SMALL CANCELLATION LABELLINGS OF SOME INFINITE GRAPHS AND APPLICATIONS

DAMIAN OSAJDA

Abstract. We construct small cancellation labellings for some infinite sequences of finite graphs of bounded degree. We use them to define infinite graphical small cancellation presentations of groups. This technique allows us to provide examples of groups with exotic properties:

- We construct the first examples of finitely generated coarsely non-amenable groups (that is, groups without Guoliang Yu's Property A) that are coarsely embeddable into a Hilbert space. Moreover, our groups act properly on CAT(0) cubical complexes.
- We construct the first examples of finitely generated groups, with expanders embedded isometrically into their Cayley graphs – in contrast, for the Gromov monster only a weak embedding is established. Such examples have important applications to the Baum-Connes conjecture.

We present further applications concerning aspherical manifolds and asymptotic dimension of groups.

1. Introduction

The main goal of this article is to present a technique of constructing finitely generated groups such that given (infinite) graphs embed isometrically into their Cayley complexes. This allows one to obtain groups with some features resembling the ones of those graphs. In particular, we construct groups without Guoliang Yu's property A that are coarsely embeddable into a Hilbert space (see Subsection 1.2 in this Introduction below), and we construct groups, into whose Cayley graphs some expanders embed isometrically (see Subsection 1.3). The latter groups are therefore not coarsely embeddable into Hilbert spaces, and various versions of the Baum-Connes conjecture fail for them. The general tool we use is the graphical small cancellation theory, and the main technical point is then finding appropriate small cancellation labellings of the graphs in question (see the next Subsection 1.1).

1.1. Small cancellation labellings of some graphs.

A labeling of a graph may be seen as an assignment of labels to directed edges; see details in Section 2. A labeling satisfies some small cancellation condition when no labeling of a long path (long with respect to the girth) appears in two

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2010 *Mathematics Subject Classification.* 20F69, 20F06, 46B85, 05C15.

Key words and phrases. Small cancellation, coarse embedding, Property A, CAT(0) cubical complex, graph coloring.

different places; see Subection 2.3. For our purposes we are interested in a finite set of labels, and in graphs being infinite disjoint unions of finite graphs with degree bounded uniformly. Examples are sequences of finite D–regular graphs, for a fixed degree $D > 2$. For such graphs the only 'small cancellation' labeling provided till now was the famous Gromov labeling of some expanders [Gro03] (cf. some explanations of this construction in [AD08, Cou14]). Gromov's labeling is in a sense generic, and as such cannot satisfy the small cancellation condition we work with (see the discussion in Subsection 2.4). Therefore Gromov's labeling defines a weak embedding but not a coarse embedding of the graphs (relators) into the corresponding group (see Subsection 2.4 for details). (Recall that a map $f\colon (X, d_X) \to (Y, d_Y)$ between metric spaces is a *coarse embedding* when $d_Y(f(x_n), f(y_n)) \to \infty$ iff $d_X(x_n, y_n) \to \infty$ for all sequences (x_n), (y_n).) We study sequences $(\Theta_n)_{n\in\mathbb{N}}$ of finite graphs of uniformly bounded degree, with growing girth, and diameters bounded in terms of girth (see Section 2 for details). For them, we construct labellings satisfying much more restrictive conditions then the Gromov labellings do.

Theorem 1 (see Theorem 2.7 in the text)**.** *For every* $\lambda > 0$ *there exists a* $C'(\lambda)$*–small cancellation labeling of* $(\Theta_n)_{n\in\mathbb{N}}$ *over a finite set of labels.*

Satisfying such strong small cancellation condition implies that for groups that we construct using this labeling, the graphs Θ_n are isometrically embedded into the Cayley graphs; see the subsections below and Section 3.

For constructing the desired labellings we use techniques coming from combinatorics (graph colorings) [AGHR02] and relying on the Lovász Local Lemma (see e.g. [AS00]). This is a novelty in the subject. Note that whereas the core of our method is probabilistic (similarly as Gromov's techniques), there is a fundamental difference with Gromov's approach: We look for any labeling with required properties, while in the other method the properties of the generic labeling are explored. This is crucial for getting stronger features, as explained in Subsection 2.4. Furthermore, for a given sequence of graphs satisfying certain conditions (see the beginning of Section 2) we prove that there exists a small cancellation labeling. In fact we show that there exists an algorithm producing the labeling; see the discussion in Subsection 3.2.2. In contrast, in Gromov's approach one claims that there exists a subsequence admitting a desired labeling. Moreover, the tools used in both approaches are different. Our argument is also relatively short (pp. 5–14 below) compared to Gromov's one as presented in [AD08].

Below we describe the actual applications of the small cancellation labellings we construct. Nevertheless, we believe that the construction itself, and the overall combinatorial technique developed in this article, are important tools that will find many applications beyond the scope presented here.

1.2. **Non-exact groups with the Haagerup property.** Property A, or *coarse amenability*, was introduced by Guoliang Yu [Yu00] for his studies on the Baum-Connes conjecture. A uniformly discrete metric space (X, d) has *property A* if for every $\epsilon > 0$ and $R > 0$ there exists a collection of finite subsets $\{A_x\}_{x \in X}$, $A_x \subseteq X \times \mathbb{N}$ for every $x \in X$, and a constant $S > 0$ such that

(1) $\frac{|A_x \triangle A_y|}{|A_x \cap A_y|} \leqslant \epsilon$ when $d(x, y) \leqslant R$, and
(2) $A_x \subseteq B(x, S) \times \mathbb{N}$.

A finitely generated group has property A if it is coarsely amenable for the word metric with respect to some finite generating set.

Property A may be seen as a weak (non-equivariant) version of amenability, and similarly to the latter notion it has many equivalent formulations and a large number of significant applications; see e.g. [Wil09, NY12]. For countable discrete groups, Property A is equivalent to: the existence of a topological amenable action on a compact Hausdorff space [HR00], to the exactness of the reduced C^*–algebra [GK02, Oza00], to nuclearity of the uniform Roe algebra [Roe03], and to few other geometric and analytic properties; see e.g. [NY12, pp. 81–82].

Property A implies coarse embeddability into a Hilbert space [Yu00]. Analogously, amenability implies the Haagerup property (that is, a-T-menability in the sense of Gromov). The following diagram depicts relations (arrows denoting implications) between those properties for groups; see e.g. [NY12, p. 124]. Observe that the notions on the right may be seen as non-equivariant counterparts of the ones on the left.

$$\begin{tikzcd}
\text{amenability} \arrow[r] \arrow[d] & \text{property A} \arrow[d] \\
\text{Haagerup property} \arrow[r] & \text{coarse embeddability into } l_2
\end{tikzcd}$$

In view of the above a natural question, which was open till now, arose: *Do groups coarsely embeddable into a Hilbert space have property A?* – see e.g. [AD02, Remark 3.8(2)], [HG04, Problem 3.4], [GK04, p. 257 & 261], [NSW08, p. 6], [AD08, footnote p. 27], [Wil09, p. 251], or [NY12, Open Question 5.3.3]. Approaches to answer this question (also in the positive) attracted much research in the area and triggered many new ideas. Following a program towards a negative answer initiated in [AO14], we prove a stronger statement.

Theorem 2 (see Theorem 6.2 in the text)**.** *There exist finitely generated groups acting properly on CAT(0) cubical complexes and not having property A.*

Acting properly on a CAT(0) cubical complex is equivalent to acting properly on a space with walls [HP98, Nic04, CN05], that is to having property PW (in a language of [Cor13]). This implies in particular the Haagerup property, and hence equivariant coarse embeddability into a Hilbert space.

