

# LOCALLY ELLIPTIC ACTIONS, TORSION GROUPS, AND NONPOSITIVELY CURVED SPACES

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ABSTRACT. Extending and unifying a number of well-known conjectures and open questions, we conjecture that locally elliptic (that is, every element has a bounded orbit) actions by automorphisms of finitely generated groups on finite dimensional nonpositively curved complexes have global fixed points. In particular, finitely generated torsion groups cannot act without fixed points on such spaces. We prove these conjectures for a wide class of complexes, including all infinite families of Euclidean buildings, Helly complexes, some graphical small cancellation and systolic complexes, uniformly locally finite Gromov hyperbolic graphs. We present consequences of these results, e.g. concerning the automatic continuity.

Our main tool are Helly graphs. On the way we prove several results concerning their automorphisms. This is of independent interest and includes a classification result: any automorphism of a Helly graph with finite combinatorial dimension is either elliptic or hyperbolic, with rational translation length. One consequence is that groups with distorted elements cannot act properly on such graphs. We also present and study a new notion of geodesic clique paths. Their local-to-global properties are crucial in our proof of ellipticity results.

## 1. INTRODUCTION

1.1. **(Locally) elliptic actions.** We consider actions of groups on complexes (e.g. simplicial complexes or graphs) by automorphisms. Below we formulate a meta-conjecture concerning such actions, being a far-going unification and generalization of several well-known conjectures and open questions.

A *locally elliptic action* is an action in which an orbit of each element is bounded. If the orbit of the whole group is bounded we say that the action is *elliptic*. The *nonpositive curvature* assumption below refers to a wide understanding of various ‘nonpositive curvature’ conditions as discussed in Section 2 – therefore we name the statement below ‘meta-conjecture’ rather than an actual conjecture. Similarly, *finite dimensionality* refers to any

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reasonable notion of dimension – it might be thought of as the maximum of (topological) dimensions of cells.

**Meta-Conjecture.** *Every locally elliptic action of a finitely generated group on a finite-dimensional nonpositively curved complex is elliptic. In particular, every action of a finitely generated torsion group on such complex is elliptic.*

Note that Serre proved that every infinitely generated group has an action on a tree without a fixed point ([Ser03, Theorem 15]), so the assumption of finite generation is crucial. The assumption on finiteness of the dimension is also essential – Grigorchuk’s infinite torsion groups as well as infinite Burnside groups (torsion groups of bounded exponent) can act with unbounded orbits on Hilbert spaces or infinite dimensional CAT(0) cubical complexes [Gri84, Osa18a]. Furthermore, the fact that we consider the combinatorial setting of complexes, rather than the one of (arbitrary) metric spaces is important as well – see Section 2 for explanations.

For nonpositively curved complexes we have in mind (see Section 2) having a bounded orbit (for a group, or a single automorphism) is usually equivalent to having a fixed point. Therefore, ‘locally elliptic’ can be thought of as ‘every group element fixes a point’ and, correspondingly, ‘elliptic’ should mean ‘having a global fixed point’. An actual conjecture being a clarification of the above Meta-Conjecture is formulated in Section 2, where we also describe in detail motivations and earlier related results. Let us just mention here that the Meta-Conjecture is related to and motivated by, notably: the Tits Alternative, property (FA), Kazhdan’s property (T), automatic continuity.

Our main aim in this paper is proving the Meta-Conjecture for a large class of nonpositively curved complexes (see Section 4 for explaining all the notions involved).

**Theorem A.** *Let a finitely generated group act on a locally finite graph  $\Gamma$  with stable intervals and of finite combinatorial dimension. If the action is locally elliptic then it is elliptic. In particular, every action of finitely generated torsion group on  $\Gamma$  is elliptic.*

Having stable intervals might be seen as a form of nonpositive curvature. For graphs, this property introduced by Lang in [Lan13] (in the context of geodesic metric spaces) coincides with a well-known falsification by fellow traveler (FFT) property by Neumann–Shapiro [NS95]. The latter is related to regularity of geodesic languages and has been established in a number of cases, e.g. for all buildings [Nos01] and Garside groups [Hol10]. We gather few consequences of Theorem A in the following (see also Theorem H in the next section). For more explanations and the description of related results see Section 2.

**Corollary B.** *Let a finitely generated group act on a complex  $X$  which is one of the following:*

- (1) *a uniformly locally finite Euclidean building of type  $\tilde{A}_n, \tilde{B}_n, \tilde{C}_n,$  or  $\tilde{D}_n$ ;*
- (2) *a uniformly locally finite Gromov hyperbolic graph;*
- (3) *a uniformly locally finite simply connected graphical  $C(4)$ - $T(4)$  small cancellation complex;*
- (4) *a uniformly locally finite quadric complex;*
- (5) *the Salvetti complex of an FC-type type Artin group.*

*If the action is locally elliptic then it is elliptic. In particular, every action of a finitely generated torsion group on  $X$  is elliptic.*

**Corollary C.** *Let  $X_1, X_2, \dots, X_k$  be complexes as in Corollary B, and let a finitely generated group act on the complex  $X_1 \times \dots \times X_k$ . If the action is locally elliptic then it is elliptic.*

The above results indicate the importance of the notion of combinatorial dimension (introduced by Dress in [Dre84]) for exploring fixed-point questions. Unlike many other notions of dimension – e.g. (virtual) cohomological dimension, geometric dimension, or asymptotic dimension – the combinatorial dimension has not been yet studied thoroughly in the context of groups. We believe that the following question is of utmost importance.

**Question D.** *Which (nonpositively curved) spaces have finite combinatorial dimension?*

In particular, we pose the following conjecture. Establishing it would extend Corollary B(1) to all (not only Euclidean or hyperbolic) locally finite buildings.

**Conjecture E.** *Buildings have finite combinatorial dimension.*

Aside from Theorem A we provide the following exemplary result confirming the Meta-Conjecture (see Section 2.3 for further examples from the literature).

**Theorem F.** *Let  $G$  be a finitely generated group acting strongly rigidly by locally elliptic automorphisms on an 18-systolic complex  $X$ . Then  $G$  acts elliptically on  $X$ .*

Although 18-systolic complexes are (Gromov) hyperbolic, observe that in the above theorem we do not assume that the dimension is finite. We believe that the assumptions on high systolicity can be weakened and the assumptions on the strong rigidity of actions can be omitted (see Conjecture in Section 2.1). Still, even in this restricted form Theorem F applies to an important class of monster groups as constructed in [Osa18b, Osa20]. Note that such groups are not hyperbolic and that groups with torsion constructed there contain elements of arbitrarily large order.

**Theorem G.** *Finitely generated torsion subgroups of  $C(18)$  graphical small cancellation groups are finite.*

**1.2. Locally elliptic actions on Helly graphs.** In fact, the main object of interest throughout the article and our main tool for proving Theorem A are Helly graphs and their automorphism groups.

Helly graphs form a very natural class of metric spaces with nonpositive curvature features, which are of increasing interest in geometric group theory. Their definition is extremely simple: a connected graph is Helly if any family of pairwise intersecting combinatorial balls has a non-empty global intersection, the so-called Helly property. See [CCG<sup>+</sup>20] for a study of group actions on Helly graphs.

Any graph embeds in an essentially unique minimal Helly graph, its Helly hull. So the study of Helly graphs is really meaningful when some finiteness conditions are assumed: for instance local finiteness, or finite combinatorial dimension.

One may think of Helly graphs as a very nice class of nonpositively curved, combinatorially defined spaces. In Section 4 we describe it in more details, providing numerous examples of groups acting nicely on Helly graphs and reminding analogies between CAT(0) and Helly, as well as advantages of the combinatorial Helly approach.

The following theorem, adding to the list in Corollary B, is in fact equivalent to Theorem A. (In the text, we conclude Theorem A from it.)

**Theorem H.** *Let  $X$  denote a locally finite Helly graph with finite combinatorial dimension, and let  $G$  denote a finitely generated group acting by automorphisms on  $X$ . Then either  $G$  stabilizes a clique in  $X$  or some element of  $G$  is hyperbolic, in which case it has infinite order.*

Note that in the case when  $X$  is a CAT(0) space, the answer is known only for very specific examples (see Section 2.3 for details), it is therefore striking that for locally finite Helly graphs with finite combinatorial dimension, we have a very simple and general answer to that question.

**Corollary I.** *Let  $G$  be a Helly group, that is a group acting geometrically on a Helly graph. Any torsion subgroup of  $G$  is finite.*

Let us state here just one corollary of the result. It concerns a notion of automatic continuity, having itself origins e.g. in the theory of Polish spaces. It is a direct (but non-obvious) consequence of Theorem H and [KMV21].

**Corollary J.** *Let  $G$  be a Helly group, that is a group acting geometrically on a Helly graph. Any group homomorphism  $\varphi: L \rightarrow G$  from a locally compact group  $L$  is continuous.*

In the case where the orthoscheme complex  $OX$  of  $X$  (see Theorem N for the definition), endowed with the piecewise  $\ell^2$  metric, is CAT(0), we can use

its visual boundary and results of [CL10] to deduce the following, including the case of infinitely generated groups.

**Theorem K.** *Let  $X$  denote a locally finite Helly graph with finite combinatorial dimension. Assume that the orthoscheme complex  $OX$  of  $X$ , endowed with the piecewise  $\ell^2$  metric, is  $CAT(0)$ . Let  $G$  denote any locally elliptic group of automorphisms of  $X$ . Then  $G$  has a fixed point in  $|OX| \cup \partial|OX|$ , where  $\partial|OX|$  denotes the  $CAT(0)$  visual boundary of  $|X|$ .*

This applies notably to  $CAT(0)$  cube complexes, and hence generalizes [LV20, Corollary B] in the locally finite case. This also applies to all locally finite Euclidean buildings of type  $\tilde{A}$ ,  $\tilde{B}$ ,  $\tilde{C}$  or  $\tilde{D}$ , see Theorem 3.1. So, for this class of buildings, we prove a conjecture by Marquis (see [Mar13, Conjecture 2]).

**Corollary L.** *Let  $\Delta$  denote a uniformly locally finite Euclidean building of type  $\tilde{A}$ ,  $\tilde{B}$ ,  $\tilde{C}$  or  $\tilde{D}$ . If  $G$  denotes a locally elliptic group of type-preserving automorphisms of  $\Delta$ , then  $G$  has a fixed point in  $\Delta \cup \partial\Delta$ , where  $\partial\Delta$  is the combinatorial bordification of  $\Delta$ .*

Note that this is a generalization of [Par03, Corollaire 3] and [NOP21, Corollary 1.2] in the case of locally finite discrete Euclidean buildings.

One should also point out that Breuillard and Fujiwara ask for a quantitative version of this result, see [BF21, Theorem 7.16].

According to Marquis (see [Mar13, Conjecture 1]), we deduce the following.

**Corollary M.** *Let  $G$  denote a locally compact group such that its group of components  $G/G^0$  is compact. Let  $\Delta$  denote a uniformly locally finite Euclidean building of type  $\tilde{A}$ ,  $\tilde{B}$ ,  $\tilde{C}$  or  $\tilde{D}$ .*

*Then any measurable action of  $G$  on  $\Delta$  fixes a point in  $\Delta$ . In other words,  $G$  has Property (FB) for the class of uniformly locally finite Euclidean buildings of type  $\tilde{A}$ ,  $\tilde{B}$ ,  $\tilde{C}$  or  $\tilde{D}$ .*

**1.3. Automorphisms of Helly graphs.** On the way to Theorem H we show a number of results concerning automorphisms of Helly graphs. They are of independent interest.

In order to study automorphisms of a Helly graph  $X$ , we are interested in finding a nice combinatorial structure on the injective hull  $EX$ . Such a description has been carried out by Lang in [Lan13], and we present a slight modification of his construction. Recall that the *combinatorial dimension* of  $X$  is the dimension of its injective hull  $EX$ , see Theorem 4.1 for the precise statement.

**Theorem N** (Orthoscheme complex of a Helly graph). *Let  $X$  denote a Helly graph with finite combinatorial dimension. For each  $N \geq 1$ , there exists a simplicial structure on the injective hull  $EX$  of  $X$ , denoted  $O_N X$  and called the ( $N^{\text{th}}$ ) orthoscheme complex of  $X$ , satisfying the following:*

- Each simplex of  $O_N X$  is isometric to the standard  $\ell^\infty$  orthosimplex with edge lengths  $\frac{1}{2N!}$ .
- The vertex set  $X'_N$  of  $O_N X$ , endowed with the induced distance, is a Helly graph (with edge lengths  $\frac{1}{2N!}$ ), containing isometrically  $X$ , called the ( $N^{\text{th}}$ ) Helly subdivision of  $X$ . Moreover, we have

$$X'_N = \left\{ p \in EX \mid \forall x \in X, d(p, x) \in \frac{1}{2N!} \mathbb{N} \right\}.$$

- To each vertex of  $X'_N$  is naturally associated a clique of  $X$ .

