x\}$ is a simplex of $\mathbf{X}$. Since the balls of $G(\mathbf{X})$ are convex, necessarily any two vertices of σ_0 are adjacent. Thus σ_0 and $\sigma \cup \sigma_0$ induce complete subgraphs of $G(\mathbf{X})$. Since $\mathbf{X}$ is a flag complex, σ_0 and $\sigma \cup \sigma_0$ are simplices. Notice that obviously $\sigma' \subseteq \sigma_0$ holds for any other simplex $\sigma' \subseteq S_i(u)$ such that $\sigma \cup \sigma' \in \mathbf{X}$. Therefore, to establish the $SD_i(u)$ property it remains to show that σ_0 is non-empty. This is obviously true if σ is a vertex. Thus we suppose that σ contains at least two vertices. Let x be a vertex of $S_i(u)$ which is adjacent to the maximum number of vertices of σ. Since $G(\mathbf{X})$ is weakly modular and σ is contained in $S_{i+1}(u)$, the vertex x must be adjacent to at least two vertices of σ. Suppose by way of contradiction that x is not adjacent to a vertex $v \in \sigma$. Pick any neighbor w of x in σ. By the triangle condition, there exists a vertex $y \in S_i(u)$ adjacent to v and w. Since

w is adjacent to $x, y \in S_i(u)$ and $w \in S_{i+1}(u)$, the convexity of $B_i(u)$ implies that x and y are adjacent. Pick any other vertex w' of σ adjacent to x. Since x is not adjacent to v and $G(\mathbf{X})$ does not contain induced 4–cycles, the vertices y and w' must be adjacent. Hence, y is adjacent to $v \in \sigma$ and to all neighbors of x in σ, contrary to the choice of x. Thus x is adjacent to all vertices of σ, i.e., $\sigma_0 \neq \emptyset$. This shows that $\mathbf{X}$ satisfies the $SD_n(u)$ property.

(v)$\Rightarrow$(vi): Pick a 5–cycle induced by the vertices x_1, x_2, x_3, x_4, x_5. Since $d(x_4, x_1) = d(x_4, x_2) = 2$, by the triangle condition there exists a vertex y adjacent to x_1, x_2, and x_4. Since $G(\mathbf{X})$ does not contain induced 4–cycles, necessarily y must be also adjacent to x_3 and x_5, yielding a 5–wheel.

(vi)$\Rightarrow$(vii): That $\mathbf{X}$ does not contain induced 4–cycles follows from the convexity of balls. To show that the flag complex $\mathbf{X}$ is simply connected, it is enough to show that every cycle in the 1–skeleton of $\mathbf{X}$ (seen as a topological loop) can be freely homotoped to a given vertex u (seen as a constant loop). By contradiction, let A be the set of cycles in $G(\mathbf{X})$, which are not freely homotopic to u, and assume that A is non-empty. For a cycle $C \in A$, let $r(C)$ denote the maximal distance $d(x, u)$ of a vertex x of C from the basepoint u. Clearly $r(C) \geq 2$ for any cycle $C \in A$ (otherwise C would be null-homotopic). Let $B \subseteq A$ be the set of cycles C with minimal $r(C)$ among cycles in A. Let $r := r(C)$ for some $C \in B$. Let $D \subseteq B$ be the set of cycles having minimal number e of edges in the r–sphere around u, i.e., with both endpoints at distance r from u. Further, let $E \subseteq D$ be the set of cycles with the minimal number m of vertices at distance r from u.

Consider a cycle $C = (x_1, x_2, \ldots, x_k, x_1) \in E$. We can assume without loss of generality that $d(x_2, u) = r$. We distinguish two cases.

Case 1: $d(x_1, u) = d(x_3, u) = r - 1$. Then, by the convexity of the ball $B_{r-1}(u)$, we have that x_1 and x_3 are adjacent. Thus the cycle $C' = (x_1, x_3, \ldots, x_k, x_1)$ is homotopic to C by a homotopy through the triangle $x_1x_2x_3$. Thus C' belongs to D and the number of its vertices at distance r from v is equal to $m - 1$. This contradicts the minimality choice of m.

Case 2: $d(x_1, u) = r$ or $d(x_3, u) = r$. Assume without loss of generality that $d(x_3, u) = r$. For $i \in \{2, 3\}$, let x_i' be a vertex in $B_{r-1}(u)$ adjacent to x_i. Since the path (x_2', x_2, x_3, x_3') has length 3, by the convexity of the ball $B_{r-1}(u)$, we have $d(x_2', x_3') \leq 2$. If $x_2' = x_3'$, then we set $C' = (x_1, x_2, x_2', x_3, \ldots, x_1)$. If $d(x_2', x_3') = 1$, then we set $C' = (x_1, x_2, x_2', x_3', x_3, \ldots, x_1)$. Observe, that in that case either x_2 is adjacent to x_3' or x_3 is adjacent to x_2'. In particular, the 4–cycle $(x_2, x_3, x_3', x_2', x_2)$ is homotopically trivial in $\mathbf{X}$. If $d(x_2', x_3') = 2$, then we set $C' = (x_1, x_2, x_2', x', x_3', x_3, \ldots, x_1)$, where $x \in B_{r-1}(u)$ is adjacent to x_2' and x_3'. Observe that in this case the 5–cycle $(x_2, x_3, x_3', x', x_2', x_2)$ is either not a full subcomplex or is included in a 5–wheel. In any case it is homotopically trivial in $\mathbf{X}$.

In each of the three cases above, the cycle C is freely homotopic to C' by a homotopy through, respectively, a triangle, a triangulated square, or a triangulated pentagon. Moreover, $C' \in B$. The number of edges of C' lying on the r–sphere around u is less than e (we removed the edge x_2x_3). This contradicts the choice of the number e. In both Cases 1 and 2 we get contradiction. It follows that the set A is empty and hence $\mathbf{X}$ is simply connected.

Finally, pick an extended 5–wheel $\widehat{W}_5$: let x_1, x_2, x_3, x_4, x_5 be the vertices of the 5–cycle, c be the center of the 5–wheel, and x_1, x_2, a be the vertices of the pendant triangle. Since x_3 and x_5 are not adjacent and the balls of $G(\mathbf{X})$ are convex, necessarily $d(a, x_4) = 2$. Let u be a common neighbor of a and x_4. If u is adjacent to one of the vertices x_2 and x_3, then to avoid induced 4–cycles (forbidden by the convexity of balls in $G(\mathbf{X})$), u will be also adjacent to the second vertex and to c. But if u is adjacent to c, then it will be adjacent to x_1 and therefore to x_5 as well. Hence, in this case u will be adjacent to all vertices x_1, x_2, x_3, x_4, x_5, and c, and we are done. So, we can suppose that u is not adjacent to any one of the vertices x_1, x_2, x_3, x_5, and c. As a result, we obtain two 5–cycles induced by the vertices a, x_2, x_3, x_4, u and a, x_1, x_5, x_4, u. Each of these cycles extends to a 5–wheel. Let v be the center of the 5–wheel extending the first cycle. To avoid a 4–cycle induced by the vertices x_2, v, x_4, c, the vertices v and c must be adjacent. Subsequently, to avoid a 4–cycle induced by the vertices c, v, a, x_1, the vertices v and x_1 must be adjacent. Finally, to avoid a 4–cycle induced by x_1, v, x_4, x_5, the vertices v and x_5 must be adjacent. In this way, we deduce that v is adjacent to all six vertices of $\widehat{W}_5$, establishing the $\widehat{W}_5$–condition.

(vii)⇒(iv): To prove this implication, as in [Che00], we will use minimal disk diagrams. Let $\mathcal{D}$ and $\mathbf{X}$ be two simplicial complexes. A map $\varphi : V(\mathcal{D}) \to V(\mathbf{X})$ is called *simplicial* if $\varphi(\sigma) \in \mathbf{X}$ for all $\sigma \in \mathcal{D}$. If $\mathcal{D}$ is a planar triangulation (i.e. the 1–skeleton of $\mathcal{D}$ is an embedded planar graph whose all interior 2–faces are triangles) and $C = \varphi(\partial\mathcal{D})$, then $(\mathcal{D}, \varphi)$ is called a *singular disk diagram* (or Van Kampen diagram) for C (for more details see [LS01, Chapter V]). According to Van Kampen's lemma ([LS01], pp.150–151), for every cycle C of a simply connected simplicial complex one can construct a singular disk diagram. A singular disk diagram with no cut vertices (i.e., its 1–skeleton is 2–connected) is called a *disk diagram.* A *minimal (singular) disk* for C is a (singular) disk diagram $\mathcal{D}$ for C with a minimum number of 2–faces. This number is called the *(combinatorial) area* of C and is denoted $\mathrm{Area}(C)$. The minimal disk diagrams $(\mathcal{D}, \varphi)$ of simple cycles C in 1–skeleta of simply connected simplicial complexes have the following properties [Che00]: (1) φ bijectively maps $\partial\mathcal{D}$ to C and (2) the image of a 2–simplex of $\mathcal{D}$ under φ is a 2–simplex, and two adjacent 2–simplices of $\mathcal{D}$ have distinct images under φ.