Theorem 2 shows that the diagram above is complete – there are no other implications between the properties there; see [NY12, p. 124]. Besides the Gromov monsters [Gro03], the groups constructed in the current paper (see also Subsection 1.3 below) are the only finitely generated groups without property A known at the moment; see e.g. [Now07], [NSW08, p. 6], [AD08, p. 28], [Wil09, p. 251 and Section 7.5], or [NY12, Open Question 4.5.4] for related remarks and questions. Note that coarsely non-amenable spaces embeddable into l_2 were constructed in [Now07] (locally finite case) and in [AGŠ12] (bounded geometry case). Our construction relies on examples constructed in [Ost12].

Let us remark here that the lack of property A for a group was believed to be an essential obstacle to various Baum-Connes conjectures by some experts. This question is clarified by Theorem 2: There are groups without property A but satisfying the Haagerup property. For such groups the strong Baum-Connes conjecture holds [HK01].

Coarsely non-amenable groups embeddable into a Hilbert space constructed in this article are given by infinite graphical small cancellation presentations (see Section 6.2 for details). The infinite family of graphs being relators consists of some coverings of regular graphs with girths growing to infinity. Relators are graphs with walls (see Section 4), and thus there is a walling for the group itself (see the proof of Theorem 6.2). Therefore, the group acts on a space with walls. This action is proper if some additional conditions are satisfied. We study such a condition – the proper lacunary walling condition – in Section 5. This is a theory of independent interest that relies on, and extends in a way, the preceding work of the author with Goulnara Arzhantseva [AO14] (cf. also [AO15]). In particular, we obtain the following analogue of [AO14, Main Theorem and Theorem 1.1].

Theorem 3 (see Theorem 5.6 in the text)**.** *Let X be a complex satisfying the proper lacunary walling condition. Then the wall pseudo-metric is proper. Consequently, a group acting properly on X acts properly on a CAT(0) cubical complex.*

A group as in Theorem 2 is constructed so that the proper lacunary walling condition is satisfied for a space acted properly upon by the group. Therefore the group acts properly on a CAT(0) cubical complex. On the other hand, by the small cancellation condition, the infinite family of relators embeds isometrically into the Cayley graph. Since, by a result of Willett [Wil11], such a family has not property A, we conclude that the whole group is coarsely non-amenable.

1.3. **Groups with expanders in Cayley graphs.**

Using his labeling of expanders Gromov constructed a finitely generated group, for which there exists a so-called 'weak embedding' of an expander [Gro03]. A weak embedding is not necessarily a coarse embedding and with Gromov's construction one cannot obtain the latter; see the discussion in Subsection 2.4. Having weakly embedded expanders is enough to claim that the group does not coarsely embed into a Hilbert space [Gro03], or that the Baum-Connes conjecture with coefficients fails for such groups [HLS02] (cf. our Corollary 3.3 and Corollary 3.4). However, in many other situations it seems to be necessary to have an actual coarse embedding of an expander to obtain desired properties; see e.g. [WY12]. Our labeling allows us to provide groups with such a property and more, as the following result shows.

Theorem 4 (see Corollary 3.3 in the text). *There exist finitely generated groups with expanders isometrically embedded into their Cayley graphs.*

The existence of such examples is crucial for some analyses of failures of the Baum-Connes conjecture with coefficients, as in [WY12, Theorem 8.3] (see Corollary 3.4) or in [BGW15, Section 7]. Besides Gromov's monsters (and groups derived from them), our examples are the only finitely generated counterexamples to the Baum-Connes conjecture with coefficients, and the only finitely generated groups not coarsely embeddable into Hilbert space, known at the moment.

As a consequence of Theorem 4 we show that there exist closed aspherical manifolds whose fundamental groups contain quasi-isometrically embedded expanders; see Corollary 3.5. Those are the first examples of this type. Furthermore, we answer an open question of Osin-Świątkowski (see [Dra08, Problem 4.5], [Osi08, Problem 6.1]) concerning asymptotic dimensions of sequences of groups:

Theorem 5 (see Corollary 3.6 in the text). *There exists a sequence* $G^1 \twoheadrightarrow G^2 \twoheadrightarrow G^3 \twoheadrightarrow \cdots$ *of finitely presented groups with the following properties. For all* i, $asdim(G_i) = 2$, *and the asymptotic dimension of the limit group* G *is infinite.*

Note that in some situations it may be necessary to have the actual isometric embedding of given graphs into groups – this happens for example in our construction of PW non-A groups; see Subsection 1.2 above and Section 6. There we need it for the delicate construction of walls. See also Corollary 3.5 for another application of the isometric embedding.

Acknowledgments. First and foremost, I would like to thank Goulnara Arzhantseva for introducing me to the subject, leading through it, and for the great collaboration preceding this work. I am grateful for encouragement, for helpful discussions, and/or for remarks improving the manuscript to Dominik Gruber, Vincent Guirardel, Frédéric Haglund, Ashot Minasyan, Piotr Nowak, Denis Osin, Mark Sapir, Ján Špakula, and Rufus Willett.

This research was partially supported by Narodowe Centrum Nauki, decision no. DEC-2012/06/A/ST1/00259, and by the ERC grant ANALYTIC no. 259527.

2. Small cancellation labellings of some graphs

The goal of this section is proving Theorem 1 from Introduction or, more precisely, Theorem 2.7 below. Considering a metric on a graph we always mean a metric on the set of vertices, being a path metric within connected components.

Throughout this paper we work with the sequence $\Theta = (\Theta_n)_{n\in\mathbb{N}}$ *of disjoint finite connected graphs of degree bounded by* $D > 0$*. Furthermore, we have* $\mathrm{girth}\,\Theta_n \overset{n\to\infty}{\longrightarrow} \infty$ *and* Θ *satisfies the following condition:*

$$\mathrm{diam}\,\Theta_n \leqslant A\,\mathrm{girth}\,\Theta_n, \tag{1}$$

where diam *denotes the diameter,* girth *is the length of the shortest simple cycle, and* A *is a universal (not depending on* n*) constant. For this section we fix a* small cancellation constant $\lambda \in (0, 1/6]$*. We also assume that* $1 < \lfloor \lambda \mathrm{girth}\,\Theta_n \rfloor < \lfloor \lambda \mathrm{girth}\,\Theta_{n+1} \rfloor$.

Observe that for a sequence $(\Theta_n)_{n\in\mathbb{N}}$ with growing girths, the last assumption can be fulfilled by passing to a subsequence – this is allowed from the point of view of our applications.

By a *labeling* (Γ, f) of an undirected graph Γ we mean the graph morphism $f\colon \Gamma \to W$ into a bouquet of finitely many loops W, that is a graph with one vertex end several edges. Usually we refer however to the following interpretation of the labeling f. Orient edges of W and decorate every directed edge (loop) by an element of a finite set S. Then the labeling f is determined by the following data: We orient every edge of Γ and we assign to it the corresponding element of the set S or an element of the set $\overline{S}$ of formal inverses of elements of S. We call the set $S \cup \overline{S}$ the *(symmetrized) set of labels*, and by $\bar{s}$ we denote the *inverse* of s. Using this interpretation we identify a labeling assigning the label s to an oriented edge vw with the labeling of wv by $\bar{s}$; see Figure 1. The labeling (Γ, f) is *reduced* if $f\colon \Gamma \to W$ is locally injective, that is, when labels of two directed edges going out of a vertex are not the same. We will usually not specify the (symmetrized)

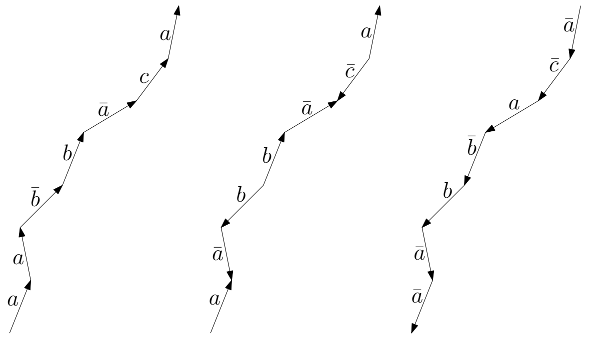

Figure 1. Three representations of the same labeling.

set of labels (although it will change often) – we will just mention that it is finite.