One key property of  $\text{CAT}(0)$  spaces is the classification of isometries into elliptic, parabolic and hyperbolic (see [BH99, Definition 6.3]).

In this article, we prove a similar classification for automorphisms of Helly graphs. We say that an automorphism (or a group of automorphisms) of a Helly graph is *elliptic* if it stabilizes a clique of  $X$  (note that this notion agrees with ellipticity defined earlier). We say that an automorphism  $g$  of a Helly graph is *hyperbolic* if the orbit map  $n \in \mathbb{Z} \mapsto g^n \cdot x$  is a quasi-isometric embedding. We refer to Section 5 for other characterizations of elliptic and hyperbolic automorphisms, and to Theorem 5.2 for the precise statement.

**Theorem O** (Classification of automorphisms of Helly graphs).

*Let  $X$  denote a Helly graph with finite combinatorial dimension  $N$ . Then any automorphism of  $X$  is either elliptic or hyperbolic.*

*More precisely, any elliptic automorphism of  $X$  fixes a vertex in the  $N^{\text{th}}$  Helly subdivision  $X'_N$  of  $X$ .*

*For every hyperbolic automorphism  $g$  of  $X$ , there exists  $k \leq 2N$  and a vertex  $x \in X'_N$  such that  $(g^{kn} \cdot x)_{n \in \mathbb{Z}}$  is a geodesic in  $X'_N$ .*

*In addition, every hyperbolic automorphism of  $X$  has rational translation length, with denominator bounded above by  $2N$ .*

This is a direct generalization (in the finite-dimensional case) of a result of Haglund stating essentially that any automorphism of a  $\text{CAT}(0)$  cube complex either fixes a point or translates a combinatorial geodesic (see [Hag07, Theorem 1.4] for the precise statement).

This also generalizes a theorem of Gromov for translation lengths of hyperbolic elements in a Gromov-hyperbolic group (see [Gro87, 8.5.S]). Since Garside groups are Helly according to [HO21], this implies a direct analogue of [LL07] for a very closely related translation length. This has consequences in particular for decision problems, following [LL07].

**Corollary P.** *Let  $G$  denote a Helly group. The following problems are solvable for  $G$ .*

- *The power problem: given  $g, h \in G$ , find  $n \geq 1$  such that  $h^n = g$ .*
- *The power conjugacy problem: given  $g, h \in G$ , find  $n \geq 1$  such that  $h^n$  is conjugate to  $g$ .*

- *The proper power problem: given  $g \in G$ , find  $h \in G$  and  $n \geq 2$  such that  $h^n = g$ .*
- *The proper power conjugacy problem: given  $g \in G$ , find  $h \in G$  and  $n \geq 2$  such that  $h^n$  is conjugate to  $g$ .*

This result also has a direct consequence concerning distortion. Recall that an element  $g$  of a finitely generated group  $G$  with a word metric  $|\cdot|_G$  is undistorted if there exists  $C > 0$  such that  $\forall n \in \mathbb{N}, |g^n|_G \geq nC$ .

**Corollary Q** (No distortion in Helly graphs).

*Let  $X$  denote a Helly graph with finite combinatorial dimension, let  $G$  denote a finitely generated group of automorphisms of  $X$  and assume that some element  $g$  of  $G$  is not elliptic. Then  $g$  is hyperbolic in  $X$ , has infinite order and is undistorted in  $G$ .*

*More generally,  $g$  is uniformly undistorted :  $\exists C > 0, \forall n \in \mathbb{N}, |g^n|_G \geq nC$ , where  $C$  depends only on the action of  $G$  on  $X$ .*

If a finitely generated group acts properly by automorphisms on a Helly graph with finite combinatorial dimension, we therefore deduce that  $G$  has *uniformly undistorted infinite cyclic subgroups* as defined by Cornuier in [Cor17, Definition 6.A.3]. This applies in particular to all discrete subgroups of semisimple Lie groups over non-Archimedean local fields of types  $A, B, C$  or  $D$ .

In particular, we deduce an obstruction to the existence of some actions on Helly graphs.

**Corollary R.** *Finitely generated groups with distorted elements do not act properly on a Helly graph with finite combinatorial dimension.*

This applies notably to nilpotent groups that are not virtually abelian, to non-uniform irreducible lattices in real semisimple Lie groups of higher rank and to Baumslag-Solitar groups.

Note that, on the other hand, any finitely generated group acts properly by automorphisms on a Helly graph, the Helly hull of any Cayley graph. In the case of a group with distorted elements, we deduce that the Helly hull of a Cayley graph has infinite combinatorial dimension.

The case of non-uniform lattices is drastically different from the uniform one. Indeed, uniform lattices in semisimple Lie groups over local fields have nice actions on Helly graphs (in the non-Archimedean case, see Theorem 3.1 and [Hae21a] for details, and [Hae21b]) and on injective metric spaces (in the Archimedean case, see [Hae21a] for details).

The arguments used in the proofs of Theorem O and Theorem H rely on a new notion of clique-paths, see Section 6 for the precise definition. The clique-paths are somewhat similar to normal cube-paths in CAT(0) cube complexes studied by Niblo and Reeves (see [NR97]) and to normal clique-paths studied by Chalopin et al. (see [CCG<sup>+</sup>20]), in both cases used in the

proof of biautomaticity. However, already in the case of a generic translation of the square grid  $\mathbb{Z}^2$ , no such normal path will be invariant. Our clique-paths have the advantage of being flexible enough to include such invariant axes (see Proposition 6.5 for a precise statement). On the other hand, they are sufficiently rigid that, even though they are locally defined, they ensure that any global path is geodesic. This key local-to-global property can be summed up in the following result, see Theorem 6.4 for the precise statement.

**Theorem S** (Local clique-paths are globally geodesic).

*Let  $X$  be a Helly graph, and assume that a bi-infinite sequence of cliques  $(\sigma_n)_{n \in \mathbb{Z}}$  is such that for all  $n \in \mathbb{Z}$ , the sequence  $(\sigma_n, \sigma_{n+1}, \sigma_{n+2})$  is a clique-path. Then, for any  $n, m \in \mathbb{Z}$ , we have  $d(\sigma_n, \sigma_m) \geq |n - m|$ .*

Note that, to our knowledge, such a construction is new even for CAT(0) cube complexes.

The other main ingredient in the proofs of Theorem O and Theorem H is the following result about linear growth of orbits in a general injective metric space.

**Theorem T** (Linear orbit growth).

*Let  $X$  denote an injective metric space, and let  $G, H$  denote elliptic groups of isometries of  $X$ . If  $x \in X^G$  is such that  $d(x, X^H) = d(X^G, X^H) = L$ , there exists a sequence  $(g_n)_{n \geq 1}$  in  $(GH)^n$  such that*

$$\forall n \geq 1, d(g_n \cdot x, x) = 2nL.$$

We believe this result could be of independent interest.

**Organization of the article:** In the following Section 2, we formulate the Conjecture being a precise version of the Meta-Conjecture, we provide motivations and list related results. The equivalence of Theorems 8.1 and H is shown in Section 3. There we also remind basics on Helly graphs and show that Theorem H implies Corollaries B and C. In Section 4, we review a refinement of Lang's cell structure on the injective hull of a Helly graph into orthosimplices. In Section 5, we prove the classification result for automorphisms of Helly graphs. In Section 6, we define the clique-paths between pairs of cliques in a Helly graph, and prove that local clique-paths are globally geodesic. In Section 7, we prove the linear growth of orbits in injective metric spaces. In Section 8, we use the clique-paths and the linear growth to study locally elliptic actions on Helly graphs and we prove Theorem H. In Section 9, we show how the existence of a CAT(0) metric on the orthoscheme complex of a Helly graph enables us to deduce Theorem K. In Section 10, we prove Theorem F about locally elliptic actions on systolic complexes and conclude Theorem G.

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2. CONJECTURES, MOTIVATIONS, AND STATE OF THE ART

**2.1. Conjectures.** We now formulate an actual conjecture, being a specification of the Meta-Conjecture. In order to do this we must specify the ‘nonpositive curvature’, and ‘finite dimensional’ assumptions. In what follows, by ‘dimension’ we mean the maximum of (topological) dimensions of cells.

**Definition 2.1** (Finite dimensional NPC spaces). The class  $\mathcal{X}$  consists of:

- (1) finite dimensional CAT(0) complexes, cf. e.g. [BH99];
- (2) finite dimensional complexes with a convex geodesic bicombing, [Lan13, DL15];
- (3) finite dimensional complexes with an injective metric [AP56, Isb64, Dre84, Lan13];
- (4) uniformly locally finite Gromov-hyperbolic graphs,
- (5) uniformly locally finite weakly modular graphs [CCHO20];
- (6) systolic complexes [Che00, JŠ06];
- (7) finite dimensional bucolic complexes [BCC<sup>+</sup>13];
- (8) finite dimensional weakly systolic complexes [CO15];
- (9) finite dimensional quadric complexes [Hod20];
- (10) graphical  $C(6)$ ,  $C(4)$ - $T(4)$ , and  $C(3)$ - $T(6)$  small cancellation complexes [OP18, CCG<sup>+</sup>20];

- (11) Salvetti complexes of Artin groups;
- (12) Deligne complexes of Artin groups;

**Conjecture.** *Let  $G$  be a finitely generated group acting by automorphisms on a complex  $X \in \mathcal{X}$ . If the action is locally elliptic then it is elliptic. In particular, if  $G$  is a torsion group then the action is elliptic.*

**Remarks. (A)** In the CAT(0) setting the conjecture concerning the part on torsion groups appears for the first time in [NOP21, Conjecture 1.5]. Then the CAT(0) version of the Conjecture appears in [21] (in a form of a question). Also in [21] the Helly version of the conjecture appears for the first time. For appearances of related questions and conjectures, see the following Subsection 2.2.

**(B)** As noted after Meta-Conjecture in Section 1.1 the assumptions on finite generation and finite dimensionality are essential. In Subsection 2.2 item (VIII) and in Subsection 2.3 we explain why restricting to the combinatorial setting of automorphisms of complexes, rather than (arbitrary) isometries of metric spaces is important.

**(C)** For item (5) we do not know a general definition of a ‘weakly modular complex’, that is, a contractible complex with 1-skeleton being a weakly modular graph – see [CCHO20, Question 9.20]. Therefore, we make a (much stronger) assumption of uniform local finiteness implying finite dimensionality of such potential complex.

**(D)** For item (6) we do not need to assume finite dimensionality. Although systolic complexes can be infinitely dimensional, they behave in many ways as (asymptotically) 2-dimensional objects (see e.g. [JŚ07, Osa08, OŚ15, Osa21]), and we believe this nature makes the Conjecture valid for them. In Theorem F we provide a partial justification for this belief.

**(E)** There are many more spaces that should be added to the class  $\mathcal{X}$ . The reader is welcomed to add her or his favorite NPC-like spaces to the list.

**2.2. Motivations.** The Meta-Conjecture and Conjecture are related to the following notions and properties, and are generalizations and unifications of the following (open) questions and conjectures:

**(I)** A conjecture stating that nonpositively curved groups satisfy the *Tits Alternative*, that is, their finitely generated subgroups are either solvable or contain non-abelian free groups. The statement is quite obvious for Gromov hyperbolic groups. Usually the conjecture is stated for CAT(0) groups; see e.g. [Bes00a, Quest 2.8],[Bri06],[Bri07, Quest 7.1],[FHT11, Prob 12],[Cap14, Sec. 5]. The question of ellipticity of locally elliptic actions may be seen as a first step towards the Tits Alternative. For a recent account on the Tits Alternative in the nonpositively curved setting see e.g. [OP21b, OP21a].

**(II)** A conjecture stating that there are no infinite torsion subgroups of CAT(0) groups, that is, groups acting geometrically on CAT(0) spaces; see e.g. [Swe99], [Bes00b, Question 2.11], [Xie04, p. 88], [Bri07, Question 8.2],

[Kap08, Problem 24], [Cap14, § IV.5]. In [NOP21, Conjecture 1.5] the conjecture that every action of a finitely generated torsion group on a finitely dimensional  $CAT(0)$  complex is elliptic was formulated for the first time.

**(III)** A conjectural equivalence of ‘locally elliptic’ and ‘elliptic’ actions, for actions on buildings, see e.g. [Par03], [Mar13, Conjecture 3.22], [Mar15, Conjecture 1.2]. This is related e.g. to questions concerning bounded subgroups of Kac-Moody groups [Cap09].

**(IV)** Numerous questions in Algebraic Geometry concerning regularizations or linearizations of certain subgroups of the groups of birational transformations of projective surfaces (see e.g. [LU21, Sec. 5.3], [Can11], [Fav10]) or subgroups of automorphism groups of  $\text{Aut}(k^n)$  (see e.g. [LP21, Introduction]).