Let C be a simple cycle in the underlying graph $G(\mathbf{X})$ of a flag simplicial complex $\mathbf{X}$ satisfying the condition (vii).

Claim 1: *If C has length 5, then the minimal disk diagram for C is a 5–wheel. If the length of C is not 5, then C admits a minimal disk diagram $\mathcal{D}$ which is a systolic complex, i.e., a plane triangulation whose all inner vertices have degrees ≥ 6.*

Proof of Claim 1: First we show that any minimal disk diagram $\mathcal{D}$ of C does not contain interior vertices of degrees 3 and 4. Let x be any interior vertex of $\mathcal{D}$. Let $x_1, \ldots, x_k$ be the cyclically ordered neighbors of x and let $\sigma_1, \sigma_2, \ldots, \sigma_k$ be the faces incident to x, where

$\sigma_i = xx_ix_{i+1(mod\ k)}$ $(i = 1, \ldots, k)$. Trivially, $k \geq 3$. Suppose by way of contradiction that $k \leq 4$. By properties of minimal disk diagrams, $\varphi(\sigma_1), \ldots, \varphi(\sigma_k)$ are distinct 2–simplices of $\mathbf{X}$.

Case 1: $k = 3$. Then the 2–simplices $\varphi(\sigma_1), \varphi(\sigma_2), \varphi(\sigma_3)$ of $\mathbf{X}$ intersect in $\varphi(x)$ and pairwise share an edge of $\mathbf{X}$. Since $\mathbf{X}$ is flag, they are contained in a 3–simplex of $\mathbf{X}$. This implies that $\delta = \varphi(x_1)\varphi(x_2)\varphi(x_3)$ is a 2–face of $\mathbf{X}$. Let $\mathcal{D}'$ be a disk triangulation obtained from $\mathcal{D}$ by deleting the vertex x and the triangles $\sigma_1, \sigma_2, \sigma_3$, and adding the 2–simplex $x_1x_2x_3$. The map $\varphi : V(\mathcal{D}') \to V(\mathbf{X})$ is simplicial, because it maps $x_1x_2x_3$ to δ. Therefore $(\mathcal{D}', \varphi)$ is a disk diagram for C, contrary to the minimality choice of $\mathcal{D}$.

Case 2: $k = 4$. Since two adjacent 2–simplices of $\mathcal{D}$ have distinct images under φ, the cycle $C' = (x_1, x_2, x_3, x_4, x_1)$ is sent to a 4–cycle $\varphi(C')$ of $\mathrm{lk}(\varphi(x), \mathbf{X})$. Since $G(\mathbf{X})$ does not contain induced 4–cycles, two opposite vertices of $\varphi(C')$, say $\varphi(x_1)$ and $\varphi(x_3)$, are adjacent. Consequently, since $\mathbf{X}$ is flag, $\delta' = \varphi(x_1)\varphi(x_3)\varphi(x_2)$ and $\delta'' = \varphi(x_1)\varphi(x_3)\varphi(x_4)$ are 2–faces of $\mathbf{X}$. Let $\mathcal{D}'$ be a disk triangulation obtained from $\mathcal{D}$ by deleting the vertex x and the triangles $\sigma_i (i = 1, \ldots, 4)$, and adding the 2–simplices $\sigma' = x_1x_3x_2$ and $\sigma'' = x_1x_3x_4$. The map φ remains simplicial, since it sends σ', σ'' to δ', δ'', respectively, contrary to the minimality choice of $\mathcal{D}$.

This establishes that the degree of each interior vertex x of any minimal disk diagram is ≥ 5. Suppose now additionally that $\mathcal{D}$ is a minimal disk diagram for C having a minimum number of inner vertices of degree 5. We will denote the vertices of $\mathcal{D}$ and their images in $\mathbf{X}$ under φ by the same symbols but specifying each time their position. Let x be any interior vertex of $\mathcal{D}$ of degree 5 and let $x_1, \ldots, x_5$ be the neighbors of x. If $C = (x_1, x_2, x_3, x_4, x_5, x_1)$ then we are done because $\mathcal{D}$ is a 5–wheel. If $C \neq (x_1, x_2, x_3, x_4, x_5, x_1)$ then one of the edges of the 5–cycle $(x_1, x_2, x_3, x_4, x_5, x_1)$, say x_1x_2, belongs in $\mathcal{D}$ to the second triangle $x_1x_2x_6$. The minimality of $\mathcal{D}$ implies that $x, x_1, x_2, x_3, x_4, x_5, x_6$ induce in $\mathbf{X}$ a $\widehat{W}_5$ or that x and x_6 are adjacent in $\mathbf{X}$. In the first case, by the $\widehat{W}_5$–condition, there exists a vertex y of $\mathbf{X}$ which is adjacent to all vertices of this $\widehat{W}_5$. Let $\mathcal{D}'$ be a disk triangulation obtained from $\mathcal{D}$ by deleting the vertex x and the five triangles incident to x as well as the triangle $x_1x_2x_6$ and replacing them by the six triangles of the resulting 6–wheel centered at y (we call this operation a *flip*). In the second case, let $\mathcal{D}'$ be a disk triangulation obtained from $\mathcal{D}$ by deleting the triangles xx_1x_2 and $x_1x_2x_6$ and replacing them by the triangles xx_1x_6 and xx_2x_6. In both cases, the resulting map φ remains simplicial. $\mathcal{D}'$ has the same number of triangles as $\mathcal{D}$, therefore $\mathcal{D}'$ is also a minimal disk diagram for C. The flip replaces in the first case the vertex x of degree 5 by the vertex y of degree 6. In the second case, it increases the degree of x from 5 to 6. In both cases, it also increases the degree of x_6 by 1 and preserves the degrees of all other vertices except the vertices x_1 and x_2, whose degrees decrease by 1. Since, by the minimality choice of $\mathcal{D}$, the disk diagram $\mathcal{D}'$ has at least as many inner vertices of degree 5 as $\mathcal{D}$, necessarily at least one of the vertices x_1, x_2, say x_1, is an inner vertex of degree at most 6 of $\mathcal{D}$. If the degree of x_1 in $\mathcal{D}$ is 5, then in $\mathcal{D}'$ the degree of x_1 will be 4, which is impossible by what has been shown above because $\mathcal{D}'$ is also a minimal disk diagram and x_1 is an interior vertex of $\mathcal{D}'$. Hence the degree of x_1 in $\mathcal{D}$ is 6 and its neighbors constitute an induced (in $\mathcal{D}$) 6–cycle $(x_6, x_2, x, x_5, u, v, x_6)$.

Case 1: x and x_6 are not adjacent in $\mathbf{X}$. Since $\mathbf{X}$ does not contain induced C_4 and the minimal disk diagrams for C do not contain interior vertices of degree 3 and 4, it can be easily shown that the images in $\mathbf{X}$ of the vertices $x_5, y, x_6, v, u, x_1, x_4$ induce a $\widehat{W}_5$ constituted by the 5–wheel centered at x_1 and the triangle x_4yx_5. By the $\widehat{W}_5$–condition, there exists a vertex z of $\mathbf{X}$ which is adjacent to all vertices of $\widehat{W}_5$. If z is adjacent in $\mathbf{X}$ with all vertices of the 7–cycle $(u, v, x_6, x_2, x_3, x_4, x_5, u)$, then replacing in $\mathcal{D}$ the 9 triangles incident to x and x_1 by the 7 triangles of $\mathbf{X}$ incident to z, we will obtain a disk diagram $\mathcal{D}''$ for C having less triangles than $\mathcal{D}$, contrary to the minimality of $\mathcal{D}$. Therefore z is different from x and is not adjacent to one of the vertices x_2, x_3. Since x_1 and x_4 are not adjacent and both x and z are adjacent to x_1, x_4, to avoid an induced C_4 we conclude that z is adjacent in $\mathbf{X}$ to x. If z is not adjacent to x_2, then, since x and x_6 are not adjacent, we will obtain a C_4 induced by x, z, x_6, x_2. Thus z is adjacent to x_2, and therefore z is not adjacent to x_3. Since both z and x_3 are adjacent to nonadjacent vertices x_2 and x_4, we will obtain a C_4 induced by z, x_2, x_3, x_4. This contradiction shows that the degree of x_1 in $\mathcal{D}$ is at least 7.