We construct the small cancellation labeling $(\Theta, m) = ((\Theta_n, m_n))_{n\in\mathbb{N}}$ in three steps. First, in Subsection 2.1 we construct a labeling $(\Theta, l) = ((\Theta_n, l_n))_{n\in\mathbb{N}}$ such that l_n–labellings of long (relative to girth Θ_n) paths in Θ_n do not appear in $(\Theta_{n'}, l_{n'})$, for $n \neq n'$; see Lemma 2.3. Then, in Subsection 2.2 we construct a labeling $(\Theta, \bar{l}) = ((\Theta_n, \bar{l}_n))_{n\in\mathbb{N}}$ with the property that, for each n, long paths in Θ_n are labeled differently; see Lemma 2.6. Finally, in Subsection 2.3 we combine (Θ, l) and $(\Theta, \bar{l})$ to obtain the required small cancellation labeling (Θ, m); see Theorem 2.7.

2.1. The labeling (Θ, l): small cancellation between graphs.

Recall the following version of the Lovász Local Lemma (see e.g. [AS00]) that can be found in [AGHR02, Lemma 1]. Here $\Pr(A)$ denotes the (discrete) probability of an event A, and $\bar{A}$ denotes the opposite event (complementary set).

Lemma 2.1 (Lovász Local Lemma)**.** *Let* $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \ldots \cup \mathcal{A}_r$ *be a partition of a finite set of events* $\mathcal{A}$, *with* $\Pr(A) = p_i$ *for every* $A \in \mathcal{A}_i$, $i = 1, 2, \ldots, r$. *Suppose that there are real numbers* $0 \leqslant a_1, a_2, \ldots, a_r < 1$ *and* $\Delta_{ij} \geqslant 0$, $i, j = 1, 2, \ldots, r$ *such that the following conditions hold:*

- *(i) for any event* $A \in \mathcal{A}_i$ *there exists a set* $\mathcal{D}_A \subseteq \mathcal{A}$ *with* $|\mathcal{D}_A \cap \mathcal{A}_j| \leqslant \Delta_{ij}$ *for all* $j = 1, 2, \ldots, r$, *such that* A *is independent of* $\mathcal{A} \setminus (\mathcal{D}_A \cup \{A\})$,
- *(ii)* $p_i \leqslant a_i \prod_{j=1}^{r} (1 - a_j)^{\Delta_{ij}}$ *for all* $i = 1, 2, \ldots, r$.

Then $\Pr(\bigcap_{A\in\mathcal{A}} \bar{A}) > 0$.

Let $\gamma_n = \lfloor \lambda \operatorname{girth} \Theta_n \rfloor$. Observe that $\lambda \operatorname{girth} \Theta_n - 1 < \gamma_n$ and thus

$$\frac{\operatorname{girth} \Theta_n}{\gamma_n} < \frac{1}{\lambda} + \frac{1}{\lambda\, \gamma_n} < \frac{2}{\lambda}. \tag{2}$$

We will find a labeling $(\Theta, l) = ((\Theta_n, l_n))_{n\in\mathbb{N}}$ with L labels such that l_n–labellings of paths of length at least γ_n do not appear as $l_{n'}$–labellings, for $n' > n$. Unless stated otherwise, we always assume that paths are without backtracking. It implies that all paths shorter than the girth are simple. Define L as follows (here e denotes the Euler constant):

$$L := \left\lceil 2De^4 D^{\frac{2A}{\lambda}+1} \right\rceil. \tag{3}$$

The number e_n of edges of Θ_n is bounded by $e_n \leqslant D^{\operatorname{diam} \Theta_n}$. Thus, by the condition (1), we have

$$e_n \leqslant D^{A \operatorname{girth} \Theta_n}. \tag{4}$$

We construct $((\Theta_n, l_n))_{n\in\mathbb{N}}$ inductively: (Θ_1, l_1) is an arbitrary labeling with L labels, and further we execute an inductive step. Assume that $(\Theta_1, l_1), \ldots, (\Theta_{n-1}, l_{n-1})$ are defined. Observe that, for $i = 1, \ldots, n-1$, we have

$$|\{\text{labellings of paths of length } \gamma_i \text{ in } (\Theta_i, l_i)\}| < e_i D^{\gamma_i}, \tag{5}$$

$$|\{\text{labellings of paths of length } \gamma_i \text{ with } L \text{ labels}\}| = L^{\gamma_i}. \tag{6}$$

The labeling (Θ_n, l_n) is then one given by the following lemma.

Lemma 2.2. *There exists a labeling (Θ_n, l_n) with L labels such that, for $i = 1, 2, \ldots, n-1$, no l_i–labeling of a path of length γ_i in Θ_i appears as an l_n–labeling of a path of length γ_i in Θ_n.*

Proof. We use the Lovász Local Lemma 2.1 following closely the proof of [AGHR02, Theorem 1]. Randomly label the edges of Θ_n by L labels. For a path p in Θ_n of length γ_i, let $A(p)$ denote the event that its l_n–labeling is the same as an l_i–labeling of some path in Θ_i of length γ_i, for $i < n$. Set $\mathcal{A}_i = \{A(p) : p \text{ is a path of length } \gamma_i \text{ in } \Theta_n\}$. Then, by (5), (6), (4), and (2), we have

$$p_i \leqslant \frac{e_i D^{\gamma_i}}{L^{\gamma_i}} \leqslant \frac{D^{A \operatorname{girth} \Theta_i + \gamma_i}}{L^{\gamma_i}} = \left(\frac{D^{\frac{A \operatorname{girth} \Theta_i}{\gamma_i} + 1}}{L} \right)^{\gamma_i} < \left(\frac{D^{\frac{2A}{\lambda} + 1}}{L} \right)^{\gamma_i}. \tag{7}$$

Each path of length γ_i shares an edge with not more than $\gamma_i \gamma_j D^{\gamma_j}$ paths of length γ_j, so that we may take $\Delta_{ij} = \gamma_i \gamma_j D^{\gamma_j}$. Let $a_i = a^{-\gamma_i}$, where $a = 2D$. Then, by using subsequently: formulas (7) and (3), the definition of a_i, the fact that $\sum_{j=1}^{\infty} j/2^j = 2$, the definitions of a, Δ_{ij}, and a_j, we obtain:

$$\begin{aligned} p_i &< \left(\frac{D^{\frac{2A}{\lambda}+1}}{L} \right)^{\gamma_i} \leqslant 2^{-\gamma_i} D^{-\gamma_i} e^{-4\gamma_i} = a_i \exp \left(-2 \sum_{j=1}^{\infty} \gamma_i \frac{j}{2^j} \right) \\ &\leqslant a_i \exp \left(-2 \sum_j \gamma_i \frac{\gamma_j}{2^{\gamma_j}} \right) = a_i \exp \left(-2 \sum_j \gamma_i \gamma_j \left(\frac{D}{a} \right)^{\gamma_j} \right) \\ &= a_i \exp \left(-2 \sum_j \Delta_{ij} a_j \right) = a_i \prod_j e^{-2 a_j \Delta_{ij}}. \end{aligned}$$

Since, by $a_j \leqslant 1/2$, we have $e^{-2a_j} \leqslant (1 - a_j)$, we obtain finally

$$p_i \leqslant a_i \prod_j (1 - a_j)^{\Delta_{ij}}.$$

Therefore the hypotheses of the Lovász Local Lemma are fulfilled, and we conclude that there exists a labeling l_n as required. □

The labeling $(\Theta, l) = ((\Theta_n, l_n))_{n \in \mathbb{N}}$ with L labels obtained by the inductive construction has the following property.