**(V)** The question of automatic continuity for groups; see e.g. [KMV21]. Except the group theory the question has origins e.g. in the theory of Polish spaces.

**(VI)** Property (FA) of triviality of group actions on trees introduced by Serre in [Ser74, Ser77, Ser03] and generalized to  $\mathbb{R}$ -trees in [MS84]. ( $\mathbb{R}$ -)trees are 1-dimensional nonpositively curved (more precisely,  $CAT(0)$ ) spaces. A generalization to higher dimensions is Farb’s property  $FA_n$  [Far09].

**(VII)** Kazhdan’s property (T), which (for discrete groups) is equivalent to the property of acting trivially on the Hilbert space, being itself  $CAT(0)$ . Notably, Kazhdan’s property was introduced in order to prove finite generation of some groups.

**(VIII)** Kaplansky’s question on unitarizability of linear groups consisting of conjugates of unitary matrices, see e.g. [Bas80, Wat88, BF21]. This question has positive answer in dimensions 2 and 3, and a negative one in dimension 4, see details in Subsection 2.3 below.

**(IX)** General questions concerning the structure and properties of infinite torsion groups. In particular, (not only) we believe (cf. e.g. [Nib98]) this is related to the following fundamental question about torsion groups: Are there finitely presented infinite torsion groups?

**(X)** Question of lilarity of groups, that is, of existence of faithful finite dimensional representations. By Schur’s theorem [Sch11], finitely generated linear torsion groups are finite. (It should be noticed that Tits Alternative takes its name from a result by Tits for linear groups [Tit72].)

**2.3. Discussion of the results and state of the art.** The property of trivial actions on nonpositively curved spaces is related to the (FA) property and Kazhdan’s property (T) – both, trees and Hilbert spaces are  $CAT(0)$ . In particular, by Serre’s result [Ser03] groups acting only trivially on NPC spaces are finitely generated. Finitely generated torsion groups have property (FA) by [Ser03]. However, such groups, even of bounded exponent (that

is, Burnside groups) can act without fixed points on infinite dimensional CAT(0) cube complexes (hence, on Hilbert spaces) by [Osa18a]. Groups with Kazhdan's property (T) have Property (FA) by [Wat82]. However, there are uniform lattices in isometry groups of affine buildings with Kazhdan's property (T).

Groups (not necessarily finitely generated) acting locally elliptically by isometries on the Euclidean spaces  $\mathbb{E}^2, \mathbb{E}^3$  or hyperbolic spaces  $\mathbb{H}^2, \mathbb{H}^3$  have fixed points [Wat88, BF21]. Answering in the negative a question of Kaplansky concerning higher dimensions, Bass [Bas80] and, independently, Waterman [Wat88] constructed examples of two-generated groups of elliptic isometries of  $\mathbb{E}^4$  and  $\mathbb{H}^4$  acting without a fixed point. Breuillard-Fujiwara [BF21] extended such examples to all Euclidean spaces  $\mathbb{E}^{2n}$ , for  $n = 2, 3, 4, \dots$

We believe that the results from the current paper make a significant progress towards the Conjecture. However, even in the CAT(0) setting the Conjecture is widely open. We believe that the case of (Gromov) hyperbolic graphs, as in Corollary B(2) has been known to the experts, and anyway could be established by more standard (that is, not via Helly techniques) methods. However we did not find such a statement in the literature (and we received signals from some colleagues that it was not known to them before). Button [But21] proved the Conjecture in the case of quasitrees, not necessarily being complexes and without further (besides being a quasitree) assumptions on finite dimensionality. Observe that any group can act without fixed points on a hyperbolic combinatorial horoball (see e.g. [GM08]) over its Cayley graph. Such horoball is locally finite, but not uniformly locally finite. This construction shows also that the assumption on finite dimensionality in our Theorem H cannot be dropped.

Imposing some further restrictions on the groups acting, special cases of the Conjecture were established e.g. for general CAT(0) spaces by Caprace-Monod [CM09, Lemma 8.1], and for buildings by Parreau [Par00, Corollaire 2], [Par03, Corollaire 3], and by Marquis [Mar13, Mar15]. Leder-Varghese [LV20] observed that it follows from an earlier work by Sageev [Sag95] that the Conjecture holds in the case of CAT(0) cube complexes. In fact, for CAT(0) cube complexes the Conjecture holds also in the infinite dimensional case when there are no infinite cubes [GLU21]. Norin-Osajda-Przytycki [NOP21] proved the Conjecture for 2-dimensional CAT(0) complexes with additional mild assumptions. In particular they proved the conjecture for all 2-dimensional (discrete) buildings, and they proved that locally elliptic actions of finitely generated torsion groups on 2-dimensional CAT(0) complexes are elliptic. Schillewaert-Struyve-Thomas [SST20] extended such results beyond the discrete case, proving them for all  $\tilde{A}_1 \times \tilde{A}_1$ ,  $\tilde{A}_2$  and  $\tilde{C}_2$  buildings.

Our Corollary B(3) proves the Conjecture (10) in the case of locally finite graphical  $C(4)$ - $T(4)$  complexes. Duda [Dud21] obtained similar results for

not necessarily locally finite classical  $C(4)$ - $T(4)$  complexes and not necessarily finitely generated groups acting on them. In particular, he showed that (not necessarily finitely generated) torsion subgroups of groups with (classical)  $C(4)$ - $T(4)$  small cancellation presentations are finite. In the ‘hyperbolic’ case of  $C'(1/4)$ - $T(4)$  complexes the conjecture was proved in [Gen21, Theorem 8.15].

Regarding Conjecture E, let us mention that for some Coxeter groups the combinatorial dimension was computed in [Eng12].

### 3. HELLY GRAPHS

A connected graph  $X$  is called *Helly* if any family of pairwise intersecting combinatorial balls of  $X$  has a non-empty global intersection. We will consider  $X$  as its vertex set, and we will endow  $X$  with induced graph metric. We refer the reader to [CCG<sup>+</sup>20] for a presentation of Helly graphs and Helly groups.

One may think of Helly graphs as a very nice class of nonpositively curved, combinatorially defined spaces. Surprisingly enough, many nonpositive curvature metric spaces and groups have a very close relationship to Helly graphs or their non-discrete counterpart, injective metric spaces.

For instance, the thickening of any CAT(0) cube complex is a Helly graph (see [BvdV91], and also [HW09, Corollary 3.6]). Lang showed that the any Gromov hyperbolic group acts properly cocompactly on the Helly hull of any Cayley graph (see [Lan13, CCG<sup>+</sup>20]). Huang and Osajda proved that any weak Garside group and any Artin group of type FC has a proper and cocompact action on a Helly graph (see [HO21]). Osajda and Valiunas proved that any group that is hyperbolic relative to Helly groups is Helly (see [OV20]). Haettel, Hoda and Petyt proved that any hierarchically hyperbolic group, and in particular any mapping class group of a surface, has a proper and cobounded action on an injective metric space, see [HHP21].

Concerning Euclidean buildings, recall the following statement.

**Theorem 3.1** (Hirai, Chalopin et al, Haettel). *The thickening of any Euclidean building of type  $\tilde{A}$  extended,  $\tilde{B}$ ,  $\tilde{C}$  or  $\tilde{D}$  is Helly.*

Hirai, and Chalopin et al. proved the case of Euclidean buildings of type  $\tilde{A}$  extended and  $\tilde{C}$ , see [Hir20] and [CCHO20]. In [Hae21a] and [Hae21b], Haettel proved the statement for all Euclidean buildings of type  $\tilde{A}$  extended,  $\tilde{B}$ ,  $\tilde{C}$  or  $\tilde{D}$ . There is an analogous result for classical symmetric spaces, see [Hae21a] for a precise statement.

The analogy between hyperbolic groups, CAT(0) groups and Helly groups is quite rich, as the following survey of results show.

**Theorem 3.2** (Analogies hyperbolic / CAT(0) / Helly). *Assume that a finitely generated group  $G$  acts properly and cocompactly by isometries on a hyperbolic space, CAT(0) space or a Helly graph. Then*

- $G$  is semi-hyperbolic in the sense of Alonso-Bridson (see [AB95] and [Lan13]), which has numerous consequences (see [BH99, III.Γ.4]).
- $G$  has finitely many conjugacy classes of finite subgroups (see [BH99] and [Lan13]).
- Asymptotic cones of  $G$  are contractible (see [CCG<sup>+</sup>20, Theorem 1.5]).
- $G$  admits an EZ-structure (see [Bes96] and [CCG<sup>+</sup>20, Theorem 1.5]).
- $G$  satisfies the Farrell-Jones conjecture (see [KR17] and [CCG<sup>+</sup>20, Theorem 1.5]).
- $G$  satisfies the coarse Baum-Connes conjecture (see [FO20] and [CCG<sup>+</sup>20, Theorem 1.5]).
- $G$  is finitely presented and has at most quadratic Dehn function (see [BH99, Proposition III.Γ.1.6] and [CCG<sup>+</sup>20, Theorem 1.5]).
- $G$  has type  $F_\infty$  (see [BH99, Proposition II.5.13] and [CCG<sup>+</sup>20, Theorem 1.5]).

Recall that a geodesic metric space is called *injective* if any family of pairwise intersecting closed balls has a non-empty global intersection. We refer the reader to [Lan13] for a presentation of injective metric space, and also the following result of Isbell.

**Theorem 3.3** ([Isb64]). *Let  $X$  denote a metric space. Then there exists an essentially unique minimal injective space  $EX$  containing  $X$ , called the injective hull of  $X$ .*

In [Lan13], Lang describes a cell structure on the injective hull of a connected graph. We describe below (in Section 4) a refinement of Lang’s cell decomposition into orthosimplices. Recall that the *standard orthosimplex* of dimension  $n$  with edge lengths  $\ell > 0$  is the simplex of  $\mathbb{R}^n$  with vertices  $(0, \dots, 0), (\ell, 0, \dots, 0), \dots, (\ell, \ell, \dots, \ell)$ , see Figure 1. We will endow this simplex with the standard  $\ell^\infty$  metric on  $\mathbb{R}^n$  (except in Section 9, where we will also consider the standard  $\ell^2$  metric).

Recall that the *combinatorial dimension* of a metric space  $X$  is the dimension of its injective hull  $EX$  (this has been defined by Dress, see [Dre84]). There are interesting examples of locally infinite Helly graphs with finite combinatorial dimension, such as thickenings of locally infinite, finite-dimensional CAT(0) cube complexes.

Lang proved a criterion ensuring that the Helly hull of a graph is locally finite. A connected graph  $X$  has *stable intervals* if there exists  $\beta \geq 0$  such that, for any  $x, y, z \in X$  such that  $y$  and  $z$  are adjacent, for any geodesics  $[x, y]$  and  $[x, z]$ , the Hausdorff distance between  $[x, y]$  and  $[x, z]$  is at most  $\beta$ . According to [CCG<sup>+</sup>20, Lemma 6.5], Helly graphs have 1-stable intervals.

**Theorem 3.4** ([Lan13, CCG<sup>+</sup>20]). *Let  $X$  be a locally finite connected graph with stable intervals. The Helly hull of  $X$  is locally finite.*

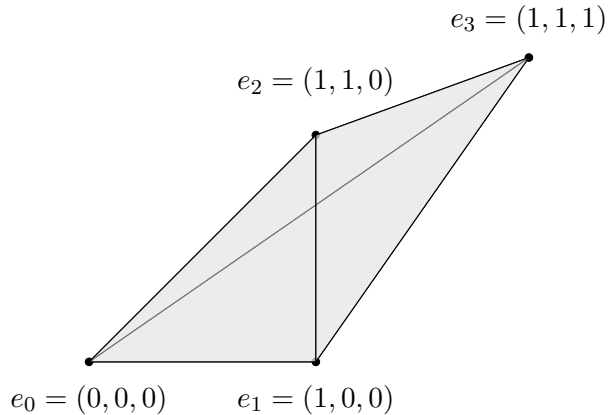


FIGURE 1. The standard orthosimplex of dimension 3 with edge lengths 1.

From this, we deduce that Theorem A is equivalent to Theorem H. Since Helly graphs have stable intervals, Theorem A implies Theorem H. Conversely, consider a finitely generated group acting locally elliptically on a locally finite graph  $X$  with stable intervals and finite combinatorial dimension. The Helly hull  $X'$  of  $X$  is a locally finite Helly graph with finite combinatorial dimension, and  $G$  acts by locally elliptic automorphisms on  $X'$ . Thus Theorem H implies Theorem A.