Case 2: x and x_6 are adjacent in $\mathbf{X}$. Again, using the fact that the minimal disk diagrams for C do not contain interior vertices of degree 3 and 4, the fact that $\mathbf{X}$ does not contain induced C_4, it can be easily shown that $d(x, u) = 2$. Therefore the vertices $x_1, x_2, x_3, x_4, x_5, x, u$ induce a $\widehat{W}_5$ constituted by the 5–wheel centered at x and the triangle x_1ux_5. Thus, by the $\widehat{W}_5$–condition, there exists a vertex $y' \neq x$ containing $\widehat{W}_5$ in its link. Then considering the minimal disk diagram obtained by the flip exchanging x and y' we conclude that the vertices u, v, x_6, x_2, y', x_1 induce a 5–wheel. Together with the vertex x_3 they induce a $\widehat{W}_5$, so that, by $\widehat{W}_5$–condition, there exists a vertex z adjacent to all the vertices $u, v, x_6, x_2, y', x_1, x_3$. If z is adjacent to x_4 and x_5 then we get a disk diagram for C having less triangles than $\mathcal{D}$, which contradicts the minimality of $\mathcal{D}$. If z is not adjacent to one of the vertices x_4, x_5 then we also get a contradiction arguing as in Case 1. Therefore, in our case the degree of x_1 in $\mathcal{D}$ is also at least 7. This final contradiction shows that all interior vertices of $\mathcal{D}$ have degrees ≥ 6, establishing Claim 1.

From Claim 1 we deduce that any simple cycle C of the underlying graph of $\mathbf{X}$ admits a minimal disk diagram $\mathcal{D}$ which is either a 5–wheel or a systolic plane triangulation. We will refer to a degree two boundary vertex v of $\mathcal{D}$ as a *corner of first type* and to a degree three boundary vertex v of $\mathcal{D}$ as a *corner of second type*. In the first case, the two neighbors of v are adjacent. In the second case, v and its neighbors in $\partial\mathcal{D}$ are adjacent to the third neighbor of v. If $\mathcal{D}$ is a 5–wheel then it has five corners of second type. Otherwise $\mathcal{D}$ is a systolic plane triangulation and we can use the Gauss-Bonnet formula "sum over interior vertices of six minus degree plus sum over boundary vertices of four minus degree equals six times Euler characteristic"; see [LS01, Ch. V.3]. From this formula we infer that $\mathcal{D}$ contains at least three corners, and if $\mathcal{D}$ has exactly three corners then they are all of first type. Furthermore, if $\mathcal{D}$ contains exactly four corners, then at least two of them are corners of first type.

Claim 2: $G(\mathbf{X})$ *is weakly modular, i.e.* $G(\mathbf{X})$ *satisfies the triangle and quadrangle conditions.*

Proof of Claim 2: To verify the triangle condition, let u, v, w be three vertices with $1 = d(v,w) \leq d(u,v) = d(u,w) = k$. We claim that if $I(u,v) \cap I(u,w) = \{u\}$, then $k = 1$. Suppose not. Pick two shortest paths P' and P'' joining the pairs u, v and u, w, respectively, such that the cycle C composed of P', P'' and the edge vw has minimal combinatorial area Area(C) among all cycles constituted by the edge vw and shortest paths connecting u with v and w (the choice of v, w implies that C is a simple cycle). Let $\mathcal{D}$ be a minimal disk diagram for C satisfying Claim 1. Then either $\mathcal{D}$ has a corner x different from u, v, w or the vertices u, v, w are the only corners of $\mathcal{D}$. In the second case, u, v, w are all three corners of first type, therefore the two neighbors of v in C will be adjacent. This means that w will be adjacent to the neighbor of v in P', contrary to $I(u,v) \cap I(u,w) = \{u\}$. Thus we can assume that a corner x exists and x is not one of u, v or w. Without loss of generality we can assume x is on the path P'. Let y and z be its neighbors on P'. Note that x cannot be of first type, since otherwise y and z are adjacent, contrary to the assumption that P' is a shortest path. Thus x is of the second type and there is a vertex p of $\mathcal{D}$ adjacent to x, y, z. If we replace in P' the vertex x by p, we will obtain a new shortest path between u and v. Together with P'' and the edge vw this path forms a cycle C' whose area is strictly smaller than Area(C), contrary to the choice of C. This establishes the triangle condition.

To verify the quadrangle condition, suppose by way of contradiction that we can find distinct vertices u, v, w, z such that $v, w \in I(u,z)$ are neighbors of z and $I(u,v) \cap I(u,w) = \{u\}$, however u is not adjacent to v and w. Again, select two shortest paths P' and P'' between u, v and u, w, respectively, so that the cycle C composed of P', P'' and the edges vz and zw has minimum area. Choose a minimal disk $\mathcal{D}$ of C as in Claim 1. From the initial hypothesis concerning the vertices u, v, w, z we deduce that $\mathcal{D}$ has at most one corner of first type located at u. Hence $\mathcal{D}$ contains at least four corners of second type. Since one corner x is distinct from u, v, w, z, then proceeding in the same way as in the triangle condition case, we will obtain a contradiction with the choice of the paths P', P''. This shows that u is adjacent to v, w, establishing the quadrangle condition. This concludes the proof of Claim 2.

By Claim 2 the graph $G(\mathbf{X})$ is weakly modular. On the other hand, by condition (vii) $G(\mathbf{X})$ does not contain induced C_4. This concludes the proof of the implication (vii)$\Rightarrow$(iv) and of the theorem. $\square$

In the analysis of his construction of locally homogeneous graphs H having a given regular graph of girth ≥ 6 (i.e., 6–large) as a link of each vertex of H, Weetman [Wee94] introduced *quasitrees* as the graphs $G = (V, E)$ satisfying the following two conditions for each vertex v : (F1) each vertex $x \in S_{i+1}(v)$ has one or two adjacent neighbors in $S_i(v)$; (F2) any two adjacent vertices $x, y \in S_{i+1}(v)$ have a common neighbor $z \in S_i(v)$. It can be easily seen that (F2) is a reformulation of the edge condition (E) (alias the triangle condition). On the other hand, (F1) is a particular case of the vertex condition (V). From Theorem 3.1(ii) we immediately obtain the following observation:

Corollary 3.2. *The simplicial complexes derived from quasitrees are weakly systolic. In particular, quasitrees are weakly bridged graphs.*

The 5–wheel is an example of a quasitree which is not a bridged graph, thus not all simplicial complexes derived from quasitrees are systolic.

4. Dismantlability of weakly bridged graphs

In this section, we show that the underlying graphs of weakly systolic complexes are dismantlable and that a dismantling order can be obtained using LexBFS. Then we use this result to deduce several consequences about collapsibility of weakly systolic complexes and fixed simplices. Other consequences of dismantling are given in subsequent sections.

Theorem 4.1 (LexBFS dismantlability)**.** *Any LexBFS ordering of a locally-finite weakly bridged graph G is a dismantling ordering.*

Proof. We will establish the result for finite weakly bridged graphs. The proof in the locally-finite case is completely similar. Let $v_n, \ldots, v_1$ be the total order returned by the LexBFS starting from the basepoint $u = v_n$. Let G_i be the subgraph of G induced by the vertices $v_n, \ldots, v_i$. For a vertex $v \neq u$ of G, denote by $f(v)$ its father in the LexBFS tree T_u, by $L(v)$ the list of all neighbors of v labeled before v, and by $\alpha(v)$ the number of v (i.e., if $v = v_i$, then $\alpha(v) = i$). We decompose the label $L(v)$ of each vertex v into two parts $L'(v)$ and $L''(v)$: if $d(v, u) = i$, then $L'(v) = L(v) \cap S_{i-1}(u)$ and $L''(v) = L(v) \cap S_i(u)$. Notice that in the lexicographic order of $L(v)$, all vertices of $L'(v)$ precede the vertices of $L''(v)$; in particular, the father of v belongs to $L'(v)$. The proof of the theorem is a consequence of the following assertion, which we call the *fellow traveler property*:

Fellow Traveler Property: *If v, w are adjacent vertices of G, then their fathers $v' = f(v)$ and $w' = f(w)$ either coincide or are adjacent. If v' and w' are adjacent and $\alpha(w) < \alpha(v)$, then w' is adjacent to v and v' is not adjacent to w.*

Indeed, if this assertion holds, then we claim that $v_n, \ldots, v_1$ is a dismantling order. To see this, it suffices to show that any vertex v_i is dominated in G_i by its father $f(v_i)$ in the LexBFS tree T_u. Pick any neighbor v_j of v_i in G_i. We assert that v_j coincides or is adjacent to $f(v_i)$. This is obviously true if $f(v_j) = f(v_i)$. Otherwise, if $f(v_i) \neq f(v_j)$, then the Fellow Traveler Property implies that $f(v_i)$ and $f(v_j)$ are adjacent and since $i < j$ that v_j is adjacent to $f(v_i)$. This shows that indeed any LexBFS order is a dismantling order.

Therefore, it remains to prove the Fellow Traveler Property which we establish in the following lemma.

Lemma 4.2. *G satisfies the Fellow Traveler Property.*

Proof of Lemma 4.2. We proceed by induction on $i+1 := \max\{d(u,v), d(u,w)\}$. We distinguish two cases: $d(u,v) < d(u,w)$ and $d(u,v) = d(u,w) = i+1$.