Lemma 2.3. *For each $n \in \mathbb{N}$, no l_n–labeling of a path of length at least $\lambda \operatorname{girth} \Theta_n$ is a labeling of a path in $(\Theta_{n'}, l_{n'})$, with $n' \neq n$.*

2.2. **The labeling $(\Theta, \bar{l})$: small cancellation within Θ_n.** For this subsection we fix n – we will work only with Θ_n. First we show that if two distinct relatively long paths in Θ_n have the same labeling then a path with a specific labeling appears; see Lemma 2.4. Then we use this observation to find a required labeling $(\Theta_n, \bar{l}_n)$, by an application of the Lovász Local Lemma, similarly as in the proof of Lemma 2.2. A shorter (but non-elementary) construction of the required labeling $(\Theta_n, \bar{l}_n)$ is also presented in Subsection 2.4, where we use properties of Gromov's labeling.

Let $\overline{v} = (v_0, v_1, \ldots, v_k)$, $\overline{w} = (w_0, w_1, \ldots, w_k)$ be two paths with the same labeling and with $k = \lfloor \lambda \operatorname{girth} \Theta_n \rfloor$ (here v_i, w_i are consecutive vertices). Denote the labeling of the directed edge $v_{i-1}v_i$ by a_i, for $i = 1, 2, \ldots, k$. We consider separately the cases when $\overline{v}$ and $\overline{w}$ share an edge, and when they do not.

Case I: $\overline{v}$ and $\overline{w}$ do not share an edge. Then there exists a path $\overline{u} = (u_0 := v_s, u_1, \ldots, u_r := w_t)$ of minimal length connecting $\overline{v}$ and $\overline{w}$. Possibly $r = 0$, that is, $\overline{u}$ is one vertex $u_0 := v_s = w_t$. Without loss of generality (subject to renaming) we may assume that $s \geqslant t \geqslant k/2$; see Figure 2. By our assumptions we have $r \leqslant \operatorname{diam} \Theta_n \leqslant A \operatorname{girth} \Theta_n$. We consider the following two cases separately.

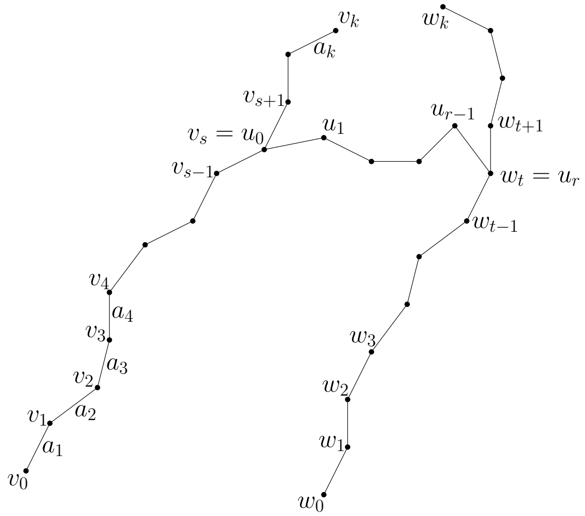

Figure 2. Case I

(Case Ia): The labeling of a directed edge $w_{i-1}w_i$ is a_i (see Figure 3 on the left). Then we have the path $p := (v_0, \ldots, v_s, u_1, \ldots, u_{r-1}, w_t, \ldots, w_0)$. By (1), its length $|p|$ may be bounded from above by

$$2k + r \leqslant 2\lambda \operatorname{girth} \Theta_n + A \operatorname{girth} \Theta_n = (2\lambda + A) \operatorname{girth} \Theta_n. \tag{8}$$

In its labeling the beginning sub-path of length t is labeled the same way – up to changing orientation – as the ending sub-path of length t, that is, it has the form (where 'repetitive' parts are underlined):

$$(\underline{a_1, a_2, \dots, a_t}, \dots, \underline{\bar{a}_t, \dots, \bar{a}_2, \bar{a}_1}), \tag{9}$$

with

$$t \geqslant k/2 > \frac{\lambda \mathrm{girth}\, \Theta_n}{4}. \tag{10}$$

(The last inequality is a rough estimate coming from $k > \lambda \mathrm{girth}\, \Theta_n - 1$.)

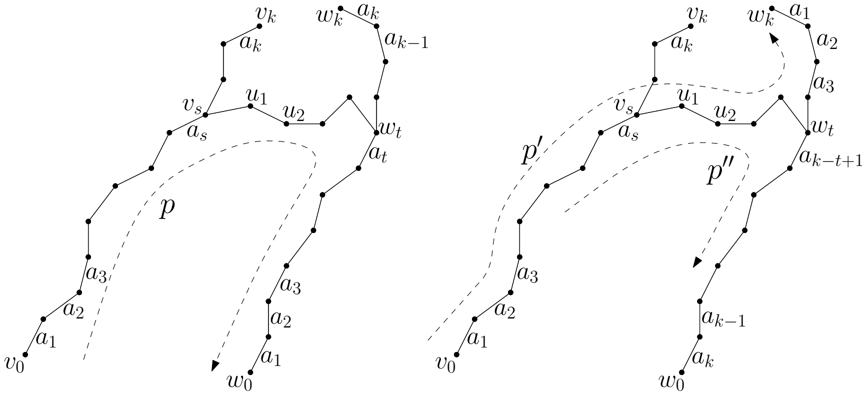

FIGURE 3. Case Ia (left) and Case Ib (right)

(Case Ib): The labeling of a directed edge $w_{i+1}w_i$ is a_{k-i} (see Figure 3 on the right). In this case again we consider separately two subcases:

(i) When $t \leqslant 3k/4$ then we consider the path $p' := (v_0, \dots, v_s, u_1, \dots, u_{r-1}, w_t, \dots, w_k)$. Its length may be again bounded from above by (8), and its labeling is of the form similar to (9):

$$(\underline{a_1, a_2, \dots, a_{k-t}}, \dots, \underline{\bar{a}_{k-t}, \dots, \bar{a}_2, \bar{a}_1}),$$

with

$$k - t \geqslant k - 3k/4 = k/4 > \frac{\lambda \mathrm{girth}\, \Theta_n}{8}. \tag{11}$$

(ii) When $t > 3k/4$ then we consider the path $p'' := (v_{k-t}, \dots, v_s, u_1, \dots, u_{r-1}, w_t, \dots, w_{k-s})$. We bound its length from above by (8), and its labeling is of the form:

$$(\underline{a_{k-t+1}, a_{k-t+2}, \dots, a_s}, \dots, \underline{a_{k-t+1}, a_{k-t+2}, \dots, a_s}),$$

with the lengths of the 'repetitive' pieces at least:

$$s - (k - t + 1) + 1 = s + t - k > \frac{k}{2} + \frac{3k}{4} - k = k/4 > \frac{\lambda \mathrm{girth}\, \Theta_n}{8}. \tag{12}$$

Case II: $\overline{v}$ shares an edge with $\overline{w}$. Then there are $r \geqslant 1$, and s, t, such that $v_{s+i} = w_{t+i}$, for $i = 1, 2, \dots, r$, and $v_i \neq w_j$ in other cases. Without loss of generality (subject to renaming) we may assume that $s \geqslant t$; see Figure 4. We consider the following two cases separately.