*Proof of Corollary B.* The result follows from Theorem H, and the aforementioned constructions of Helly graphs associated with the corresponding complexes for: buildings [CCHO20, Hir20, Hae21a, Hae21b], hyperbolic graphs [Lan13, CCG<sup>+</sup>20], small cancellation complexes [CCG<sup>+</sup>20], Salvetti complexes [HO21].  $\square$

*Proof of Corollary C.* This follows from the fact that the direct product of Helly graphs is Helly (see e.g. [CCHO20, Proposition 5.1]).  $\square$

#### 4. ORTHOSCHEME COMPLEXES AND HELLY GRAPHS

We now present a refinement of Lang’s description of the cell structure on the injective hull of a connected graph (see [Lan13]).

**Theorem 4.1.** *Let  $X$  denote a Helly graph with finite combinatorial dimension. For each  $N \geq 1$ , there exists a simplicial structure on the injective hull  $EX$  of  $X$ , denoted  $O_N X$  and called the ( $N^{\text{th}}$ ) orthoscheme complex of  $X$ , satisfying the following:*

- *Each simplex of  $O_N X$  is isometric to the standard  $\ell^\infty$  orthosimplex with edge lengths  $\frac{1}{2^{N!}}$ .*
- *The vertex set  $X'_N$  of  $O_N X$ , endowed with the induced distance, is a Helly graph (with edge lengths  $\frac{1}{2^{N!}}$ ), containing isometrically  $X$ ,*

called the ( $N^{\text{th}}$ ) Helly subdivision of  $X$ . Moreover, we have

$$X'_N = \left\{ p \in EX \mid \forall x \in X, d(p, x) \in \frac{1}{2N!}\mathbb{N} \right\}.$$

- There exists a map  $\phi$  from the set of vertices of  $X'_N$  to the set of cliques in  $X$ , equivariant with respect to automorphisms of  $X$ .
- For any  $p \in O_N X$  and for any simplex of  $O_N X$  containing  $p$  with vertices  $x_1, \dots, x_n$  in  $X'_N$ , there exist  $t_1, \dots, t_n \geq 0$  such that  $t_1 + \dots + t_n = 1$  and

$$\forall q \in O_N X, d(p, q) = \sum_{i=1}^n t_i d(x_i, q).$$

*Proof.* According to [Lan13, Theorem 4.5], the injective hull  $EX$  may be realized as an isometric subset of  $\mathbb{R}^X$ , and the injective hull  $EX$  of  $X$  has a natural cell decomposition satisfying the following. For each cell  $C$  of  $EX$ , there is a finite set of vertices  $x_1, \dots, x_n$  of  $X$  such that the map

$$\begin{aligned} C &\rightarrow \mathbb{R}^n \\ p &\mapsto (d(p, x_1), \dots, d(p, x_n)) \end{aligned}$$

is an isometry (with the  $\ell^\infty$  metric on  $\mathbb{R}^n$ ) onto the compact convex subspace of  $\mathbb{R}^n$  defined by inequalities of the type

$$\pm d(\cdot, x_i) \pm d(\cdot, x_j) \leq D,$$

for some  $1 \leq i < j \leq n$  and  $D \in \mathbb{Z}$ , and also of the type

$$\pm d(\cdot, x_i) \leq D',$$

for some  $1 \leq i \leq n$  and  $D' \in \frac{1}{2}\mathbb{Z}$ . In particular there is an affine structure on  $C$ . Moreover, for any  $p \in EX$ , for any  $p_1, \dots, p_k \in C$  and  $t_1, \dots, t_k \geq 0$  such that  $t_1 + \dots + t_k = 1$ , we have

$$d(p, \sum_{i=1}^k t_i p_i) = \sum_{i=1}^k t_i d(p, p_i).$$

Note that the hyperplanes of  $\mathbb{R}^n$

$$\left\{ \pm x_i \pm x_j = D \mid 1 \leq i < j, D \in \frac{1}{N!}\mathbb{Z} \right\} \text{ and } \left\{ x_i = D' \mid 1 \leq i \leq n, D' \in \frac{1}{2N!}\mathbb{Z} \right\}$$

partition  $\mathbb{R}^n$  into (open) standard orthosimplices with edge lengths  $\frac{1}{2N!}$ , see Figure 2.

We may consider the refinement of Lang's cell decomposition of  $EX$ , obtained by considering all possible hyperplanes  $\{d(\cdot, x) \pm d(\cdot, y) = D\}$ , for  $x, y \in X$  and  $D \in \frac{1}{N!}\mathbb{Z}$ , and  $\{d(\cdot, x) = D'\}$ , for  $x \in X$  and  $D' \in \frac{1}{2N!}\mathbb{Z}$ . Each cell from Lang's decomposition is now refined into a finite union of orthoscheme simplices with edge lengths  $\frac{1}{2N!}$ . Let us denote by  $O_N X$  the corresponding simplicial complex. Note that the geometric realization of  $O_N X$  is naturally identified with  $EX$ .



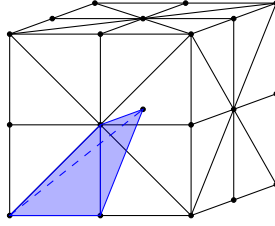


FIGURE 2. The partition of a cube in  $\mathbb{R}^3$  into standard orthosimplices.

The vertex set of  $O_N X$  will be denoted  $X'_N$ , and called the Helly subdivision of  $X$ . When  $EX$  is realized as an isometric subset of  $\mathbb{R}^X$ , the vertex set  $X'_N$  is naturally identified with

$$X' = EX \cap \left( \frac{1}{2N!} \mathbb{N} \right)^X = \left\{ p \in EX \mid \forall x \in X, d(p, x) \in \frac{1}{2N!} \mathbb{N} \right\}.$$

According to [CCG<sup>+</sup>20, Theorem 4.4],  $X'_N$  is a Helly graph (with edge length  $\frac{1}{2N!}$ ).

To each vertex  $p \in O_N X$ , we will associate a clique  $\phi(p)$  in  $X$ . Let us define

$$\phi(p) = \bigcap_{x \in X} B(x, \lceil d(x, p) \rceil).$$

Since  $X$  is a Helly graph, we know that  $\phi(p) \neq \emptyset$ . For any  $y \in \phi(p)$ , since  $p \in O_N X$  is extremal (see [Lan13]), there exists  $x \in X$  such that  $d(x, p) + d(p, y) = d(x, y)$ , and since  $d(x, y) \leq \lceil d(x, p) \rceil$  we deduce that  $d(p, y) < 1$ . Then for any  $y, z \in \phi(p)$ , we have  $d(y, z) \leq d(y, p) + d(p, z) < 2$ , so  $\phi(p)$  is a clique of  $X$ .  $\square$

We now use the orthoscheme complex to study fixed point sets of automorphisms.

**Lemma 4.2.** *Let  $X$  denote a Helly graph with finite combinatorial dimension  $N - 1$ , and let  $g$  denote an automorphism of  $X$ . If  $g$  fixes a point  $p \in EX$ , then  $g$  fixes pointwise the minimal simplex of  $O_N X$  containing  $p$ . In particular,  $g$  fixes a vertex of  $X'_N$  and stabilizes a clique in  $X$ .*

*Proof.* Let  $C$  denote the minimal simplex of  $O_1 X$  containing  $p$ , and let  $x_1, \dots, x_n$  denote the vertices of  $C$ , with  $n \leq N$ . Since  $g \cdot p = p$ , we deduce that  $g$  induces a permutation of  $x_1, \dots, x_n$ . Let  $A$  denote the partition of  $\{1, \dots, n\}$  into  $g$ -orbits.

According to Theorem 4.1, let us consider the unique  $t_1, \dots, t_n > 0$  such that  $t_1 + \dots + t_n = 1$  and  $p = t_1 x_1 + \dots + t_n x_n$ . Since  $g \cdot p = p$ , we deduce that, for any  $I \in A$  and for any  $i, j \in I$ , we have  $t_i = t_j$ .

For each  $I \in A$ , let  $x_I = \frac{1}{\#I} \sum_{i \in I} x_i$ : since  $\#I \leq n \leq N$ , it divides  $N!$ . For each  $i \in I$  and  $x \in X$  we have  $d(x, x_i) \in \frac{1}{2}\mathbb{N}$ , so  $d(x, x_I) \in \frac{1}{2N!}\mathbb{N}$ . In particular,  $x_I$  is a vertex of  $O_N X$ . Then  $p$  is a convex combination of  $\{x_I\}_{I \in A}$  with positive coefficients, so the minimal simplex of  $O_N X$  containing  $p$  is contained in the convex hull  $C_A$  of  $\{x_I\}_{I \in A}$ . It is clear that each vertex  $x_I$ , for  $I \in A$ , is fixed by  $g$ , so  $C_A$  is fixed pointwise by  $g$ . In particular, the minimal simplex of  $O_N X$  containing  $p$  is fixed pointwise by  $g$ .

Furthermore, for any  $I \in A$ , the vertex  $x_I$  is a vertex of  $O_N X$ , so according to Theorem 4.1 we deduce that  $g$  stabilizes the clique  $\phi(x)$  in  $X$ .  $\square$

We now state a simple lemma about fixed point sets of two elliptic groups, that will be used in Section 7.

**Lemma 4.3.** *Let  $X$  denote a Helly graph with finite combinatorial dimension  $N - 1$ , and let  $G, H$  denote automorphism groups of  $X$  which have fixed points in  $EX$ . Then the distance between  $EX^G$  and  $EX^H$  is realized by vertices in the Helly subdivision  $X'_N$  of  $X$ .*

*Proof.* Let  $p \in EX^G$  and  $p' \in EX^H$ . Denote by  $C, C'$  the minimal simplices of  $O_N X$  containing  $p, p'$  respectively. According to Lemma 4.2, we know that  $C \subset EX^G$  and  $C' \subset EX^H$ .

According to Theorem 4.1, the distance  $d(p, p')$  is a convex combination of distances between vertices of  $C$  and of  $C'$ , so in particular there exist vertices  $x \in C$  and  $x' \in C'$  such that  $d(x, x') \leq d(p, p')$ .

We deduce that the distance  $d(EX^G, EX^H)$  is attained, and it is realized by vertices of  $X'_N$ .  $\square$

## 5. CLASSIFICATION OF AUTOMORPHISMS OF HELLY GRAPHS

We now turn to the study of automorphisms of Helly graphs, and the proof of the classification Theorem O.

Fix a Helly graph  $X$ . An automorphism  $g$  of  $X$  is called:

- *elliptic* if  $g$  has bounded orbits in  $X$ .
- *hyperbolic* if, for some vertex  $x \in X$ , the map  $n \in \mathbb{Z} \mapsto g^n \cdot x \in X$  is a quasi-isometric embedding.

We now give several simple equivalent characterizations of elliptic groups of automorphisms.

**Proposition 5.1.** *Let  $G$  denote a group of automorphisms of a Helly graph  $X$ . The following are equivalent:*

- (1)  $G$  stabilizes a clique in  $X$ ,
- (2)  $G$  fixes a point in the injective hull  $EX$  of  $X$  and
- (3)  $G$  has a bounded orbit in  $X$ .

Furthermore, if  $X$  a finite combinatorial dimension  $N - 1$ , these properties are also equivalent to:

(4)  $G$  fixes a vertex in the Helly subdivision  $X'_N$  of  $X$ ,

Such a group is called an elliptic group of automorphisms of  $X$ .

*Proof.*

1.  $\Rightarrow$  3. If  $G$  stabilizes a clique in  $X$ , it is clear that  $G$  has a bounded orbit in  $X$ .
3.  $\Rightarrow$  2. According to [Lan13, Proposition 1.2], if  $G$  has a bounded orbit in  $X$ , then  $G$  has a fixed point in  $EX$ .
2.  $\Rightarrow$  1. Let  $p \in EX$  denote a point fixed by  $G$ , and let

$$\phi(p) = \bigcap_{x \in X} B(x, \lceil d(x, p) \rceil).$$

According to the proof of Theorem 4.1 (which does not require any finiteness assumption),  $\phi(p)$  is a clique of  $X$ . Since  $p$  is fixed by  $G$ , we deduce that  $\phi(p)$  is stabilized by  $G$ .

4.  $\Rightarrow$  3. If  $G$  fixes a vertex in  $X'_N$ , then  $G$  has a bounded orbit in  $EX$ , so  $G$  has a bounded orbit in  $X$ .
2.  $\Rightarrow$  4. Assume that  $X$  has finite combinatorial dimension  $N - 1$ . Let  $p \in EX$  denote a point fixed by  $G$ , and let  $C$  denote the minimal simplex of  $O_N X$  containing  $p$ . According to Lemma 4.2,  $G$  fixes  $C$  pointwise. In particular, if  $x$  denote a vertex of  $C$ , then  $x$  is a vertex of  $X'_N$  that is fixed by  $G$ .

□

We deduce the following important classification of automorphisms of Helly graphs.