Case 1: $d(u,v) < d(u,w)$. Then $v, w' \in I(w,u)$ and since G is thin, we conclude that v and $w' = f(w)$ either coincide or are adjacent. In the first case we are done because v (and therefore w') is adjacent to its father $v' = f(v)$. If v and w' are adjacent, since $i = d(u,v) = d(u,w')$, the vertices v' and $f(w')$ coincide or are adjacent by the induction assumption. Again, if $v' = f(w')$, the assertion is immediate. Now suppose that v' and $f(w')$ are adjacent. Since $w' = f(w)$ was labeled before v (otherwise the father of w is v and not w'), $f(w')$ must be labeled before v', therefore by the induction hypothesis we deduce that $v' = f(v)$ must be adjacent to $w' = f(w)$. This concludes the analysis of the case $d(u,v) < d(u,w)$.

Case 2: $d(u,v) = d(u,w) = i+1$. Suppose, without loss of generality that $\alpha(w) < \alpha(v)$. If the vertices $v' = f(v)$ and $w' = f(w)$ coincide, then we are done. If the vertices v' and w' are adjacent, then the vertices v, w, w', v' define a 4–cycle. Since G is weakly bridged, by Theorem 3.1 this cycle cannot be induced. Since v was labeled before w, the vertex v' must be labeled before w'. Therefore, if v' is adjacent to w, then LexBFS will label w from v' and not from w', giving a contradiction. Thus v' and w are not adjacent, showing that w' must be adjacent to v, establishing the required assertion. So, assume by way of contradiction that the vertices v' and w' are not adjacent in G. Then w' is not adjacent to v, otherwise $w', v' \in B_i(u)$ and $v \in I(v', w') \cap S_{i+1}(u)$, contrary to the convexity of the ball $B_i(u)$ (similarly, v' is not adjacent to w).

Since G is weakly modular by Theorem 3.1(iii), the triangle condition applied to the vertices v, w, and u implies that there exists a common neighbor s of v and w located at distance i from u. Denote by S the set of all such vertices s. From the property $SD_i(u)$ we infer that S is a simplex of $\mathbf{X}$ (i.e., its vertices are pairwise adjacent in G). Since v' and w' do not belong to S, necessarily all vertices of S have been labeled later than v' and w' (but obviously before v and w). Pick a vertex s in S with the largest label $\alpha(s)$ and set $z := f(s)$. By the induction assumption applied to the pairs of adjacent vertices $\{v', s\}$ and $\{s, w'\}$, we conclude that the vertices of each of the pairs $\{f(v'), z\}$ and $\{z, f(w')\}$ either coincide or are adjacent. Moreover, in all cases, the vertex z must be adjacent to the vertices v' and w'.

Claim 1: $L'(v') = L'(s) = L'(w')$ and z is the father of v' and w'.

Proof of Claim 1: Since s was labeled later than v' and w', it suffices to show that $L'(v') = L'(s)$. Indeed, if this is the case, then necessarily z is the father of v'. Then, as z is adjacent to w' and $\alpha(w') < \alpha(v')$, necessarily z is also the father of w'. Now, if $L'(w')$ and $L'(s) = L'(v')$ do not coincide, since $L'(v')$ lexicographically precedes $L''(v')$ and $L'(w')$ precedes $L''(w')$, the fact that LexBFS labeled v' before w' means that $L'(v')$ lexicographically precedes $L'(w')$. Since $L'(s) = L'(v')$, then necessarily LexBFS would label s before w', a contradiction. This shows that the equality of the two labels $L'(s)$ and $L'(v')$ implies the equality of the three labels $L'(v'), L'(s)$, and $L'(w')$.

To show that $L'(v') = L'(s)$, since $\alpha(s) < \alpha(v')$, it suffices to establish only the inclusion $L'(v') \subseteq L'(s)$. Suppose by way of contradiction that there exists a vertex in $L'(v') \setminus L'(s)$ i.e., a vertex $x \in S_{i-1}(u)$ which is adjacent to v' but is not adjacent to s. Let x be the vertex of $L'(v') \setminus L'(s)$ having the largest label $\alpha(x)$. Since s was labeled by LexBFS later than v', necessarily any vertex of $L'(s) \setminus L'(v')$ must be labeled later than x. Notice that x cannot be adjacent to w', since otherwise we would obtain an induced 4–cycle formed by the vertices v', s, w', x. On the other hand x is adjacent to z because both vertices belong to the convex ball $B_{i-1}(u)$ and both are adjacent to the vertex $v' \in S_i(u)$. Since x is not adjacent to v, w, and s, we conclude that the vertices v, w, w', z, c', s, x induce an extended 5–wheel $\widehat{W}_5$. By the $\widehat{W}_5$–condition, there exists a vertex t adjacent to all vertices of this $\widehat{W}_5$. Notice that $t \in S_i(u)$ because it is adjacent to $x \in S_{i-1}(u)$ and $v \in S_{i+1}(u)$. Hence $t \in S$. By definition, t is adjacent to z. Further, t must be adjacent to any other vertex z' belonging to $L'(v') \cap L'(s)$, otherwise we obtain a forbidden 4–cycle. This means that LexBFS will label t before s. Since t belongs to S and $\alpha(t) > \alpha(s)$, we obtain a contradiction with the choice of the vertex s. This contradiction concludes the proof of Claim 1.

We continue with the analysis of Case 2. Since v' and w' are not adjacent and G does not contain induced 4–cycles, any vertex $s' \neq s$ adjacent to v' and w' is also adjacent to s. In particular, this shows that $L''(v') \cap L''(w') \subseteq L''(s)$. Therefore, if $L''(w') \subseteq L''(v')$, then $L''(w') \subseteq L''(s)$. Since $v' \in L''(s) \setminus L''(w')$ and $L'(s) = L'(w')$ by Claim 1, we conclude that the vertex s must be labeled before w', contrary to the assumption that $\alpha(s) < \alpha(w')$. Therefore the set $B := L''(w') \setminus L''(v')$ is nonempty. Then, since v' was labeled before w' and $L'(v') = L'(w')$ by Claim 1, we conclude that the set $A := L''(v') \setminus L''(w')$ is nonempty as well. Let p be the vertex of A with the largest label $\alpha(p)$ and let q be the vertex of B with the largest label $\alpha(q)$. Since LexBFS labeled v' before w' and $L'(v') = L'(w')$ holds, necessarily $\alpha(q) < \alpha(p)$ holds. Since $p \in L''(v')$, we obtain that $\alpha(w') < \alpha(v') < \alpha(p)$. Since $v' = f(v)$ and $w' = f(w)$, this shows that p cannot be adjacent to the vertices v and w. If s is adjacent to p, then $p \in L''(s)$. But then from Claim 1 and the inclusion $L''(v') \cap L''(w') \subseteq L''(s)$ we could infer that LexBFS must label s before w', contrary to the assumption that $\alpha(s) < \alpha(w')$. Therefore p is not adjacent to s either. On the other hand, since $\alpha(v') < \alpha(p)$, by the induction hypothesis applied to the adjacent vertices p and v', we infer that $z = f(v')$ must be adjacent to p. Hence the vertices v, w, w', z, v', s, p induce an extended 5–wheel. By the $\widehat{W}_5$–condition, there exists a vertex t adjacent to all these vertices. Since $C := L'(v') = L'(w')$ and $d(v', w') = 2$, to avoid induced 4–cycles, the vertex t must be adjacent to any vertex of C. For the same reason, t must be adjacent to any vertex of $L''(v') \cap L''(w')$. Since additionally t is adjacent to the vertex p of A with the highest label, necessarily t will be labeled by LexBFS before w' and s. Since t is adjacent to v and w, this contradicts the assumption that $w' = f(w)$. This shows that the initial assumption that v' and w' are not adjacent lead to a final contradiction. Hence the order returned by LexBFS is indeed a dismantling order of the weakly bridged graph G. This completes the proof of the lemma and of the theorem. $\square$

Corollary 4.3. *Locally finite weakly systolic complex* $\mathbf{X}$ *and every its Rips complex* $\mathbf{X}_k$ *are LC-contractible and therefore collapsible.*

Proof. Again we consider only the finite case. To show that any finite weakly systolic complex $\mathbf{X}$ is LC-contractible it suffices to notice that, since $\mathbf{X}$ is a flag complex, the LC-contractibility of $\mathbf{X}$ is equivalent to the dismantlability of its graph $G = G(\mathbf{X})$, and hence the result follows from Theorem 4.1.