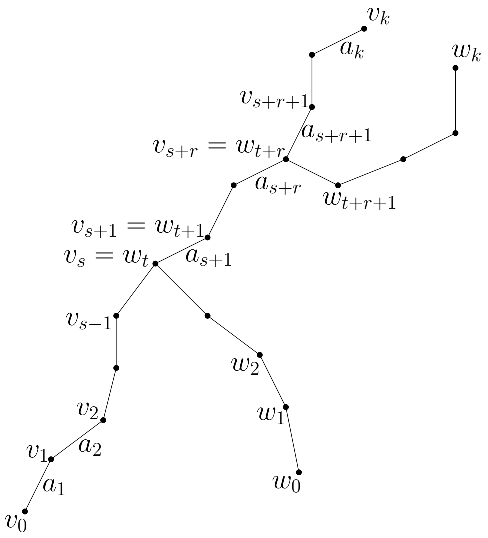

FIGURE 4. Case II

(Case IIa): The labeling of a directed edge $w_{i-1}w_i$ *is* a_i (see Figure 5 on the left). In this case we consider separately two subcases:

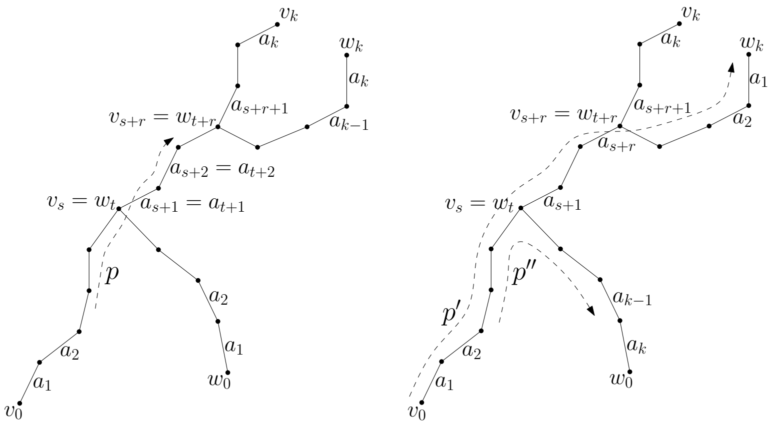

FIGURE 5. Case IIa (left) and Case IIb (right)

(i) If $s = t$ then we consider the path (v_{s-1}, v_s, w_{s-1}), if $s > 0$, or the path $(v_{s+r+1}, v_{s+r}, w_{s+r+1})$ otherwise. We obtain the labeling:

$$(\underline{a_s}, \underline{\bar{a}_s}) \text{ or } (\underline{\bar{a}_{s+r+1}}, \underline{a_{s+r+1}}).$$

(ii) If $s > t$ then we obtain a path $p := (v_t, v_{t+1}, \ldots, w_t, w_{t+1}, \ldots, w_s)$ of length bounded from above by

$$2k \leqslant 2\lambda \operatorname{girth} \Theta_n. \tag{13}$$

Its labeling has the form:

$$(\underline{a_{t+1}, a_{t+2}, \ldots, a_s}, \underline{a_{t+1}, a_{t+2}, \ldots, a_s}). \tag{14}$$

(The two above cases are 'repetitive' labellings as in [AGHR02].)

(Case IIb): The labeling of a directed edge $w_{i+1}w_i$ is a_{k-i} (see Figure 5 on the right). In this case we consider separately three subcases:

(i) If $s > k/3$ and $t < 2k/3$ then we consider the path $p' := (v_0, \ldots, v_s, w_{t+1}, \ldots, w_k)$. Its length is bounded from above by (13), and its labeling has the form:

$$(\underline{a_1, a_2, \ldots, a_q}, \ldots, \underline{\bar{a}_q, \ldots, \bar{a}_2, \bar{a}_1}),$$

with

$$q > k/3. \tag{15}$$

(ii) If $t \geqslant 2k/3$, then $s \geqslant t \geqslant 2k/3$. In this case we consider the path $p'' := (v_{k-t}, \ldots, v_s, w_{t-1}, \ldots, w_{k-s})$. Its length is bounded from above by (13), and its labeling has the form:

$$(\underline{a_{k-t+1}, \ldots, a_s}, \underline{a_{k-t+1}, \ldots, a_s}). \tag{16}$$

(iii) If $s \leqslant k/3$ then for $s + r < 2k/3$ we are in one of the previous cases (with $s > k/3$) after changing indexes i to $k - i$ and renaming. Thus we may assume that $s + r \geqslant 2k/3$. Then we consider the path $p' := (v_0, v_1, \ldots, v_s, w_{t+1}, \ldots, w_k)$. Its length is bounded from above by (13), and its labeling has the form:

$$(\underline{a_1, a_2, \ldots, a_q}, \ldots, \underline{\bar{a}_q, \ldots, \bar{a}_2, \bar{a}_1}), \tag{17}$$

with

$$q \geqslant k/3. \tag{18}$$

Lemma 2.4. *Let $E := \lambda/(16\lambda + 8A)$ and $F := (2\lambda + A)\operatorname{girth} \Theta_n$. Assume that there are two different paths in (Θ_n, m_n), of length at least $\lambda \operatorname{girth} \Theta_n$, with the same labeling. Then one of the following situations happens:*

- (A) *there is a path p with the labeling $(\underline{a_{i_1}, a_{i_2}, \ldots, a_{i_q}}, \ldots, \underline{a_{i_1}, a_{i_2} \ldots, a_{i_q}})$, for $|p| \leqslant F$ and $q \geqslant E|p|$;*
- (B) *there is a path p with the labeling $(\underline{a_{i_1}, a_{i_2}, \ldots, a_{i_q}}, \ldots, \underline{\bar{a}_{i_q}, \ldots, \bar{a}_{i_2}, \bar{a}_{i_1}})$, for $|p| \leqslant F$ and $q \geqslant E|p|$.*
- (C) *there is a path p with the labeling $(\underline{a_i}, \underline{\bar{a}_i})$.*

Proof. We show that all the cases analyzed earlier in this section lead to (A), (B) or (C). This covers all the possible configurations.

(A) corresponds to the cases: Ib(ii), IIa(ii) and IIb(ii). The estimates on $|p|$ and q follow then from: formula (8) and formula (12), or from (14), or from (16), and from the fact that

$$E = \frac{\lambda}{16\lambda + 8A} < \frac{1}{16}. \tag{19}$$

(B) corresponds to one of the cases: Ia, Ib(i), IIb(i) or IIb(iii). The estimates on $|p|$ and q follow then from: (8) and (10) or (11), or from (13) and (15) or (18), using (19).