**Theorem 5.2.** *Let  $X$  be a Helly graph with finite combinatorial dimension. Then any automorphism of  $X$  is either elliptic or hyperbolic.*

*Proof.* Let  $N - 1$  denote the combinatorial dimension of  $X$ . Fix an automorphism  $g$  of  $X$ . Let  $D = \inf_{p \in EX} d(g \cdot p, p)$ . Consider any  $p \in EX$  such that  $d(p, g \cdot p) \leq D + 1$ , and let  $C$  denote the minimal simplex of the orthoscheme complex  $O_1 X$  of  $X$  containing  $p$ . Since simplices in  $O_1 X$  have diameter at most  $\frac{1}{2}$ , vertices of  $C$  and  $g \cdot C$  are at most  $D + 2$  apart.

According to Theorem 4.1, for each point  $q \in C$ , the distance function  $d(q, \cdot)$  on  $EX$  is a convex combination of distance functions to vertices of  $C$ . The simplex  $C$  has dimension at most  $N - 1$ , so  $C$  has at most  $N$  vertices. The pairwise distances between vertices of  $C$  and  $g \cdot C$  form a collection of at most  $\binom{2N}{2}$  half-integers between 0 and  $D + 2$ . Each such collection of distances determines a unique minimal distance between  $C$  and  $g \cdot C$ . So the infimum  $D$  is realized: in other words, the isometry  $g$  of  $EX$  is semisimple.

Assume that  $D = 0$ , and let  $p \in EX$  such that  $g \cdot p = p$ . According to Proposition 5.1,  $g$  is elliptic.

Assume now that  $D > 0$ , and let  $p \in EX$  such that  $d(p, g \cdot p) = D$ . According to [Lan13, Proposition 3.8],  $EX$  has a conical geodesic bicombing. So according to [DL16, Proposition 4.2], for any  $n \geq 1$ , we have  $\min_{q \in EX} d(g^n \cdot q, q) = nD$ . In particular, for any  $n \in \mathbb{N}$ , we have  $d(p, g^n \cdot p) = nD$ . So the orbit map  $n \in \mathbb{Z} \mapsto g^n \cdot p \in EX$  is a quasi-isometric embedding:  $g$  is hyperbolic.

This concludes the proof that any automorphism of  $X$  is either elliptic or hyperbolic.  $\square$

We deduce the following equivalent characterizations of hyperbolic automorphisms.

**Proposition 5.3.** *Let  $g$  denote an automorphism of a Helly graph  $X$  with finite combinatorial dimension  $N$ . The following are equivalent:*

- (1)  $g$  is hyperbolic, i.e for some vertex  $x \in X$ , the map  $n \in \mathbb{Z} \mapsto g^n \cdot x \in X$  is a quasi-isometric embedding.
- (2)  $g$  has a geodesic axis in the injective hull  $EX$  of  $X$ .
- (3) There exists a vertex  $x$  of the Helly subdivision  $X'$  of  $X$  and integers  $1 \leq a \leq 2N$  and  $L \in \mathbb{N} \setminus \{0\}$  such that  $\forall n \in \mathbb{N}, d(x, g^{an} \cdot x) = nL$ .
- (4)  $g$  has unbounded orbits in  $X$ .

*Proof.*

1.  $\Rightarrow$  2. According to [Lan13, Proposition 3.8],  $EX$  has a conical, geodesic bicombing. According to [DL16, Proposition 4.2], the isometry  $g$  of  $EX$  is hyperbolic, so the isometry  $g$  itself is hyperbolic, and it has a geodesic axis in  $EX$ .
2.  $\Rightarrow$  3. Let  $D = \min_{p \in EX} d(g \cdot p, p) > 0$ , and let  $p \in EX$  such that  $d(p, g \cdot p) = D$ . Since  $g$  has a geodesic axis in  $EX$ , we may assume that  $p$  lies in a simplex  $C$  of  $O_1X$  of codimension at least 1.

Let  $x_1, \dots, x_n$  denote the vertices of  $C$ : we have  $n \leq (N+1) - 1 = N$ . For each  $k \geq 2$ , let  $A_k = \{1, 2, \dots, n\}^k$ . For each  $a \in A_k$ , let us define

$$f(a) = \sum_{i=1}^{k-1} d(g^{i-1} \cdot x_{a_i}, g^i \cdot x_{a_{i+1}}).$$

Let us also define

$$\alpha = \inf \left\{ \frac{f(a)}{k-1} \mid k \geq 2, a \in A_k, a_1 = a_k \right\}.$$

One can interpret these quantities in terms of lengths of paths in a graph. Consider the finite graph  $\Gamma$  with vertices labeled  $1, \dots, n$ , such that given any two vertices  $i, j$ , there exists one oriented edge from  $i$  to  $j$  with length  $d(x_i, g \cdot x_j)$ . The set  $A_k$  is the set of oriented paths of  $k+1$  vertices in  $\Gamma$ , and  $f(a)$  is the length of the path  $a$ . Finally,  $\alpha$  is the minimal average length of an oriented loop.

We claim that  $\alpha$  is attained by some element  $a \in A_k$  with  $a_1 = a_k$  such that  $k \leq n + 1$ . Consider some  $k \geq 3$  and  $a \in A_k$  with  $a_1 = a_k$  such that, for any  $k' < k$  and  $a' \in A_{k'}$  with  $a'_1 = a'_{k'}$ , we have  $\frac{f(a')}{k'-1} > \frac{f(a)}{k-1}$ . We will prove that  $k \leq n + 1$ . By contradiction, if  $k > n + 1$ , since there are  $n$  vertices  $x_1, \dots, x_n$ , there exists a strict subloop  $a'$  of  $a$  consisting of  $k'$  vertices, with  $k' < k$ . Since  $\frac{f(a')}{k'-1} > \frac{f(a)}{k-1}$ , removing the loop  $a'$  decreases the average length of the loop, which contradicts the assumption. Hence  $k \leq n + 1$ . Since  $A_{n+1}$  is finite, we also deduce that  $\alpha$  is attained.

Let  $2 \leq k \leq n + 1$  and  $a \in A_k$  with  $a_1 = a_k$  such that  $\frac{f(a)}{k-1} = \alpha$ . In particular, since  $a_k = a_1$ , we have

$$d(a_1, g^{k-1} \cdot a_1) \leq \sum_{i=1}^{k-1} d(g^{i-1} \cdot a_i, g^i \cdot a_{i+1}) = f(a) = (k-1)\alpha.$$

Hence  $(k-1)D \leq (k-1)\alpha$ , so  $D \leq \alpha$ . We will prove that  $D = \alpha$ .

Fix an integer  $h > n$ . For any  $a \in A_h$ , there exists a subloop consisting of at least  $h - n$  vertices, hence  $f(a) \geq (h - n)\alpha$ . Let  $t_1, \dots, t_n \in \mathbb{R}_+$  such that  $t_1 + \dots + t_n = 1$  and  $p = t_1 x_1 + \dots + t_n x_n$ . We have

$$\begin{aligned} & \sum_{a \in A_h} t_{a_1} t_{a_2} \cdots t_{a_h} f(a) \\ &= \sum_{a \in A_h} t_{a_1} t_{a_2} \cdots t_{a_h} (d(x_{a_1}, g \cdot x_{a_2}) + d(g \cdot x_{a_2}, g^2 \cdot x_{a_3}) + \\ & \quad \cdots + d(g^{h-2} \cdot x_{a_{h-1}}, g^{h-1} \cdot x_{a_h})) \\ &= (h-1) \sum_{a \in A_h} t_{a_1} t_{a_2} \cdots t_{a_h} d(x_{a_1}, g \cdot x_{a_2}) \\ &= (h-1) \sum_{a \in A_2} t_{a_1} t_{a_2} d(x_{a_1}, g \cdot x_{a_2}) \\ &= (h-1) d(p, g \cdot p) = (h-1)D. \end{aligned}$$

For any  $a \in A_h$ , we have  $f(a) \geq (h - n)\alpha$ , hence  $(h - 1)D \geq (h - n)\alpha$ . So  $D \geq \frac{h-n}{h-1}\alpha$ . Since this holds for any  $h > n$ , we conclude that  $D \geq \alpha$ . Hence  $D = \alpha$ .

Any two vertices of  $X'$  have distance in  $\frac{1}{2}\mathbb{N}$ , so we deduce that  $\alpha \in \frac{1}{2(k-1)}\mathbb{N}$ . In particular,  $D$  is rational, and its denominator is at most  $2n \leq 2N$ .

3.  $\Rightarrow$  4. This is immediate.

4.  $\Rightarrow$  1. If  $g$  has unbounded orbits in  $X$ , by definition  $g$  is not elliptic. According to Theorem 5.2,  $g$  is hyperbolic.

□

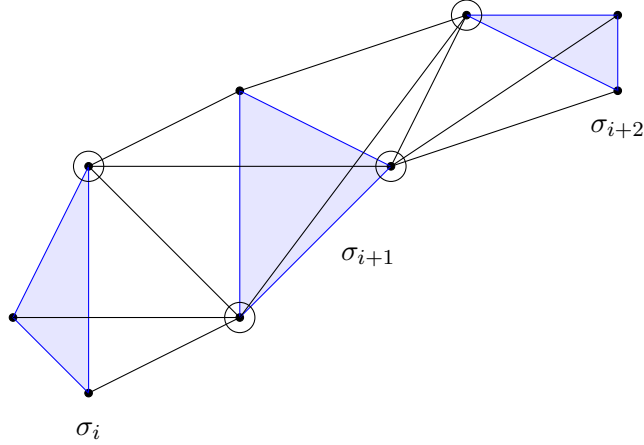


FIGURE 3. A 1-clique-path  $\sigma_i, \sigma_{i+1}, \sigma_{i+2}$ , where vertices connecting to neighbouring cliques are circled.

We deduce the following interesting corollary about translations lengths in Helly graphs, which directly generalizes the analogous theorem by Gromov about translation lengths in Gromov-hyperbolic groups (see [Gro87, 8.5.S]). Since Garside groups are Helly according to [HO21], this implies a direct analogue of [LL07] for a very closely related translation length.

**Corollary 5.4.** *Let  $X$  denote a Helly graph with finite combinatorial dimension  $N$ . Then any hyperbolic automorphism of  $X$  has rational translation length in  $X$ , with denominator uniformly bounded by  $2N$ .*

## 6. CLIQUE-PATHS

We now turn to the description of clique-paths in Helly graphs. They will be a subtle variation of normal clique-paths from [CCG<sup>+</sup>20] and from normal cube paths from [NR97]. They will be flexible enough to allow any hyperbolic automorphism to admit an invariant clique-path (see Proposition 6.5), and still allow a local-to-global rigidity (see Theorem 6.4).

Let  $X$  be a graph, and let  $\sigma, \tau$  be cliques in  $X$ . We will say that  $\sigma, \tau$  are at *transverse distance*  $n \geq 0$ , denoted  $\sigma \leq \tau = n$ , if the following hold:

- for every  $x \in \sigma$  and  $y \in \tau$ , we have  $d(x, y) \geq n$ ,
- there exists  $x \in \sigma$  such that  $\forall y \in \tau, d(x, y) = n$  and
- there exists  $y \in \tau$  such that  $\forall x \in \sigma, d(x, y) = n$ .

Say that a sequence of pairwise disjoint cliques  $\sigma_0, \dots, \sigma_n$  is an  *$L$ -clique-path* if for some  $L \geq 1$  the following hold:

- For every  $0 \leq i < j \leq n$ , we have  $\sigma_i \leq \sigma_j = (j - i)L$ .
- For any  $1 \leq i \leq n - 1$ ,  $\sigma_i$  is a clique satisfying  $\sigma_i \leq \sigma_{i+1} = L$  and  $\sigma_i \leq \sigma_{i-1} = L$ , and  $\sigma_i$  is a maximal such clique.

Say that a sequence of cliques  $\sigma_0, \dots, \sigma_n$  is a *local  $L$ -clique-path* if, for any  $0 \leq i \leq n-2$ , the sequence  $\sigma_i, \sigma_{i+1}, \sigma_{i+2}$  is an  $L$ -clique-path. In other words:

- For any  $0 \leq i \leq n-1$ ,  $\sigma_i \leq \sigma_{i+1} = L$ .
- For any  $1 \leq i \leq n-1$ ,  $\sigma_i$  is a clique satisfying  $\sigma_i \leq \sigma_{i+1} = L$  and  $\sigma_i \leq \sigma_{i-1} = L$ , and  $\sigma_i$  is a maximal such clique.
- For any  $0 \leq i \leq n-2$ ,  $\sigma_i \leq \sigma_{i+2} = 2L$ .

Observe that an  $L$ -clique-path is a local  $L$ -clique-path. In Theorem 6.4 below we will prove a version of a converse statement.

If  $L = 1$ , we will also simply say (local) clique-path. See Figure 3.