To show that the Rips complex $\mathbf{X}_k$ is LC-contractible, since $\mathbf{X}_k$ is a flag complex, it suffices to show that its graph $G(\mathbf{X}_k)$ is dismantlable. From the definition of $\mathbf{X}_k$, the graph $G(\mathbf{X}_k)$ coincides with the kth power G^k of the underlying graph G of $\mathbf{X}$. Now notice that if a vertex v is dominated in G by a vertex v', then v' also dominates v in the graph G^k. Indeed, pick any vertex x adjacent to v in G^k. Then $d(v,x) \leq k$ in G. Let y be the neighbor of v on some shortest path P connecting the vertices v and x in G. Since v' dominates v, necessarily v' is adjacent to y. Hence $d(v',x) \leq k$ in G, therefore v' is adjacent to x in G^k. This shows that v is dominated by v' in the graph G^k as well. Therefore the dismantling order of G returned by LexBFS is also a dismantling order of G^k, establishing that the Rips complex $\mathbf{X}_k$ is LC-contractible. $\square$

Corollary 4.4. *Graphs of Rips complexes* $\mathbf{X}_n$ *of locally-finite systolic and weakly systolic complexes are dismantlable.*

For a locally-finite weakly bridged graph G and integer k we denote by G_k the subgraph of G induced by the first k labeled vertices $v_1, \ldots, v_k$ in a LexBFS order with basepoint u, i.e., by the vertices of G with k lexicographically largest labels. For each k, let v_k be the last labeled vertex of G_k (notice that $v_1 = u$).

Corollary 4.5. *Any* G_k *is an isometric weakly bridged subgraph of* G.

Proof. First we show that every G_k is an isometric subgraph of G. Pick two arbitrary vertices x, y of G_k. For a shortest path P in G between x and y let $i(P)$ be the least integer i such that P is completely contained in the subgraph G_i. From the definition of $i(P)$ it follows that P passes via the vertex v_i of G_i. Among all shortest paths between x and y, let P^* has minimal index $i(P^*)$. Let $k' = i(P^*)$. If $k' \leq k$, then P^* is contained in G_k and we are done. So, suppose that $k' > k$. Since x, y belong to G_k, $v_{k'} \neq x, y$. Let x', y' be the neighbors of $v_{k'}$ in P^* such that x' belongs to the portion of P^* between x and $v_{k'}$. Let $v' = f(v_{k'})$ be the vertex (of $G_{k'-1}$) dominating $v_{k'}$ in the dismantling order of G. Then v' is adjacent to x' and y'. Therefore, the path Q of G consisting of the portion of P^* between x and x', the path of length 2 (x', v', y'), and the subpath of P^* between between y' and y, is a shortest path between x and y. Since $i(Q) < i(P^*)$, we obtain a contradiction with minimality of $i(P^*)$. This contradiction shows that each G_k is an isometric subgraph of G. In particular, this implies that any interval $I(x, y)$ in G_k is contained in the interval between x and y in G. Since G is a thin graph, each G_k is also thin. Moreover, since G is weakly bridged and weakly bridged graphs do not contain embedded isometric cycles of length > 5, G_k, as an isometric subgraph of G, does not contain such isometric cycles either. All balls of a graph are convex if and only if this graph is thin and does not contain embedded isometric cycles of length > 5; cf. [SC83, Theorem 2] and [FJ87, Theorem 2.2]. Hence, each G_k is a graph with convex balls.

To complete the proof that each graph G_k is weakly bridged, by Theorem 3.1(vi) it remains to show that any induced 5–cycle C of G_k is included in a 5–wheel. Suppose by the induction assumption that this is true for G_{k-1}. Therefore C must contain the last labeled vertex of G_k. Denote this vertex by v and let x, y be the neighbors of v in C. Since C is induced, necessarily v' is adjacent to x and y but distinct from these vertices. Denote by C' the 5–cycle obtained by replacing in C the vertex v by v'. If C' is not induced, then v' will be adjacent to a third vertex of C, and since G_k does not contain induced 4–cycles, v' will be adjacent to all vertices of C, showing that C extends to a 5–wheel. So, suppose that C' is induced. Applying the induction hypothesis to G_{k-1}, we conclude that C' extends to a 5–wheel in G_{k-1}. Let w be the central vertex of this wheel. To avoid a 4–cycle induced by the vertices x, y, v, and w, necessarily v and w must be adjacent. Hence C extends in G_k to a 5–wheel centered at w. This establishes that G_k is indeed weakly bridged. □

A *simplicial map* on a simplicial complex $\mathbf{X}$ is a map $\varphi : V(\mathbf{X}) \to V(\mathbf{X})$ such that for all $\sigma \in \mathbf{X}$ we have $\varphi(\sigma) \in \mathbf{X}$. A *homomorphism* of a graph $G = (V, E)$ is a simplicial map on a one-dimensional simplicial complex G. A simplicial map fixes a simplex $\sigma \in \mathbf{X}$ if $\varphi(\sigma) = \sigma$. Every simplicial map on $\mathbf{X}$ is a homomorphism of its graph $G(\mathbf{X})$. Every homomorphism of a graph G is a simplicial map on its clique complex $\mathbf{X}(G)$. Therefore, if $\mathbf{X}$ is a flag complex, then the set of simplicial maps of $\mathbf{X}$ coincides with the set of homomorphisms of its graph $G(\mathbf{X})$. It is well know (see, for example, [HN04, Theorem 2.65]) that any homomorphism of a finite dismantlable graph to itself fixes some clique. From Theorem 4.1 we know that the graphs of weakly systolic complexes as well as the graphs of their Rips complexes are dismantlable. Therefore from the preceding discussion we obtain:

Corollary 4.6. *Let $\mathbf{X}$ be a finite weakly systolic complex. Then any simplicial map of $\mathbf{X}$ to itself or of its Rips complex $\mathbf{X}_k$ to itself fixes some simplex of the respective complex. Any homomorphism of $G = G(\mathbf{X})$ to itself fixes some clique.*

5. Fixed point theorem

In this section, we establish the fixed point theorem (Theorem C from Introduction). We start with two auxiliary results. The first one is an easy corollary of Theorem 4.1:

Lemma 5.1 (Strictly dominated vertex)**.** *Let $\mathbf{X}$ be a finite weakly systolic complex. Then either $\mathbf{X}$ is a single simplex or it contains two vertices v, w such that v is strictly dominated by w, i.e., $B_1(v) \subsetneq B_1(w)$.*

Proof. Let v be the last vertex of $\mathbf{X}$ labeled by LexBFS which started at vertex u (see Theorem 4.1). If $d(u,v)=1$, then the construction of our ordering implies that $B_1(u)=V(\mathbf{X})$. Hence, either there exists a vertex w, such that $B_1(w) \subsetneq V(\mathbf{X}) = B_1(u)$, and we are done, or every two vertices of $\mathbf{X}$ are adjacent, i.e., $\mathbf{X}$ is a simplex. Now suppose that $d(u,v) \geq 2$. Let w be the father of v and let z be the father of w. From Theorem 4.1 we know that $B_1(v) \subseteq B_1(w)$. Since $d(u,v) = d(u,w)+1 \geq 2$, we conclude that $u \neq w$ and that $z \in B_1(w) \setminus B_1(v)$. Hence $B_1(v)$ is a proper subset of $B_1(w)$. $\square$

Lemma 5.2 (Elementary LC-reduction)**.** *Let $\mathbf{X}$ be a finite weakly systolic complex. Let v, w be two vertices such that $B_1(v)$ is a proper subset of $B_1(w)$. Then the full subcomplex $\mathbf{X}_0$ of $\mathbf{X}$ spanned by all vertices of $\mathbf{X}$ except v is weakly systolic.*

Proof. It is easy to see that $\mathbf{X}_0$ is simply connected (see also the discussion in Section 2.3). Thus, by condition (vii) of Theorem 3.1, it suffices to show that $\mathbf{X}_0$ does not contain induced 4–cycles and satisfies the $\widehat{W}_5$–condition. Since, by Theorem 3.1, $\mathbf{X}$ does not contain induced C_4, the same is true for its full subcomplex $\mathbf{X}_0$. Let $\widehat{W}_5 \subseteq \mathbf{X}_0$ be a given 5–wheel plus a triangle as defined in Section 3. By Theorem 3.1 there exists a vertex $v' \in \mathbf{X}$ adjacent in $\mathbf{X}$ to all vertices of $\widehat{W}_5$. If $v' \neq v$ then $v' \in \mathbf{X}_0$ and if $v' = v$ then $\widehat{W}_5 \subseteq \mathrm{lk}(w, \mathbf{X}_0)$. In both cases all vertices of $\widehat{W}_5$ are adjacent to a vertex of $\mathbf{X}_0$: $\widehat{W}_5$ is coned to v in one case and to w in the other. Thus $\mathbf{X}_0$ also satisfies the $\widehat{W}_5$–condition and hence the lemma follows. $\square$

Theorem 5.3 (The fixed point theorem)**.** *Let G be a finite group acting by simplicial automorphisms on a locally-finite weakly systolic complex $\mathbf{X}$. Then there exists a simplex $\sigma \in \mathbf{X}$ which is invariant under the action of G.*

Proof. Let $\mathbf{X}'$ be the subcomplex of $\mathbf{X}$ spanned by the convex hull of the set $Gz = \{gz : g \in G\}$, for an arbitrary vertex z. Since Gz is finite and, by Theorem 3.1(v), balls in $\mathbf{X}$ are convex, $\mathbf{X}'$ is a bounded full subcomplex of $\mathbf{X}$. Since $\mathbf{X}$ is locally-finite, $\mathbf{X}'$ is finite. Moreover, as a convex subcomplex of a weakly systolic complex, $\mathbf{X}'$ is itself weakly systolic. Clearly $\mathbf{X}'$ is also G–invariant. Thus there exists a minimal finite non-empty G–invariant subcomplex $\mathbf{X}_0$ of $\mathbf{X}$, that is itself weakly systolic. We assert that $\mathbf{X}_0$ must be a single simplex.