(C) corresponds to Case IIa(i). □

Now we show, similarly as in the preceding Subsection 2.1, that there exists a labeling $(\Theta_n, \bar{l}_n)$ such that none of the patterns (A), (B) or (C) from Lemma 2.4 appears. This will imply that no two different paths in Θ_n of length at least $\lambda \, \mathrm{girth}\, \Theta_n$ have the same $\bar{l}_n$–labeling. This will also mean that $\bar{l}_n$ is reduced. The labeling $\bar{l}_n$ will use $\bar{L}$ labels. Define $\bar{L}$ as (here e denotes the Euler constant)

$$\bar{L} := \left\lceil (4De^4)^{\frac{1}{E}} \right\rceil, \tag{20}$$

where $E = \lambda/(16\lambda + 8A)$ is the constant from Lemma 2.4. Call a labeling of a path p *bad* if it is of the form (A), (B) or (C) as in Lemma 2.4. Observe that

$$|\{\text{bad labellings of a path of length } i \text{ with } \bar{L} \text{ labels}\}| \leqslant 2\bar{L}^{(1-E)i}, \tag{21}$$

$$|\{\text{labellings of a path of length } i \text{ with } \bar{L} \text{ labels}\}| = \bar{L}^i. \tag{22}$$

Lemma 2.5. *There exists a labeling $(\Theta_n, \bar{l}_n)$ with $\bar{L}$ labels such that, for $2 \leqslant i \leqslant F = (2\lambda + A)\mathrm{girth}\, \Theta_n$ no $\bar{l}_n$–labeling of a path of length i is bad.*

Proof. We use the Lovász Local Lemma 2.1 as in the proof of Lemma 2.2. Randomly label the edges of Θ_n with $\bar{L}$ labels. For a path p in Θ_n of length i, let $A(p)$ denote the event that its labeling is bad. Set $\mathcal{A}_i = \{A(p) : p \text{ is a path of length } i \text{ in } \Theta_n\}$. Then, by (21) and (22), we have

$$p_i \leqslant \frac{2\bar{L}^{(1-E)i}}{\bar{L}^i} \leqslant \left(\frac{2}{\bar{L}^E}\right)^i. \tag{23}$$

Each path of length i shares an edge with not more than ijD^j paths of length j, so that we may take $\Delta_{ij} = ijD^j$. Let $a_i = a^{-i}$, where $a = 2D$. Then, by using subsequently: formulas (23) and (20), the definition of a_i, the fact that $\sum_{j=1}^{\infty} j/2^j = 2$, the definitions of a, Δ_{ij}, and a_j, we obtain:

$$\begin{aligned} p_i &\leqslant \left(\frac{2}{\bar{L}^E}\right)^i \leqslant 2^{-i}D^{-i}e^{-4i} = a_i \exp\left(-2\sum_{j=1}^{\infty} i\frac{j}{2^j}\right) \\ &< a_i \exp\left(-2\sum_j ij\left(\frac{D}{a}\right)^j\right) = a_i \exp\left(-2\sum_j \Delta_{ij}a_j\right) \\ &= a_i \prod_j e^{-2a_j\Delta_{ij}} \end{aligned}$$

Since, by $a_j \leqslant 1/2$, we have $e^{-2a_j} \leqslant (1-a_j)$, we obtain finally

$$p_i \leqslant a_i \prod_j (1-a_j)^{\Delta_{ij}}.$$

Therefore the hypotheses of the Lovász Local Lemma are fulfilled, and we conclude that there exists a labeling $\bar{l}_n$ as required. □

Lemma 2.6 ($C'(\lambda)$–small cancellation labeling of Θ_n)**.** *The labeling* $(\Theta_n, \bar{l}_n)$ *with* $\bar{L}$ *labels is reduced and no two paths in* Θ_n *of length at least* $\lambda\, \mathrm{girth}\, \Theta_n$ *have the same* $\bar{l}_n$*–labeling.*

Proof. The labeling $(\Theta_n, \bar{l}_n)$ is reduced because the situation (C) from Lemma 2.4 does not appear. The second assertion follows from Lemma 2.4 and the fact that none of the situations (A) and (B) appears for $\bar{l}_n$, by Lemma 2.5. □

2.3. **Small cancellation labeling of** Θ**.** Let $(\Theta, l) = ((\Theta_n, l_n))_{n\in\mathbb{N}}$ and $(\Theta, \bar{l}) = ((\Theta_n, \bar{l}_n))_{n\in\mathbb{N}}$ be the labellings with, respectively, L and $\bar{L}$ labels given by Lemma 2.3 and Lemma 2.6. Let $(\Theta, m) = ((\Theta_n, m_n))_{n\in\mathbb{N}}$ be a labeling being the product of (Θ, l) and $(\Theta, \bar{l})$. That is, to every directed edge e in Θ_n we assign a pair $(l(e), \bar{l}(e))$. By Lemma 2.3 and Lemma 2.6 we obtain the following main technical result of the paper (see Theorem 1 in Introduction).

Theorem 2.7 ($C'(\lambda)$–small cancellation labeling of Θ)**.** *The labeling* (Θ, m) *is reduced and no* m_n*–labeling of a path of length at least* $\lambda\, \mathrm{girth}\, \Theta_n$ *in* Θ_n *appears as the* m*–labeling of some other path in* Θ*.*

2.4. **Remarks on the Gromov labeling.** In this subsection we recall Gromov's construction of a 'small cancellation' labeling of some expanders [Gro03], following its exposition presented in [AD08]. We show a simplified (but non-elementary) construction of a labeling as in Lemma 2.6, relying on Gromov's construction. (Note however that our construction from Subsection 2.2 works in much more general scope then Gromov's one.) Further, we explain why one cannot obtain the small cancellation labeling out of the one of Gromov, that is, why Lemma 2.3 does not hold for the generic labeling.

For primes $p \neq q$ congruent to 1 modulo 4 and with the Legendre symbol $\left(\frac{p}{q}\right) = -1$, let $X^{p,q}$ be the Cayley graph of the projective linear group $PGL_2(q)$, for some particular set of $(p+1)$ generators, as in [AD08, Section 7.2]. Fix p. Throughout this subsection we consider subsequences of the sequence $\Theta = (\Theta_n)_{n\in\mathbb{N}}$, where $\Theta_n = X^{p,q_n}$, with q_n denoting the n–th prime. Then the family Θ is an expander with the constant degree $D := p+1$, with girth $\Theta_n \to \infty$, as $n \to \infty$, and for which there exists a constant A such that (1) holds. Gromov [Gro03] constructs a labeling $(\Theta, \bar{l}) = ((\Theta_n, \bar{l}_n))_{n\in\mathbb{N}}$ (also for a class of expanders) satisfying some small cancellation conditions.

Let G_0 be the free group generated by a finite set S. The labeling $\bar{l}$ is a map $\bar{l}\colon \Theta \to W$ onto the bouquet of $|S|$ oriented loops labeled by S (whose fundamental group is G_0). The labeling $(\Theta, \bar{l})$ is obtained inductively.

We begin with Θ_1 and we find a labeling $\bar{l}_1\colon \Theta_1 \to W$ satisfying some small cancellation conditions. We obtain a hyperbolic group G_1 being the quotient of the free group G_0 by the normal subgroup generated by images of $\bar{l}_1$. At the inductive step, having a hyperbolic group G_{n-1} generated by S, a random (generic) labeling $\bar{l}_n\colon \Theta_n \to W$ satisfies the *very small cancellation conditions* for the small cancellation constant arbitrarily close to 0 by [AD08, Proposition 5.9].

The following lemma shows that $\bar{l}$ has the properties as in Lemma 2.6.

Lemma 2.8. *A labeling of a path of length at least* $\lambda\,\mathrm{girth}\,\Theta_n$ *in* $(\Theta_n, \bar{l}_n)$ *does not occur in any different place in* $(\Theta_n, \bar{l}_n)$.