**Lemma 6.1.** *Fix a locally finite Helly graph  $X$ . Consider any two cliques  $\sigma, \tau$  in  $X$  such that  $\sigma \leq \tau = nL$ , for some integers  $n, L \in \mathbb{N} \setminus \{0\}$ . There exists an  $L$ -clique-path  $\sigma_0 = \sigma, \sigma_1, \dots, \sigma_n = \tau$  between  $\sigma$  and  $\tau$  of length  $n$ . Moreover, if  $x_0 \in \sigma$  and  $x_n \in \tau$  are such that  $\tau \subset B(x_0, nL)$  and  $\sigma \subset B(x_n, nL)$ , one can choose a  $L$ -clique-path  $\sigma_0 = \sigma, \sigma_1, \dots, \sigma_n = \tau$  such that  $\sigma_1 \subset B(x_0, L)$  and  $\sigma_{n-1} \subset B(x_n, L)$ .*

*Proof.* We will first prove the statement for  $L = 1$ , and by induction on  $n \geq 1$ . For  $n = 1$  there is nothing to prove.

By induction, assume that  $n \geq 2$ , and assume that the statement is true for  $n-1$ . Consider cliques  $\sigma, \tau$  such that  $\sigma \leq \tau = n$ . Let  $x_0 \in \sigma_0 = \sigma$  be such that  $\sigma_n = \tau \subset B(x_0, n)$ , and let  $x_n \in \sigma_n$  be such that  $\sigma_0 \subset B(x_n, n)$ .

The balls  $B(t_0, n-1)$ , for  $t_0 \in \sigma_0$ , and  $B(x_n, 1)$  pairwise intersect, so as  $X$  is a Helly graph there exists  $x \in \cap_{t_0 \in \sigma_0} B(t_0, n-1) \cap B(x_n, 1)$ . Now the balls  $B(t_n, 1)$ , for  $t_n \in \sigma_n$ ,  $B(x_0, n-1)$  and  $B(x, 1)$  pairwise intersect, so there exists  $y \in \cap_{t_n \in \sigma_n} B(t_n, 1) \cap B(x_0, n-1) \cap B(x, 1)$ . So the clique  $\sigma'_{n-1} = \{x, y\}$  satisfies  $\sigma'_{n-1} \leq \sigma_0 = n-1$  and  $\sigma_{n-1} \leq \sigma_n = 1$ .

By induction, there exists a clique-path  $\sigma'_0, \sigma'_1, \dots, \sigma'_{n-1}$  from  $\sigma'_0 = \sigma_0$  to  $\sigma'_{n-1}$  of length  $n-1$ .

Let  $\sigma'_n = \sigma_n$ . For any  $0 \leq i \leq n-1$ , we have  $\sigma'_i \leq \sigma'_{i+1} = 1$ . Therefore, since  $X$  is locally finite, we may consider a maximal family of cliques  $\sigma_1, \dots, \sigma_{n-1}$  containing  $\sigma'_1, \dots, \sigma'_{n-1}$  respectively, such that for any  $1 \leq i \leq n-1$ , we have  $\sigma_i \leq \sigma_{i-1} = 1$  and  $\sigma_i \leq \sigma_{i+1} = 1$ .

We will prove that  $\sigma_0, \sigma_1, \dots, \sigma_n$  is a 1-clique-path. For any  $0 \leq i < j \leq n$ , let  $y_i \in \sigma_i$  and  $y_j \in \sigma_j$ . Since  $d(x_0, y_i) + d(y_i, y_j) + d(y_j, x_n) \geq d(x_0, x_n) = n$ , we deduce that  $d(y_i, y_j) \geq j - i$ . So we deduce that  $\sigma_i \leq \sigma_j = j - i$ .

Hence  $\sigma_0, \sigma_1, \dots, \sigma_{n-1}, \sigma_n$  is a 1-clique-path from  $\sigma$  to  $\tau$ .

Now consider the case  $L \geq 2$ , and consider cliques  $\sigma, \tau$  such that  $\sigma \leq \tau = nL$ . According to the case  $L = 1$ , there exists a 1-clique path  $\sigma'_0 = \sigma, \sigma'_1, \dots, \sigma'_{nL} = \tau$  from  $\sigma$  to  $\tau$  of length  $nL$ . For every  $0 \leq i \leq n-1$ , we have  $\sigma'_{iL} \leq \sigma'_{iL+1} = L$ . Since  $X$  is locally finite, we may consider a maximal family of cliques  $\sigma_1, \dots, \sigma_{n-1}$  containing  $\sigma'_L, \dots, \sigma'_{(n-1)L}$  respectively, such

that for any  $1 \leq i \leq n-1$ , we have  $\sigma_i \leq \sigma_{(i-1)L} = L$  and  $\sigma_i \leq \sigma_{i+1} = L$ . We then argue as before that, for any  $0 \leq i < j \leq n$ , we have  $\sigma_i \leq \sigma_j = (j-i)L$ . So  $\sigma_0 = \sigma, \sigma_1, \dots, \sigma_n = \tau$  is a  $L$ -clique path from  $\sigma$  to  $\tau$  of length  $n$ .  $\square$

We will now prove the main local-to-global property of clique-paths, starting with the case of local 1-clique-paths.

**Proposition 6.2.** *Fix a Helly graph  $X$ ,  $n \geq 2$ , and consider a local clique-path  $\sigma_0, \dots, \sigma_n$  in  $X$ . Let  $x_0 \in \sigma_0$  such that  $\sigma_1 \subset B(x_0, 1)$ , and let  $x_n \in \sigma_n$  such that  $\sigma_{n-1} \subset B(x_n, 1)$ . Then  $\{x_0\}, \sigma_1, \sigma_2, \dots, \sigma_{n-1}, \{x_n\}$  is a clique-path.*

*Proof.* We will prove the statement by induction on  $n \geq 2$ . For  $n = 2$  any local clique-path is a clique-path, so fix  $n \geq 3$  and assume that the statement holds for  $n-1$ . Fix a local clique-path  $\sigma_0, \dots, \sigma_n$ .

By induction, we only need to prove that  $\{x_0\} \leq \sigma_{n-1} = n-1$ ,  $\{x_n\} \leq \sigma_0 = n-1$  and  $d(x_0, x_n) = n$ . The first two equalities are consequences of  $d(x_0, x_n) = n$ : we will prove it.

By contradiction, assume that  $d(x_0, x_n) \leq n-1$ . The balls  $B(x_0, n-2)$ ,  $B(x_n, 1)$ ,  $B(y_{n-2}, 1)$  for  $y_{n-2} \in \sigma_{n-2}$  and  $B(y_{n-1}, 1)$  for  $y_{n-1} \in \sigma_{n-1}$  pairwise intersect: since  $X$  is a Helly graph, we may consider

$$z \in B(x_0, n-2) \cap B(x_n, 1) \cap \bigcap_{y_{n-2} \in \sigma_{n-2}} B(y_{n-2}, 1) \cap \bigcap_{y_{n-1} \in \sigma_{n-1}} B(y_{n-1}, 1).$$

We will prove that  $z \in \sigma_{n-1}$ .

The vertex  $z$  is adjacent to  $\sigma_{n-1}$ , let  $\sigma'_{n-1} = \sigma_{n-1} \cup z$ . Since  $\sigma_{n-2} \leq \sigma_n = 2$ , we know that  $\sigma'_{n-1}$  is disjoint from  $\sigma_{n-2}$  and  $\sigma_n$ .

We have  $\sigma'_{n-1} \leq \sigma_n = 1$  and  $\sigma'_{n-1} \leq \sigma_{n-2} = 1$ . So by maximality of  $\sigma_{n-1}$ , we deduce that  $\sigma'_{n-1} = \sigma_{n-1}$ , hence  $z \in \sigma_{n-1}$ .

Since  $\sigma_{n-2} \subset B(z, 1)$ , by induction we know that  $\{x_0\}, \sigma_1, \dots, \sigma_{n-2}, \{z\}$  is a clique-path. In particular  $d(x_0, z) = n-1$ , which contradicts  $d(z, x_0) \leq n-2$ .

So we conclude that  $d(x_0, x_n) = n$ . This implies that  $\{x_0\}, \sigma_1, \sigma_2, \dots, \sigma_{n-1}, \{x_n\}$  is a clique-path.  $\square$

We now prove a very similar result for local  $L$ -clique-paths.

**Proposition 6.3.** *Fix a locally finite Helly graph  $X$ ,  $L \geq 1$  and  $n \geq 2$ . Consider a local  $L$ -clique-path  $\sigma_0, \dots, \sigma_n$  in  $X$ . There exist  $x_0 \in \sigma_0$  and  $x_n \in \sigma_n$  such that  $\sigma_1 \subset B(x_0, L)$ ,  $\sigma_{n-1} \subset B(x_n, L)$  and  $\{x_0\}, \sigma_1, \sigma_2, \dots, \sigma_{n-1}, \{x_n\}$  is a  $L$ -clique-path.*

*Proof.* According to Lemma 6.1, for each  $0 \leq i \leq n-1$ , there exists a 1-clique-path between  $\sigma_i$  and  $\sigma_{i+1}$ . So there exists a sequence of cliques  $\tau_0, \tau_1, \dots, \tau_{nL}$  such that:

- For each  $0 \leq i \leq n$ , we have  $\tau_{iL} = \sigma_i$ .



- For each  $0 \leq i \leq n-1$ , the sequence  $\tau_{iL}, \tau_{iL+1}, \dots, \tau_{(i+1)L}$  is a 1-clique-path.

We will prove that  $\tau_0, \tau_1, \dots, \tau_{nL}$  is a local 1-clique-path.

We first prove that, for any  $1 \leq i \leq n-1$ , we have  $\tau_{iL-1} \leq \tau_{iL+1} = 2$ . Fix  $y_{iL-1} \in \tau_{iL-1}$  and  $y_{iL+1} \in \tau_{iL+1}$ . Let  $x_{i-1} \in \sigma_{i-1}$  such that  $\tau_{iL-1} \subset B(x_{i-1}, L-1)$  and  $x_{i+1} \in \sigma_{i+1}$  such that  $\tau_{iL+1} \subset B(x_{i+1}, L-1)$ . Then  $d(x_{i-1}, y_{iL-1}) + d(y_{iL-1}, y_{iL+1}) + d(y_{iL+1}, x_{i+1}) \geq d(x_{i-1}, x_{i+1}) = 2L$ , so  $d(y_{iL-1}, y_{iL+1}) \geq 2$ . Hence  $\tau_{iL-1} \leq \tau_{iL+1} = 2$ .

We now prove that, for any  $1 \leq i \leq n-1$ ,  $\tau_{iL}$  is a maximal clique such that  $\tau_{iL} \leq \tau_{iL-1} = 1$  and  $\tau_{iL} \leq \tau_{iL+1} = 1$ . Assume that  $\tau'_{iL}$  is a clique containing  $\tau_{iL}$  such that  $\tau'_{iL} \leq \tau_{iL-1} = 1$  and  $\tau'_{iL} \leq \tau_{iL+1} = 1$ . Since  $\sigma_{i-1} \leq \sigma_{i+1} = 2L$ , we deduce that  $\sigma_{i-1} \leq \tau'_{iL} = L$  and  $\sigma_{i+1} \leq \tau'_{iL} = L$ . Since  $\sigma_0, \sigma_1, \dots, \sigma_n$  is a local  $L$ -clique-path, by maximality of  $\sigma_i$  we deduce that  $\tau'_{iL} = \sigma_i = \tau_{iL}$ .

So we have proved that  $\tau_0, \tau_1, \dots, \tau_{nL}$  is a local 1-clique-path. Let  $x_0 \in \tau_0 = \sigma$  such that  $\tau_1 \subset B(x_0, 1)$ , and let  $x_n \in \tau_{nL} = \tau$  such that  $\tau_{nL-1} \subset B(x_n, 1)$ . According to Proposition 6.2,  $\{x_0\}, \tau_1, \dots, \tau_{n-1}, \{x_n\}$  is a  $L$ -clique-path.

Then for any  $1 \leq i < j \leq n-1$ , we have  $\tau_{iL} \leq \tau_{jL} = (j-i)L$ , so  $\sigma_i \leq \sigma_j = (j-i)L$ . We deduce that  $\{x_0\}, \sigma_1, \sigma_2, \dots, \sigma_{n-1}, \{x_n\}$  is a  $L$ -clique-path.  $\square$

So we immediately deduce from Proposition 6.3 the following crucial local-to-global result.

**Theorem 6.4.** *In a locally finite Helly graph, for any  $L \geq 1$ , any bi-infinite local  $L$ -clique-path is a  $L$ -clique-path.*

We also deduce that any hyperbolic automorphism of a Helly graph has an invariant clique-path.