Assume by way of contradiction that $\mathbf{X}_0$ is not a simplex. Then, by Lemma 5.1, $\mathbf{X}_0$ contains two vertices v, w such that $B_1(v) \subsetneq B_1(w)$ (i.e., v is a strictly dominated vertex). Since the strict inclusion of 1–balls is a transitive relation and $\mathbf{X}_0$ is finite, there exists a finite set S of strictly dominated vertices of $\mathbf{X}_0$ with the following property: for a vertex $x \in S$ there is no vertex y with $B_1(y) \subsetneq B_1(x)$. Let $\mathbf{X}_0'$ be the full subcomplex of $\mathbf{X}$ spanned by $V(\mathbf{X}_0) \setminus S$. It is clear that $\mathbf{X}_0'$ is a non-empty G–invariant proper subcomplex of $\mathbf{X}_0$. By Lemma 5.2, $\mathbf{X}_0'$ is weakly systolic. This contradicts the minimality of $\mathbf{X}_0$ and thus shows that $\mathbf{X}_0$ has to be a simplex. $\square$

Corollary 5.4 (Conjugacy classes of finite subgroups)**.** *Let G be a group acting geometrically by automorphisms on a weakly systolic complex $\mathbf{X}$ (i.e. G is weakly systolic). Then G contains only finitely many conjugacy classes of finite subgroups.*

Proof. Suppose by way of contradiction that we have infinitely many conjugacy classes of finite subgroups represented by $H_1, H_2, \ldots \leq G$. Since G acts geometrically on $\mathbf{X}$, there exists a compact subset $K \subseteq V(\mathbf{X})$ with $\bigcup_{g \in G} gK = \mathbf{X}$. For $i = 1, 2, \ldots$, let σ_i be an H_i–invariant simplex of $\mathbf{X}$ (whose existence is assured by the fixed point Theorem 5.3) and let $g_i \in G$ be such that $g_i(\sigma_i) \cap K \neq \emptyset$. Then $g_i(\sigma_i)$ is $g_i H_i g_i^{-1}$ invariant and $\bigcup_i g_i H_i g_i^{-1}$ is infinite. But for every element $g \in \bigcup_i g_i H_i g_i^{-1}$ we have $g(B_1(K)) \cap B_1(K) \neq \emptyset$, a contradiction with the properness of the G–action on $\mathbf{X}$. □

6. Contractibility of the fixed point set

The aim of this section is to prove that for a group acting on a weakly systolic complex its fixed point set is contractible (Proposition 6.6). As explained in the Introduction, this result implies Theorem E asserting that weakly systolic complexes are models for $\underline{E}G$ for groups acting on them properly.

Our proof closely follows Przytycki's proof of an analogous result for the case of systolic complexes [Prz09]. There are however minor technical difficulties. In particular, since balls around simplices in weakly systolic complexes need not to be convex, we have to work with other convex objects that are defined as follows. For a simplex σ of a simplicial complex $\mathbf{X}$, set $K_0(\sigma) = \sigma$ and $K_i(\sigma) = \bigcap_{v \in \sigma} B_i(v)$ for $i = 1, 2, \ldots$.

Lemma 6.1 (Properties of $K_i(\sigma)$)**.** *Let σ be a simplex of a weakly systolic complex $\mathbf{X}$. Then, for $i = 0, 1, 2, ...$, $K_i(\sigma)$ is convex and $K_{i+1}(\sigma) \subseteq B_1(K_i(\sigma))$.*

Proof. Trivially, $K_0(\sigma) = \sigma$ is convex. For $i > 0$, $K_i(\sigma)$ is the intersection of the balls $B_i(v), v \in \sigma$. By Theorem 3.1, balls around vertices are convex, whence $K_i(\sigma)$ is convex as well. To establish the inclusion $K_{i+1}(\sigma) \subseteq B_1(K_i(\sigma))$, pick any vertex $w \in K_{i+1}(\sigma)$. Let $l = d(w, \sigma) - 1$ and denote by σ_0 the metric projection of w in σ. By the property $SD_l(w)$, there exists a vertex $z \in S_l(w)$ adjacent to all vertices of the simplex σ_0. Let w' be a neighbor of w in the interval $I(w, z)$. Then obviously $d(w', \sigma) = l$ and therefore σ_0 is the metric projection of w' in σ. Since $d(w', v) = d(w, v) - 1$ for any vertex $v \in \sigma$ and $w \in K_{i+1}(\sigma)$, we conclude that $w' \in K_i(\sigma)$, whence $w \in B_1(w') \subseteq B_1(K_i(\sigma))$. □

We recall now two general results that were proved in [Prz09] and which will be important in the proof of Proposition 6.6.

Proposition 6.2 ([Prz09, Proposition 4.1])**.** *If $\mathcal{C}, \mathcal{D}$ are posets and $F_0, F_1 \colon \mathcal{C} \to \mathcal{D}$ are functors such that for each object c of $\mathcal{C}$ we have $F_0(c) \leq F_1(c)$, then the maps induced by F_0, F_1 on the geometric realizations of $\mathcal{C}, \mathcal{D}$ are homotopic. Moreover this homotopy can be chosen to be constant on the geometric realization of the subposet of $\mathcal{C}$ of objects on which F_0 and F_1 agree.*

Proposition 6.3 ([Prz09, Proposition 4.2])**.** *Let $F_0 \colon \mathcal{C}' \to \mathcal{C}$ be the functor from the flag poset $\mathcal{C}'$ of a poset $\mathcal{C}$ into the poset $\mathcal{C}$, assigning to each object of $\mathcal{C}'$, which is a chain of* *objects of $\mathcal{C}$, its minimal element. Then the map induced by F_0 on geometric realizations of $\mathcal{C}', \mathcal{C}$ (that are homeomorphic in a canonical way) is homotopic to identity.*

The following property of flag complexes will be crucial in the definition of expansion by projection below. It says that in the weakly systolic case we can define projections on convex subcomplexes the same way as projections on balls.

Lemma 6.4 (Projections on convexes)**.** *Let $\mathbf{X}$ be a weakly systolic complex and let Y be its convex subset. If a simplex σ belongs to $S_1(Y)$, i.e. $\sigma \subseteq B_1(Y)$ and $\sigma \cap Y = \emptyset$, then $\tau := \mathrm{lk}(\sigma, \mathbf{X}) \cap Y$ is a single nonempty simplex.*

Proof. First, assuming that τ is nonempty, we show that it is a single simplex. By definition of links, τ consists of all vertices v of Y adjacent in $G(\mathbf{X})$ to all vertices of σ. Since the set Y is convex and σ is disjoint from Y, necessarily the vertices of τ are pairwise adjacent. As $\mathbf{X}$ is a flag complex, τ is a simplex of $\mathbf{X}$.

By induction on the dimension m of σ we will prove that τ is nonempty. The claim is clear for $m = 0$. Now we show it for $m = 1$. Let σ be an edge x_1x_2. Let $y_1, y_2 \in Y$ be adjacent to x_1, x_2, respectively. By convexity of Y we have that $d(y_1, y_2) \leq 2$. If $y_1 = y_2$, then $y_1 \in \tau$ and we are done. If $d(y_1, y_2) = 1$ then, since there are no induced 4–cycles in $\mathbf{X}$ (cf. Theorem 3.1), y_1 is adjacent to x_2 or y_2 is adjacent to x_1. Consequently, one of the vertices y_1, y_2 belongs to τ. If $d(y_1, y_2) = 2$, then there exists $y \in Y$ adjacent to both y_1 and y_2. By Theorem 3.1, the 5–cycle $(x_1, x_2, y_2, y, y_1, x_1)$ is either not a full subcomplex, or it is contained in a 5–wheel. In both cases one easily finds a vertex in Y adjacent to both x_1, x_2 and thus belonging to τ.