Proof. If there is such a labeling then we obtain a contradiction with the very small cancellation condition. Here we give a precise argument, following the notation from [AD08, Section3.2]. Let $(v_1, v_2, \ldots, v_l)$ be a path of length at least $\lambda\,\mathrm{girth}\,\Theta_n$ in Θ_n whose $\bar{l}_n$–labeling is the same as the one of a distinct path $(w_1, w_2, \ldots, w_l)$. Let $u = (u_1 := v_1, u_2, \ldots, u_s := w_1)$ be a path. The labeling $\bar{l}_n$ induces a $\pi_1(\Theta_n)$–equivariant simplicial map $\bar{l}_n\colon T \to Cay(G_{n-1})$ from the covering tree T of Θ_n into the Cayley graph of G_{n-1}. Let Y denote the image of T, and let $\widetilde{v}_i, \widetilde{w}_i, \widetilde{u}_i$ denote lifts of, respectively, v_i, w_i, u_i to T, such that $\widetilde{v}_1$ is the vertex corresponding to $1_{G_{n-1}}$, and there is a path $\widetilde{u} = (\widetilde{u}_1 := \widetilde{v}_1, \widetilde{u}_2, \ldots, \widetilde{u}_s := \widetilde{w}_1)$. Then the element of G_{n-1} corresponding to $\widetilde{u}$, denoted $g_{\widetilde{u}}$, is not in the image of $\pi_1(\Theta_n)$, and $gY \cap Y$ contains a path $\widetilde{w} := (\widetilde{w}_1, \widetilde{w}_2, \ldots, \widetilde{w}_l)$. This path has length at least $\lambda\,\mathrm{girth}\,\Theta_n$. This contradicts the very small cancellation condition for an appropriate choice of the small cancellation constant. □

The labeling $(\Theta, \bar{l})$ may be non-reduced. To remedy this one may proceed in the following way constructing a reduced labeling, for which Lemma 2.8 still holds. Fix an orientation on Θ. For any vertex v, assign to edges containing v different labels from a set of cardinality D (the degree). Then the label of an oriented edge e is a triple (a, b, c) where a is the $\bar{l}$–label of e, and b, c are labels of e assigned from, respectively, the start- and the end-vertex of e; see Figure 6.

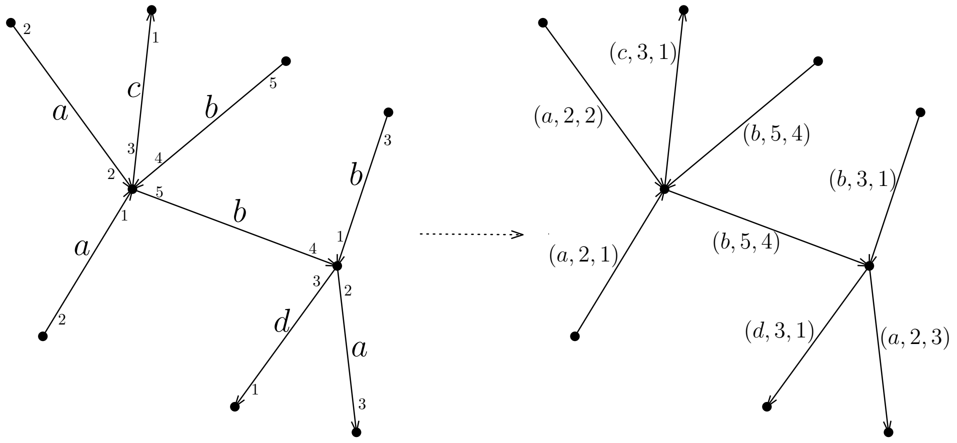

FIGURE 6. Obtaining a reduced labeling from $(\Theta, \bar{l})$.

Finally, let us explain why one cannot derive the required small cancellation labeling as in Theorem 2.7 from Gromov's construction. Since at each step Gromov's labeling is the generic labeling appearing as girth $\Theta_n \to \infty$, it is clear that the following holds: For any fixed labeling of a path of a fixed length, with overhelming probability this labeling will appear among labellings of Θ_n as $n \to \infty$. In particular, labellings of all cycles in graphs obtained at earlier inductive steps will appear as labellings of paths in later steps. This is the reason why Gromov's labeling is not a graphical small cancellation labeling, and why this labeling does not define a coarse embedding of Θ into the resulting group G being the limit of $(G_n)_{n\in\mathbb{N}}$. There is only a weak embedding or, stronger, a map $f\colon \Theta \to G$ satisfying the following condition: for $x, y \in \Theta_n$ one has $d_G(f(x), f(y)) \geqslant B d_\Theta(x, y) - c_n$, where B is a universal constant, and additive constants $c_n > 0$ grow to infinity with $n \to \infty$; see [Gro03, Section 4.6] and [AD08, Theorem 7.7].

3. GROUPS WITH Θ IN CAYLEY GRAPHS

In this section we construct groups, such that Θ embeds isometrically into their Cayley graphs – this means that the vertex set of every connected component Θ_n embeds isometrically. The groups are defined by *graphical small cancellation presentations* using the sequence Θ as follows. Let Γ be a finite graph and let $(\varphi_n\colon \Theta_n \to \Gamma)_{n\in\mathbb{N}}$ be a family of local isometries of graphs. They form a *graphical presentation*

$$\langle \Gamma \mid \Theta \rangle, \tag{24}$$

defining a group $G := \pi_1(\Gamma)/\langle\langle \varphi_*(\pi_1(\Theta_n))_{n\in\mathbb{N}} \rangle\rangle$. In our case we choose Γ to be a bouquet of loops with local isometries φ_n corresponding to the labellings m_n.

3.1. **$C'(\lambda)$–small cancellation complexes.** This subsection follows closely [AO14, Section 2]. Here we describe the spaces that we will work with further. Let $(\varphi_i\colon r_i \to X^{(1)})_{i\in\mathbb{N}}$ be a family of local isometries of finite graphs r_i. We will call these finite graphs *relators*. We assume that $\varphi_i \neq \varphi_j$, for $i \neq j$. The *cone* over the relator r_i is the quotient space $\operatorname{cone} r_i := (r_i \times [0,1])/\{(x,1) \sim (y,1)\}$. The main object of our study in this section is the *coned-off space*:

$$X := X^{(1)} \cup_{(\varphi_i)} \bigcup_{i\in\mathbb{N}} \operatorname{cone} r_i,$$

where φ_i is the map $r_i \times \{0\} \to X^{(1)}$. We assume that X is simply connected. The space X has a natural structure of a CW complex and we call X a 'complex'. If not specified otherwise, we consider the *path metric*, denoted by $d(\cdot,\cdot)$, defined on the 0–skeleton $X^{(0)}$ of X by (combinatorial) paths in the 1–skeleton $X^{(1)}$. *Geodesics* are the shortest paths in $X^{(1)}$ for this metric. (Otherwise, a geodesic between vertices $p, q \in X^{(0)}$ is a shortest sequence $p_0 := p, p_1, \ldots, p_k := q$ of vertices such that p_i and p_{i+1} are connected by an edge in $X^{(1)}$.)

A subgraph $p \hookrightarrow X^{(1)}$ is a *piece* if there are relators r_i, r_j such that $p \hookrightarrow X$ factors as $p \hookrightarrow r_i \xrightarrow{\varphi_i} X$ and as $p \hookrightarrow r_j \xrightarrow{\varphi_j} X$, but there is no isomorphism $r_i \to r_j$ that makes the following diagram commutative.

$$\begin{tikzcd}
p \arrow[r] \arrow[d] & r_j \arrow[d] \\
r_i \arrow[r] \arrow[ur] & X
\end{tikzcd}$$

This means that p occurs in r_i and r_j in two essentially distinct ways.