**Proposition 6.5.** *Fix a locally finite Helly graph  $X$ , and let  $g$  denote a hyperbolic automorphism of  $X$ . There exist integers  $a, L \in \mathbb{N} \setminus \{0\}$  and a bi-infinite  $L$ -clique path  $(\sigma_n)_{n \in \mathbb{Z}}$  in the Helly subdivision  $X'$  of  $X$  such that  $\forall n \in \mathbb{Z}, g^a \cdot \sigma_n = \sigma_{n+1}$ .*

*Proof.* According to Proposition 5.3, there exist integers  $a, L \in \mathbb{N} \setminus \{0\}$  and a vertex  $x \in X'$  of the Helly subdivision  $X'$  of  $X$  such that  $\forall n \in \mathbb{N}, d(x, g^{an} \cdot x) = nL$ .

Since  $X$  is locally finite, we may consider a maximal family of cliques  $(\sigma'_n)_{n \in \mathbb{Z}}$  of  $X'$  such that:

- for every  $n \in \mathbb{Z}$ , we have  $g^{an} \cdot x \in \sigma'_n$  and
- for every  $n \in \mathbb{Z}$ , we have  $\sigma'_n \leq \sigma'_{n+1} = L$ .

Since  $X$  is locally finite, there exists  $m \geq 1$  such that  $g^{am} \cdot \sigma'_0 = \sigma'_m$ ,  $g^{am} \cdot \sigma'_1 = \sigma'_{m+1}$  and  $g^{am} \cdot \sigma'_2 = \sigma'_{m+2}$ . For each  $n = mq + r \in \mathbb{Z}$ , where

$q \in \mathbb{Z}$  and  $r \in [0, m - 1]$ , let us define  $\sigma_n = g^{amq} \cdot \sigma'_r$ . Then  $(\sigma_n)_{n \in \mathbb{Z}}$  is a local  $L$ -clique-path, and for every  $n \in \mathbb{Z}$ , we have  $g^a \cdot \sigma_n = \sigma_{n+1}$ .

According to Theorem 6.4,  $(\sigma_n)_{n \in \mathbb{Z}}$  is a  $L$ -clique path.  $\square$

## 7. LINEAR ORBIT GROWTH

We will now prove a simple result about elliptic groups of isometries of an injective metric space ensuring that orbits grow linearly.

**Lemma 7.1.** *Let  $X$  be an injective metric space, and let  $G, H$  denote elliptic isometry groups of  $X$ , i.e. with fixed points in  $X$ . For any  $n \in \mathbb{N}$  and any  $x \in X^G$ , there exists  $y \in X^G$  such that*

$$\begin{aligned} \text{diam}((GH)^{2^{n-1}} \cdot y) &\leq \frac{1}{2} \text{diam}((GH)^{2^n} \cdot x) \text{ if } n \geq 1 \\ \text{diam}(H \cdot y) &\leq \frac{1}{2} \text{diam}((GH)^{2^n} \cdot x) \text{ if } n = 0, \end{aligned}$$

where  $(GH)^k$  denotes  $(GH)^k = \{g_1 h_1 g_2 h_2 \cdots g_k h_k \mid g_1, \dots, g_k \in G, h_1, \dots, h_k \in H\}$ .

*Proof.* Assume that  $2L = \text{diam}((GH)^{2^n} \cdot x)$  is finite, otherwise the conclusion holds trivially. Then, for any  $u, v \in (GH)^{2^n}$ , we have  $d(u \cdot x, v \cdot x) \leq 2L$ . In particular, since  $X$  is injective, the intersection  $\bigcap_{u \in (GH)^{2^n}} B(u \cdot x, L)$  is non-empty. Since this intersection is bounded and  $G$ -invariant, according to [Lan13, Proposition 1.2], there exists  $z \in \bigcap_{u \in (GH)^{2^n}} B(u \cdot x, L)$  that is fixed by  $G$ .

So for every  $u \in (GH)^{2^n}$ , we have  $d(z, u \cdot x) \leq L$ . Hence we deduce that for every  $u, v \in K_n$  we have  $d(u \cdot z, v \cdot x) \leq L$ , where  $K_n = (GH)^{2^{n-1}}$  if  $n \geq 1$  and  $K_n = H$  if  $n = 0$ .

According to [Lan13, Proposition 3.8], there exists a conical geodesic reversible bicombing  $\gamma$  on  $X$ , equivariant with respect to isometries of  $X$ . Let  $y = \gamma(x, z, \frac{1}{2})$  denote the midpoint of  $x$  and  $z$  with respect to  $\gamma$ .

For any  $u, v \in K_n$ , we have  $u \cdot y = \gamma(u \cdot x, u \cdot z, \frac{1}{2})$  and  $v \cdot y = \gamma(v \cdot z, v \cdot x, \frac{1}{2})$ . By conicality, we deduce that

$$d(u \cdot y, v \cdot y) \leq \frac{1}{2} d(u \cdot x, v \cdot z) + \frac{1}{2} d(u \cdot z, v \cdot x) \leq L.$$

So the diameter of  $K_n \cdot y$  is at most  $L$ . And by equivariance of  $\gamma$ ,  $y$  is fixed by  $G$ .  $\square$

**Proposition 7.2.** *Let  $X$  be an injective metric space, and let  $G, H$  denote elliptic isometry groups of  $X$  such that  $d(X^G, X^H) = L > 0$ . Assume that there exists  $x \in X^G$  such that  $d(x, X^H) = L$ . Then, for any  $n \in \mathbb{N}$ , there exist  $g_1, \dots, g_n \in G$  and  $h_1, \dots, h_n \in H$  such that*

$$x_0 = x, x_1 = g_1 h_1 \cdot x, x_2 = g_1 h_1 g_2 h_2 \cdot x, \dots, x_n = g_1 h_1 \cdots g_n h_n \cdot x$$

lie on a geodesic. More precisely, for any  $0 \leq i \leq j \leq n$ , we have

$$d(x_i, x_j) = (j - i)2L.$$

*Proof.* Note that it is sufficient to prove the statement when  $n$  is a power of 2.

Let  $2L'$  denote the diameter of  $H \cdot x$ , we will prove that  $L' = L$ . For any  $h, h' \in H$ , we have  $d(h \cdot x, h' \cdot x) \leq 2L'$ , so since  $X$  is injective the intersection  $\bigcap_{h \in H} B(h \cdot x, L')$  is non-empty. Since this intersection is furthermore  $H$ -invariant, according to [Lan13, Proposition 1.2], there exists  $y \in \bigcap_{h \in H} B(h \cdot x, L')$  that is fixed by  $H$ . Since  $d(x, y) \leq L'$ , we deduce that  $L \leq L'$ . And if  $z \in X^H$  is such that  $d(x, z) = L$ , for any  $h \in H$  we have  $d(x, h \cdot x) \leq d(x, z) + d(z, h \cdot x) \leq 2L$ , so  $\text{diam}(H \cdot x) \leq 2L$ . In conclusion  $L' = L$ , and the diameter of  $H \cdot x$  equals  $2L$ .

For any  $y \in X^G$ , as in the previous paragraph, we may argue that the diameter of  $H \cdot y$  is at least  $2L$ . According to Lemma 7.1, by induction on  $m \geq 0$ , we deduce that the diameter of  $(GH)^{2^m} \cdot x$  is at least  $2^{m+2}L$ .

In particular, for any  $m \geq 0$ , there exist  $g_1, \dots, g_{2^{m+1}} \in G$  and  $h_1, \dots, h_{2^{m+1}} \in H$  such that  $d(x, g_1 h_1 \cdots g_{2^{m+1}} h_{2^{m+1}} \cdot x) \geq 2^{m+2}L$ . Since for any  $1 \leq i \leq 2^{m+1}$  we have  $d(x, g_i h_i \cdot x) \leq 2L$ , we deduce that  $d(x, g_1 h_1 \cdots g_{2^{m+1}} h_{2^{m+1}} \cdot x) = 2^{m+2}L$ , and furthermore that for any  $0 \leq i \leq j \leq 2^{m+1}$ , we have

$$d(g_1 h_1 \cdots g_i h_i \cdot x, g_1 h_1 \cdots g_j h_j \cdot x) = (j - i)2L.$$

□

## 8. LOCALLY ELLIPTIC ACTIONS

We now turn to one of the main results of this article, namely that locally elliptic actions on Helly graphs are globally elliptic. Theorem H is an immediate consequence of the following result.

**Theorem 8.1.** *Let  $X$  be a locally finite Helly graph with finite combinatorial dimension, and assume that  $G, H$  are elliptic automorphism groups of  $X$ . Then either  $\langle G, H \rangle$  is elliptic, or there exists a hyperbolic element in  $\langle G, H \rangle$ .*

*Proof.* By assumption,  $G$  and  $H$  have fixed points in  $EX$ . Let  $N \geq 1$  denote the combinatorial dimension of  $X$ . According to Lemma 4.3, up to replacing  $X$  with its Helly subdivision  $X'_N$ , we may assume that the minimal distance between  $(EX)^G$  and  $(EX)^H$  is realized by vertices of  $X$ . Let  $x \in X^G$  and  $y \in X^H$  denote vertices such that  $d(x, y) = L = d(EX^G, EX^H)$ .

If  $L = 0$ , then  $x = y$  is fixed by  $G$  and  $H$ , so  $\langle G, H \rangle$  is elliptic.

Assume that  $L \geq 1$ . Let us denote  $S = GH$ . According to Proposition 7.2, for every  $N \geq 1$ , there exists a sequence  $(s_n^N)_{1 \leq n \leq N}$  in  $S$  such that  $x_0^N = x, x_1^N = s_1^N \cdot x, x_2^N = s_1^N s_2^N \cdot x, \dots, x_N^N = s_1^N s_2^N \cdots s_N^N \cdot x$  lie on a geodesic ray in  $X$ : more precisely, for any  $0 \leq n \leq m \leq N$ , we have  $d(x_n^N, x_m^N) = (m - n)2L$ .

Since  $X$  is locally finite, we may find a sequence  $(N_k)_{k \in \mathbb{N}}$  going to  $+\infty$  such that, for any  $n \in \mathbb{N}$ , the sequence  $(x_n^{N_k})_k$  (defined for  $k$  large enough)

is eventually constant, equal to  $x_n$ . By induction on  $n \in \mathbb{N}$ , there exists a sequence  $(s_n)_{n \geq 1}$  in  $S$  such that, for any  $n \geq 1$ , we have  $x_n = s_1 s_2 \cdots s_n \cdot x$ .

Let  $\sigma_0 = \{x\}$ . Since  $X$  is locally finite, we may consider a maximal family  $(\sigma_n)_{n \geq 1}$  of cliques of  $X$  such that:

- for all  $n \geq 0$ ,  $x_n = s_1 s_2 \cdots s_n \cdot x \in \sigma_n$ ,
- for all  $n \geq 0$ ,  $\sigma_n \leq \sigma_{n+1} = 2L$  and
- for all  $n \geq 0$ , there exists  $m \geq n$  such that, for any  $y \in \sigma_n$ , we have  $d(x, y) = n(2L)$  and  $d(y, x_m) = (m - n)(2L)$ .

We claim that, for each  $n \geq 0$ , we have  $\sigma_n \leq \sigma_{n+2} = 4L$ . Indeed it is sufficient to prove that any two vertices of  $\sigma_n$  and  $\sigma_{n+2}$  are at distance at least  $4L$ : fix  $y_n \in \sigma_n$  and  $y_{n+2} \in \sigma_{n+2}$ . Let  $m \geq n + 2$  be such that  $x$ ,  $y_{n+2}$  and  $x_m$  lie on a geodesic of  $X$ . Since  $d(x, y_n) + d(y_n, y_{n+2}) + d(y_{n+2}, x_m) \geq d(x, x_m) = m(2L)$ , we deduce that

$$\begin{aligned} d(y_n, y_{n+2}) &\geq m(2L) - d(x, y_n) - d(y_{n+2}, x_m) \\ &= m(2L) - n(2L) - (m - n - 2)(2L) = 4L. \end{aligned}$$

So  $\sigma_n \leq \sigma_{n+2} = 4L$ .

We now prove that, for each  $n \geq 1$ , the clique  $\sigma_n$  is maximal such that  $\sigma_n \leq \sigma_{n-1} = 2L$  and  $\sigma_n \leq \sigma_{n+1} = 2L$ . Let  $u$  denote a vertex in  $\sigma_n$  or adjacent to  $\sigma_n$  such that the clique  $\sigma'_n = \sigma_n \cup \{u\}$  satisfies  $\sigma'_n \leq \sigma_{n-1} = 2L$  and  $\sigma'_n \leq \sigma_{n+1} = 2L$ , we will prove that  $u \in \sigma_n$ . Let  $y_{n+1}$  denote a vertex of  $\sigma_{n+1}$  such that every vertex of  $\sigma'_n$  is at distance  $2L$  from  $y_{n+1}$ . Let  $m \geq n + 1$  be such that  $d(x, y_{n+1}) = (n + 1)(2L)$  and  $d(y_{n+1}, x_m) = (m - n - 1)(2L)$ . Since all vertices of  $\sigma'_n$  lie on a geodesic between  $x$  and  $y_{n+1}$ , we deduce that for any vertex  $y \in \sigma'_n$  we have  $d(x, y) = n(2L)$  and  $d(y, x_m) = (m - n)(2L)$ . By maximality of  $\sigma_n$ , we deduce that  $u \in \sigma_n$ .