Now, we turn to the induction step. Assume that $\mathrm{lk}(\sigma, \mathbf{X}) \cap Y \neq \emptyset$ for all σ of dimension at most $m-1$. We show that it is true for dimension $m \geq 2$. Let $x_1, x_2, \ldots, x_{m+1}$ be the vertices of σ. Let $\sigma_i := \sigma \setminus \{x_i\}, i = 1, \ldots, m+1$. By the induction assumption, for each σ_i there exists a vertex $y_i \in Y$ such that $\sigma_i \cup \{y_i\}$ is a simplex of $\mathbf{X}$. Pick any two indices $i \neq j$. Then $\sigma_i \cap \sigma_j \neq \emptyset$ because $m + 1 \geq 3$. Since $y_i, y_j \in Y$ and Y is convex, this implies that y_i and y_j are adjacent. Then either y_i is adjacent to x_i or y_j is adjacent to x_j, otherwise the vertices x_i, x_j, y_i, y_j induce a forbidden 4–cycle. In both cases we will obtain a vertex of Y adjacent to all vertices of σ, showing that $\tau \neq \emptyset$. □

We will call the simplex τ as in the lemma above the *projection of σ on Y*. Now we are in position to define the following notion introduced (in a more general version) by Przytycki [Prz09, Definition 3.1] in the systolic case. Let Y be a convex subset of a weakly systolic complex $\mathbf{X}$ and let σ be a simplex in $B_1(Y)$. The *expansion by projection* $e_Y(\sigma)$ of σ is a simplex in $B_1(Y)$ defined in the following way: if $\sigma \subseteq Y$, then $e_Y(\sigma) = \sigma$, otherwise $e_Y(\sigma)$ is the join of $\sigma \cap S_1(Y)$ and its projection on Y. A version of the following simple lemma was proved in [Prz09] in the systolic case. Its proof given there is valid also in our case.

Lemma 6.5 ([Prz09, Lemma 3.8])**.** *Let Y be a convex subset of a weakly systolic complex $\mathbf{X}$ and let $\sigma_1 \subseteq \sigma_2 \subseteq \ldots \subseteq \sigma_n \subseteq B_1(Y)$ be an increasing sequence of simplices. Then the intersection $(\bigcap_{i=1}^n e_Y(\sigma_i)) \cap Y$ is nonempty.*

Let σ be a simplex of a weakly systolic complex $\mathbf{X}$. As in [Prz09], we define an increasing sequence of full subcomplexes $\mathbf{D}_{2i}(\sigma)$ and $\mathbf{D}_{2i+1}(\sigma)$ of the barycentric subdivision $\mathbf{X}'$ of $\mathbf{X}$ in the following way. Let $\mathbf{D}_{2i}(\sigma)$ be the subcomplex spanned by all vertices of $\mathbf{X}'$ corresponding to simplices of $\mathbf{X}$ which have all their vertices in $K_i(\sigma)$. Let $\mathbf{D}_{2i+1}(\sigma)$ be the subcomplex spanned by all vertices of $\mathbf{X}'$ which correspond to those simplices of $\mathbf{X}$ that have all their vertices in $K_{i+1}(\sigma)$ and at least one vertex in $K_i(\sigma)$. The proof of the main proposition in this section follows closely the proof of [Prz09, Proposition 1.4].

Proposition 6.6 (Contractibility of the fixed point set)**.** *Let H be a group acting by simplicial automorphisms on a weakly systolic complex $\mathbf{X}$. Then the complex $\mathrm{Fix}_H\mathbf{X}'$ is contractible or empty.*

Proof. Assume that $\mathrm{Fix}_H\mathbf{X}'$ is nonempty and let σ be a maximal H–invariant simplex. By $\mathbf{D}_i$ we will denote here $\mathbf{D}_i(\sigma)$. We will prove the following three assertions.

(i) $\mathbf{D}_0 \cap \mathrm{Fix}_H\mathbf{X}'$ is contractible;

(ii) the inclusion $\mathbf{D}_{2i} \cap \mathrm{Fix}_H\mathbf{X}' \subseteq \mathbf{D}_{2i+1} \cap \mathrm{Fix}_H\mathbf{X}'$ is a homotopy equivalence;

(iii) the identity on $\mathbf{D}_{2i+2}\cap\mathrm{Fix}_H\mathbf{X}'$ is homotopic to a mapping with image in $\mathbf{D}_{2i+1}\cap\mathrm{Fix}_H\mathbf{X}' \subseteq \mathbf{D}_{2i+2} \cap \mathrm{Fix}_H\mathbf{X}'$.

As in the proof of [Prz09, Proposition 1.4], the three assertions imply that $\mathbf{D}_k \cap \mathrm{Fix}_H\mathbf{X}'$ is contractible for every k, thus the proposition holds. To show (i), note that $\mathbf{D}_0 \cap \mathrm{Fix}_H\mathbf{X}'$ is a cone over the barycenter of σ and hence it is contractible.

To prove (ii), let $\mathcal{C}$ be the poset of H–invariant simplices in $\mathbf{X}$ with vertices in $K_{i+1}(\sigma)$ and at least one vertex in $K_i(\sigma)$. Its geometric realization is $\mathbf{D}_{2i+1} \cap \mathrm{Fix}_H\mathbf{X}'$. Consider a functor $F\colon \mathcal{C} \to \mathcal{C}$ assigning to each object of $\mathcal{C}$ (i.e., each simplex of $\mathbf{X}$), its subsimplex spanned by its vertices in $K_i(\sigma)$. By Proposition 6.2, the geometric realization of F is homotopic to identity (which is the geometric realization of the identity functor). Moreover, this homotopy is constant on $\mathbf{D}_{2i} \cap \mathrm{Fix}_H\mathbf{X}'$. The image of the geometric realization of F is contained in $\mathbf{D}_{2i} \cap \mathrm{Fix}_H\mathbf{X}'$. Hence $\mathbf{D}_{2i} \cap \mathrm{Fix}_H\mathbf{X}'$ is a deformation retract of $\mathbf{D}_{2i+1} \cap \mathrm{Fix}_H\mathbf{X}'$, as desired.

To establish (iii), let $\mathcal{C}$ be the poset of H–invariant simplices of $\mathbf{X}'$ with vertices in $K_{i+1}(\sigma)$ and let $\mathcal{C}'$ be its flag poset. Let also $F_0\colon \mathcal{C}' \to \mathcal{C}$ be the functor assigning to each object of $\mathcal{C}'$ its minimal element; cf. Proposition 6.3. Now we define another functor $F_1\colon \mathcal{C}' \to \mathcal{C}$. For any object c' of $\mathcal{C}'$, which is a chain of objects $c_1 < c_2 < \ldots < c_k$ of $\mathcal{C}$, recall that c_j are some H–invariant simplices in $K_{i+1}(\sigma)$. Let $c_j' = e_{K_i(\sigma)}(c_j)$. Then by Lemma 6.5 the intersection $\bigcap_{j=1}^k c_j'$ contains at least one vertex in $K_i(\sigma)$. Thus $\bigcap_{j=1}^k c_j'$ is an H–invariant non-empty simplex and hence it is an object of $\mathcal{C}$. We define $F_1(c')$ to be this object. In the geometric realization of $\mathcal{C}$, which is $\mathbf{D}_{2i+2}\cap\mathrm{Fix}_H\mathbf{X}'$, the object $F_1(c')$ corresponds to a vertex of $\mathbf{D}_{2i+1} \cap \mathrm{Fix}_H \mathbf{X}'$. It is obvious that F_1 preserves the partial order. Notice that for any object c' of $\mathcal{C}'$ we have $F_0(c') \subseteq F_1(c')$, hence, by Proposition 6.3, the geometric realizations of F_0 and F_1 are homotopic. We have that F_0 is homotopic to the identity and that F_1 has image in $\mathbf{D}_{2i+1} \cap \mathrm{Fix}_H \mathbf{X}'$, thus establishing (iii). □

7. Some remarks on systolic complexes

In this final section, we restrict to the case of systolic complexes and present some further results in that case. First, using Lemma 3.10 and Theorem 3.11 of Polat [Pol02] for bridged graphs, we prove a stronger version of the fixed point theorem for systolic complexes. Namely, Polat [Pol02] established that for any subset $\overline{Y}$ of vertices of a graph with finite intervals, there exists a minimal isometric subgraph of this graph which contains $\overline{Y}$. Moreover, if $\overline{Y}$ is finite and the graph is bridged, then [Pol02, Theorem 3.11(i)] shows that this minimal isometric (and hence bridged) subgraph is also finite. We continue with two lemmata which can be viewed as G–invariant versions of these two results of Polat [Pol02].