We deduce that  $(\sigma_n)_{n \in \mathbb{N}}$  is a local  $2L$ -clique-path. According to Theorem 6.4, we conclude that  $(\sigma_n)_{n \in \mathbb{N}}$  is a  $2L$ -clique-path.

Since  $X$  is locally finite, the family  $\{(s^{-1} \cdot x, t \cdot x) \mid s, t \in S\}$  is a finite subset of  $X^2$ . So there exist  $i \geq 1$  and  $j \geq i + 3$  such that  $s_i^{-1} \cdot x = s_j^{-1} \cdot x$  and  $s_{i+1} \cdot x = s_{j+1} \cdot x$ . As a consequence, if we denote

$$g = (s_1 \cdots s_j)(s_1 \cdots s_i)^{-1} \in \langle G, H \rangle,$$

we have  $g \cdot x_{i-1} = x_{j-1}$ ,  $g \cdot x_i = x_j$  and  $g \cdot x_{i+1} = x_{j+1}$ .

Since  $X$  is locally finite, there are only finitely many cliques containing any given vertex. So up to choosing larger  $i, j$ , we may furthermore assume that  $g \cdot \sigma_{i-1} = \sigma_{j-1}$ ,  $g \cdot \sigma_i = \sigma_j$  and  $g \cdot \sigma_{i+1} = \sigma_{j+1}$ .

Let  $p = j - i$ . For each  $k = qp + r \in \mathbb{Z}$ , with  $q \in \mathbb{Z}$  and  $r \in [0, p - 1]$ , let  $\tau_k = g^q \cdot \sigma_{i+r}$ . The sequence of cliques  $(\tau_k)_{k \in \mathbb{Z}}$  is a local  $2L$ -clique-path that is invariant under  $g$ . According to Theorem 6.4, we deduce that it is a  $2L$ -clique-path. In particular, for any  $n \in \mathbb{N}$ , we have  $d(g^n \cdot x_i, x_i) = n(2L)$ . So according to Proposition 5.3,  $g$  is hyperbolic.  $\square$

9. ADDING A CAT(0) METRIC

We now give a proof of Theorem K in the introduction. Let  $X$  denote a locally finite Helly graph with finite combinatorial dimension. Assume that the piecewise  $\ell^2$  metric on the orthoscheme complex  $O_1X$  of  $X$  is CAT(0). Let  $G$  denote a group of automorphisms of  $X$ , such that each element of  $G$  stabilizes a clique in  $X$ .

Consider the family  $(G_\alpha)_{\alpha \in A}$  of all finitely generated subgroups of  $G$ . According to Theorem 8.1, for each  $\alpha \in A$ , the finitely generated group  $G_\alpha$  of  $G$  is elliptic, so it stabilizes a clique in  $X$ . In particular, it has a non-empty fixed point  $X_\alpha$  in  $|O_1X|$ , which is closed and convex with respect to the CAT(0) metric.

According to [CL10, Theorem 1.1], since the CAT(0) space  $|O_1X|$  has finite dimension, either the filtering family  $(X_\alpha)_{\alpha \in A}$  has non-empty intersection, or the intersection of the family  $(\partial X_\alpha)_{\alpha \in A}$  is non-empty in the visual boundary  $\partial|O_1X|$  of  $|O_1X|$ .

In the former case, the group  $G$  has a fixed point in  $|O_1X|$ . In the latter case, the group  $G$  has a fixed point in the boundary  $\partial|O_1X|$ .

10. SYSTOLIC COMPLEXES AND GRAPHICAL SMALL CANCELLATION COMPLEXES

Recall that, for  $k \geq 6$ , a flag simplicial complex  $X$  is  $k$ -large if every cycle in (the 1-skeleton of)  $X$  of length smaller than  $k$  has a *diagonal*, that is, there exists an edge in  $X$  connecting two non-consecutive vertices of the cycle. A simply connected  $k$ -large complex is  $k$ -systolic, and 6-systolic complexes are usually called simply *systolic*. Observe that  $k$ -systolicity implies  $l$ -systolicity, for  $k \geq l$ . Such complexes were first considered by Chepoi [Che00] in relation with *bridged graphs*, which are their 1-skeleta. The study of groups acting geometrically on systolic complexes was initiated by Januszkiewicz-Świątkowski [JS06] and by Haglund [Hag03].

In this section we consider the following class of group actions on systolic complexes. We say that  $G$  acts *strongly rigidly* on  $X$  if the only  $g \in G$  stabilizing a (possibly infinite) simplex of dimension at least 1 is the identity. Such actions appear naturally as actions induced by actions of small cancellation groups on their Cayley graphs as explained below.

In what follows, unless stated otherwise, we consider a group  $G$  acting strongly rigidly on a 18-systolic complex  $X$ .

**Lemma 10.1.** *Let  $g \in G \setminus \{1\}$  fix a vertex in  $X$ . Then  $Fix(g)$  is a vertex in  $X$ .*

*Proof.* Suppose there are two distinct vertices  $v, w \in X$  in  $Fix(g)$ . Consider the interval  $I := I(v, w)$  between them. Let  $\sigma = I \cap X_v$ , where  $X_v$  denotes the link of  $v$  in  $X$ . It is a non-empty (possibly infinite) simplex, see e.g.

[Osa07, Lemma 2.3]. We have  $gI = I$ , hence  $g\sigma = \sigma$ . By the strong rigidity of the action we have  $g = 1$ , contradiction.  $\square$

**Lemma 10.2.** *Let  $w$  be a vertex in the link  $X_v$ , for  $v = \text{Fix}(g)$ . Then there exists a natural number  $k > 0$  such that  $d_{X_v}(w, g^k w) \geq 5$ .*

*Proof.* Suppose not, that is, suppose that for every  $k$  we have  $d_{X_v}(w, g^k w) \leq 4$ . Consider the intersection  $Z := \bigcap_k B_4(g^k w, X_v)$  of balls of radius 4 around  $g^k w$  in the link  $X_v$ . It is nonempty and it is contained in  $B_8(w, X_v)$ . By [Osa07, Lemma 2.5] the ball  $B_8(w, X_v)$  is itself a systolic complex and the balls in  $X_v$  of the form  $B_4(g^k w, X_v)$  are balls in  $B_8(w, X_v)$ , hence are convex subcomplexes of  $B_8(w, X_v)$ . Therefore  $Z$  is a convex subcomplex of a systolic complex  $B_8(w, X_v)$ , hence it is systolic itself. Furthermore,  $Z$  is  $\langle g \rangle$ -invariant and bounded. It follows that there exists a (possibly infinite) simplex  $\sigma$  in  $Z$  fixed by  $g$ . It follows that the simplex of  $X$  spanned by  $\sigma$  and  $v$  is fixed by  $g$ , contradicting the strict rigidity of the action.  $\square$

**Lemma 10.3.** *Let  $g, g' \in G \setminus \{1\}$  such that  $\text{Fix}(g) \neq \text{Fix}(g')$ . Then there exist natural numbers  $k, k' > 0$  such that  $g^k g'^{k'}$  is an element in  $G$  acting with an unbounded orbit on  $X$ .*

*Proof.* Choose a 1-skeleton geodesic  $\gamma$  between  $v := \text{Fix}(g)$  and  $v' := \text{Fix}(g')$ . Let  $w$  and  $w'$  denote vertices on  $\gamma$  adjacent to, respectively,  $v$  and  $v'$ . By Lemma 10.2 there exist  $k, k'$  such that  $d_{X_v}(w, g^k w) \geq 5$  and  $d_{X_{v'}}(w', g'^{k'} w') \geq 5$ . We claim that  $f := g^k g'^{k'}$  is not elliptic.

Suppose not, that is, suppose that the orbit of  $f$  is bounded. Consider the path (see Figure 4)

$$(1) \quad \alpha := \bigcup_{i \in \mathbb{Z}} f^i(\gamma \cup g^k \gamma).$$

Since the orbits of  $f$  are bounded, there exist vertices  $u, u'$  in  $\alpha$  whose distance in  $\alpha$  is strictly bigger than their distance in  $X$ . We can choose  $u, u'$  with the minimal distance among such pairs. Let  $\alpha'$  be the path being part of  $\alpha$  between  $u$  and  $u'$ , and let  $\beta$  be a geodesic in  $X$  between  $u'$  and  $u$ . Consider the closed path  $\alpha'\beta$ . Consider a minimal (singular) disk diagram  $D$  for  $\alpha'\beta$  (see Figure 4). Such a diagram is systolic. Recall, that the *defect* of a vertex  $z$  on the boundary  $\alpha'\beta$  of  $D$  is equal to three minus the number of triangles of  $D$  containing  $z$ . By the combinatorial Gauss-Bonnet Theorem and since  $D$  is systolic we have that the sum of defects of the boundary vertices of  $D$  is at least 6. By the choice of  $u, u'$  the sum of their defects is at most 4. By the choice of  $k, k'$  the defects in the points  $f^i v, f^i v'$  are at most  $-1$ . The sum of defects along interior vertices of a geodesic (in particular,  $f^i \gamma$  or  $f^i g^k \gamma$  or  $\beta$ ) is at most 1. Since there is a shortcut between  $u$  and  $u'$ , the sum of defects of vertices of  $\alpha'$  different than  $u, u'$  is at most 0. Therefore, the sum of defects along the boundary vertices is at most 5, a contradiction.  $\square$

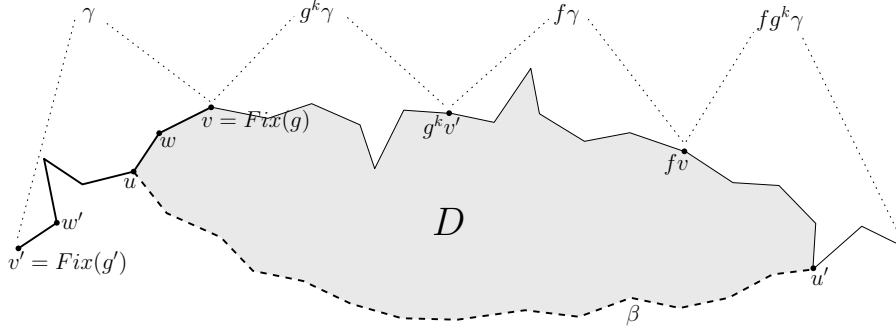


FIGURE 4. The path  $\alpha$ , a shortcut  $\beta$  (dashed), and a disk diagram  $D$  (shaded).

*Proof of Theorem F.* If all the generators of  $G$  have a common fixed point then the  $G$ -action is elliptic. If not, then by Lemma 10.3 there exists a non-elliptic element leading to a contradiction.  $\square$

In the rest of the section we turn towards actions on graphical small cancellation complexes. We do not give all the definitions here directing the reader to [OP18], whose notation we use in what follows.

Let  $Y$  be a simply connected graphical  $C(18)$ -small cancellation complex. Let  $G$  be a finitely generated group acting on  $Y$  by automorphisms such that the action induces a free action on the 1-skeleton  $Y^{(1)}$  of  $Y$ . Recall [OP18] that the *Wise complex*  $W(Y)$  of  $Y$  is the nerve of the covering of  $Y$  by (closed) 2-cells. We assume here every edge in  $Y$  is contained in some 2-cell – this is not restrictive since one can always add equivariantly ‘artificial’ 2-cells preserving small cancellation conditions. Since  $Y$  is  $C(18)$  its Wise complex  $W(Y)$  is 18-systolic by [OP18, Theorem 7.10].

**Lemma 10.4.** *The induced  $G$ -action on the Wise complex  $W(Y)$  is strongly rigid.*

*Proof.* Let  $g \in G$ . suppose that there exists a simplex  $\sigma = \{z_0, z_1, \dots\}$  of dimension at least 1 in  $W(Y)$  with  $g\sigma = \sigma$ . It follows that for the intersection  $Z := \bigcap z_i \subseteq Y$  we have  $gZ = Z$ . Such intersection is a non-empty bounded tree, hence there exists a vertex or an edge fixed by  $g$ . By the assumption on freeness of the action we have that  $g = 1$ .  $\square$

*Proof of Theorem G.* The action of a  $C(18)$  group on the graphical small cancellation complex derived from the Cayley graph is free on the 1-skeleton. Hence the theorem follows from Lemma 10.4 and Theorem F.  $\square$

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