Lemma 7.1 (Minimal subcomplex)**.** *Let a group G act by simplicial automorphisms on a systolic complex $\mathbf{X}$. Let $\overline{Y}$ be a G–invariant set of vertices of $\mathbf{X}$. Then there exists a minimal G–invariant subcomplex $\mathbf{Y}$ of $\mathbf{X}$ containing $\overline{Y}$, which is itself a systolic complex.*

Proof. Let Σ be a chain (with respect to the subcomplex relation) of G–invariant subcomplexes of $\mathbf{X}$, which contain $\overline{Y}$ and induce isometric subgraphs of the underlying graph of $\mathbf{X}$ (and thus are systolic complexes themselves). Then, as in the proof of [Pol02, Lemma 3.10], by Zorn lemma we conclude that the subcomplex $\mathbf{Y} = \bigcap \Sigma$ is a minimal G–invariant subcomplex of $\mathbf{X}$, containing $\overline{Y}$ and which is itself a systolic complex. □

Lemma 7.2 (Minimal finite subcomplex)**.** *Let a group G act by simplicial automorphisms on a systolic complex X. Let $\overline{Y}$ be a finite G–invariant set of vertices of X. Then there exists a minimal (as a simplicial complex) finite G–invariant subcomplex Y of X, which is itself a systolic complex.*

Proof. Let $\mathrm{conv}(\overline{Y})$ be the convex hull of $\overline{Y}$ in $\mathbf{X}$. The full subcomplex $\mathbf{Z}$ of $\mathbf{X}$ spanned by $\mathrm{conv}(\overline{Y})$ is a bounded systolic complex. By Lemma 7.1, there exists a minimal G–invariant subcomplex $\mathbf{Y}$ of $\mathbf{Z}$ containing the set $\overline{Y}$ and which itself is a systolic complex. Then from the minimality of $\mathbf{Y}$ (and as in the proof of [Pol02, Theorem 3.11]) we conclude that all dominated vertices of $\mathbf{Y}$ are contained in $\overline{Y}$. Thus $\mathbf{Y}$ contains finitely many dominated vertices. Since additionally $\mathbf{Y}$ is bounded and does not contain infinite simplices, by [Pol02, Theorem 3.8], $\mathbf{Y}$ is finite. □

Theorem 7.3 (The fixed point theorem)**.** *Let G be a finite group acting by simplicial automorphisms on a systolic complex $\mathbf{X}$. Then there exists a simplex $\sigma \in \mathbf{X}$ which is invariant under the action of G.*

Proof. Let $\overline{Y} = Gv = \{gv| \ g \in G\}$, for some vertex $v \in \mathbf{X}$. Then $\overline{Y}$ is a finite G–invariant set of vertices of $\mathbf{X}$ and thus, by Lemma 7.2, there exists a minimal finite G–invariant subcomplex $\mathbf{Y}$ of $\mathbf{X}$, which is itself a systolic complex. Then, the same way as in the proof of Theorem 5.3, we conclude that there exists a simplex in $\mathbf{Y}$ that is G–invariant. □

Remark 7.4. We believe that, as in the systolic case, the stronger version of Theorem 5.3 holds also for weakly systolic complexes, i.e., one can drop the assumption on the local finiteness of $\mathbf{X}$ in Theorem 5.3. This needs extensions of some results of Polat (in particular, Theorems 3.8 and 3.11 from [Pol02]) to the class of weakly bridged graphs.

Zawiślak [Zaw04] initiated another approach to the fixed point theorem in the systolic case based on the following notion of round subcomplexes. A systolic complex $\mathbf{X}$ of finite diameter k is *round* (cf. [Prz08]) if $\cap\{B_{k-1}(v) : v \in V(\mathbf{X})\} = \emptyset$. Przytycki [Prz08] established that all round systolic complexes have diameter at most 5 and used this result to prove that for any finite group G acting by simplicial automorphisms on a systolic complex there exists a subcomplex of diameter at most 5 which is invariant under the action of G. Zawiślak [Zaw04, Conjecture 3.3.1] and Przytycki (Remark 8.1 of [Prz08]) conjectured that in fact the diameter of round systolic complexes must be at most 2. Zawiślak [Zaw04, Theorem 3.3.1] showed that if this is true, then it implies that G has an invariant simplex, thus paving another way to the proof of Theorem 7.3. We will show now that the positive answer to the question of Zawiślak and Przytycki directly follows from an earlier result of Farber [Far89] on diameters and radii of finite bridged graphs.

Proposition 7.5 (Round systolic complexes)**.** *Any round systolic complex* $\mathbf{X}$ *has diameter at most 2.*

Proof. Let $\mathrm{diam}(\mathbf{X})$ and $\mathrm{rad}(\mathbf{X})$ denote the diameter and the radius of a systolic complex $\mathbf{X}$, i.e., the diameter and radius of its underlying bridged graph $G = G(\mathbf{X})$. Recall that $\mathrm{rad}(\mathbf{X})$ is the smallest integer r such that there exists a vertex c of $\mathbf{X}$ (called a central vertex) so that the ball $B_r(c)$ of radius r and centered at c covers all vertices of $\mathbf{X}$, i.e., $B_r(c) = V(\mathbf{X})$.

Farber [Far89, Theorem 4] proved that if G is a finite bridged graph, then $3\mathrm{rad}(G) \leq 2\mathrm{diam}(G) + 2$. We will show first that this inequality holds for infinite bridged graphs G of finite diameter $\mathrm{diam}(G)$. Set $k := \mathrm{rad}(G) \leq \mathrm{diam}(G)$. By definition of $\mathrm{rad}(G)$ the intersection of all balls of radius $k-1$ of G is empty. Then using an argument of Polat (personal communication) presented below, we can find a finite subset of vertices Y of G such that the intersection of the balls $B_{k-1}(v)$, v running over all vertices of Y, is still empty. By [Pol02, Theorem 3.11], there exists a finite isometric bridged subgraph H of G containing Y. From the choice of Y we conclude that the radius of H is at least k, while the diameter of H is at most the diameter of G. As a result, applying Farber's inequality to H, we obtain $3\mathrm{rad}(G) \leq 3\mathrm{rad}(H) \leq 2\mathrm{diam}(H) + 2 \leq 2\mathrm{diam}(G) + 2$, whence $3\mathrm{rad}(G) \leq 2\mathrm{diam}(G) + 2$.

To show the existence of a finite set Y such that $\cap\{B_{k-1}(v) : v \in Y\} = \emptyset$, we use an argument of Polat. According to Theorem 3.9 of [Pol98], any graph $G = (V, E)$ without isometric rays (in particular, any bridged graph of finite diameter) can be endowed with a topology, called geodesic topology, so that the resulting topological space is compact. A vertex x of G *geodesically dominates* a subset A of V if, for every finite $S \subseteq V - \{x\}$, there exists an element a of $A - \{x\}$ such that the interval $I(x,a)$ between x and a is disjoint from S. A set $A \subseteq V$ is *geodesically closed* if it contains all vertices which geodesically dominate A. Then the *geodesic topology* on V consists of all geodesically closed sets. It is shown in [Pol04, Corollary 6.26] that any convex set of a bridged graph containing no infinite simplices is closed in the geodesic topology. As a result, the balls of a bridged graph G of finite diameter containing no infinite simplices are compact convex sets. Hence any family of balls with an empty intersection contains a finite subfamily with an empty intersection, showing that such a finite set Y indeed exists.

Now suppose that $\mathbf{X}$ is a round systolic complex and let $k := \mathrm{diam}(\mathbf{X})$. Since $\mathbf{X}$ is round, one can easily deduce that $\mathrm{rad}(\mathbf{X}) = k$: indeed, if $\mathrm{rad}(\mathbf{X}) \leq k-1$ and c is a central vertex, then c will belong to the intersection $\cap\{B_{k-1}(v) : v \in V(\mathbf{X})\}$, which is impossible. Applying Farber's inequality to the (bridged) underlying graph of $\mathbf{X}$, we conclude that $3k \leq 2k+2$, whence $k \leq 2$. □

Remark 7.6. It would be interesting to extend Proposition 7.5 and the relationship of [Far89] between radii and diameters to weakly systolic complexes.

Osajda-Przytycki [OP09] constructed a Z–set compactification $\overline{\mathbf{X}} = \mathbf{X} \cup \partial\mathbf{X}$ of a systolic complex $\mathbf{X}$. The main result there ([OP09, Theorem 6.3]) together with Theorem E from Introduction, suggest that for a group G acting geometrically by simplicial automorphisms on a systolic complex $\mathbf{X}$ the following result holds:

The compactification $\overline{\mathbf{X}} = \mathbf{X} \cup \partial\mathbf{X}$ of $\mathbf{X}$ satisfies the following properties:

1. *$\overline{\mathbf{X}}$ is a Euclidean retract (ER);*
2. *$\partial\mathbf{X}$ is a Z–set in $\overline{\mathbf{X}}$;*
3. *for every compact set $K \subseteq \mathbf{X}$, $(gK)_{g\in G}$ is a null sequence;*
4. *the action of G on $\mathbf{X}$ extends to an action, by homeomorphisms, of G on $\overline{\mathbf{X}}$;*
5. *for every finite subgroup F of G, the fixed point set $\mathrm{Fix}_F\overline{\mathbf{X}}$ is contractible;*
6. *for every finite subgroup F of G, the fixed point set $\mathrm{Fix}_F\mathbf{X}$ is dense in $\mathrm{Fix}_F\overline{\mathbf{X}}$.*

This asserts that $\overline{\mathbf{X}}$ is an *EZ–structure*, sensu Rosenthal [Ros03], for a systolic group G; for details, see [OP09]. The existence of such a structure implies, by [Ros03], the Novikov conjecture for G.

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