For $\lambda \in (0,1)$, we say that the complex X satisfies the *$C'(\lambda)$–small cancellation condition* (or that X is a *$C'(\lambda)$–complex*) if every piece $p \hookrightarrow X$ factorizing through $p \hookrightarrow r_i \xrightarrow{\varphi_i} X$ has diameter strictly less than $\lambda \operatorname{girth} r_i$.

Lemma 3.1 ([AO14, Lemma 2.1], cf. [Oll06, Theorem 1]). *If X is a $C'(1/24)$–complex, then the maps $\varphi_i\colon r_i \to X$ are isometric embeddings.*

3.2. **The groups.** In this section we use the labeling (Θ, m) as in Theorem 2.7, obtained for $\lambda \leqslant 1/24$.

Theorem 3.2 (Groups containing Θ). *Let G be the group defined by the graphical presentation $\langle \Gamma \mid \Theta \rangle$, where the local isometries $\Theta_n \to \Gamma$ are defined by labellings m_n. Then Θ embeds isometrically into the Cayley graph of G given by $\langle \Gamma \mid \Theta \rangle$.*

Proof. Consider the *coned-off space* obtained by gluing, using $\Theta_n \to \Gamma$, cones over graphs Θ_n to Γ. The fundamental group of this space is G. The Cayley graph is the 1–skeleton $X^{(1)}$ of the universal cover X of the coned-off space. By Lemma 2.7 X satisfies the $C'(\lambda)$–small cancellation condition, with relators (r_i) being copies of graphs (Θ_n). By Lemma 3.1, every r_i embeds isometrically into $X^{(1)}$. □

In the following subsections we study more specific applications of Theorem 3.2.

3.2.1. *Groups containing expanders.* When Θ is an expander, we obtain an analogue of [Gro03, 1.2.C] and [HLS02, Section 7] (cf. [AD08, Corollaries 7.10 & 7.12]).

Corollary 3.3. *If Θ is an expanding sequence of graphs then the group $\langle \Gamma \,|\, \Theta \rangle$ is not coarsely embeddable into a Hilbert space, and it does not satisfy the Baum-Connes conjecture with coefficients.*

The next result has been proved in [WY12] for groups with coarsely embedded expanders. As explained in Subsection 2.4, for Gromov's monster only the weak embedding is established. Therefore, our construction provides the first examples of groups, for which the conclusion of the following corollary holds.

Corollary 3.4 ([WY12, Corollary 1.7])**.** *Let G be a group defined by the graphical presentation $\langle \Gamma \,|\, \Theta \rangle$, where the local isometries $\Theta_n \to \Gamma$ are defined by labellings m_n, and where Θ is the sequence of expanding graphs with growing girth. Let X be the image of the isometric embedding of Θ into the Cayley graph Y of G. For each $n \in \mathbb{N}$, let $X_n = \{y \in Y \,|\, d_Y(y, X) \leqslant n\}$. Let $A_n = l^\infty(X_n, \mathcal{K})$ and $A = \lim_{n\to\infty} l^\infty(X_n, \mathcal{K})$, where $\mathcal{K}$ is the algebra of compact operators on a given infinite dimensional separable Hilbert space. Then the right action of G on Y gives A the structure of a G–C^*–algebra and:*

1. *the Baum-Connes assembly map for G with coefficients in A is injective;*
2. *the Baum-Connes assembly map for G with coefficients in A is not surjective;*
3. *the maximal Baum-Connes assembly map for G with coefficients in A is an isomorphism.*

Similarly, the existence of groups with coarsely embedded expanders is crucial for [BGW15, Section 7].

3.2.2. *Exotic aspherical manifolds.* Sapir [Sap14] developed a technique of embedding groups with combinatorially aspherical recursive presentation complexes into groups with finite combinatorially aspherical presentation complexes. The presentation (24) defined by the labeling (Θ, m) from Theorem 2.7 is aspherical; see e.g. [Oll06]. It is also recursive – the brute force algorithm can be used to find the labeling (Θ, l) (see Subsection 2.1), the labeling $(\Theta, \bar{l})$ (see Subsection 2.2) and, consequently, (Θ, m). By embedding the group $\langle \Gamma \,|\, \Theta \rangle$ from Corollary 3.3 into a finitely presented group we obtain the first examples of such groups coarsely containing expanders. Furthermore, Sapir's embedding is quasi-isometric – this may be proved similarly as in [BORS02].[1] Therefore, using Sapir's techniques and Theorem 3.2 we obtain the first examples of manifolds as follows.

Corollary 3.5. *There exist closed aspherical manifolds of dimension* 4 *and higher whose fundamental groups contain quasi-isometrically embedded expanders.*

3.2.3. *Infinite asymptotic dimension.* Let $\Theta^i = (\Theta_n)_{n\leqslant i}$ be the finite subsequence of Θ, for every i. The corresponding groups $G^i = \langle \Gamma \mid \Theta^i \rangle$ are finitely presented graphical $C'(1/6)$–small cancellation groups. Therefore, by [Oll06], they are hyperbolic, with asymptotic dimension at most two, that is $\mathrm{asdim}(G^i) \leqslant 2$. If G^i is not free, then we have $\mathrm{asdim}(G^i) = 2$. If the degree of Θ is at least three then, by [Wil11], $\mathrm{asdim}(\Theta) = \infty$, and hence $\mathrm{asdim}(G) = \infty$, where $G = \langle \Gamma \mid \Theta \rangle$. Therefore we obtain the following.

Corollary 3.6. *There exists a sequence* $G^1 \twoheadrightarrow G^2 \twoheadrightarrow G^3 \twoheadrightarrow \cdots$ *of finitely presented groups with the following properties. For all* i*,* $asdim(G_i) = 2$*, and the asymptotic dimension of the limit group* G *is infinite.*

This answers a question of Osin-Świątkowski; see [Dra08, Problem 4.5], [Osi08, Problem 6.1].

Despite the group G may have infinite asymptotic dimension, it behaves in many ways as a (asymptotically) two-dimensional object. In particular, G is asymptotically hereditarily aspherical (AHA) [OŚ15] – this coarse property allows e.g. to exclude many groups from being subgroups of G.

4. Walls

In this section and in the next Section 5 we develop a theory that will allow us in Section 6 to show that the group we construct there acts properly on a space with walls. We use here the notation from Section 3.1 concerning $C'(\lambda)$–complexes. The current section is very similar to [AO14, Section 3].

Recall, that for a set Y and a family $\mathcal{W}$ of partitions (called *walls*) of Y into two parts, the pair $(Y, \mathcal{W})$ is called a *space with walls* [HP98] if the following holds. For every two distinct points $x, y \in Y$ the number of walls separating x from y (called the *wall pseudo-metric*), denoted by $d_{\mathcal{W}}(x, y)$, is finite.

In this section, following the method of Wise [Wis11] (see also [Wis12]), we equip the 0–skeleton of a $C'(\lambda)$–complex with the structure of space with walls. To be able to do it we have to make some assumptions on relators.

A *wall* in a graph Γ is a collection w of edges such that removing all open edges of w decomposes Γ in exactly two connected components. We call Γ a *graph with walls*, if every edge belongs to a unique wall. This is a temporary abuse of notations with respect to 'walls' defined as above, which will be justified later.

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[1] This was pointed out to us by Mark Sapir.

If not stated otherwise, we assume that for a $C'(1/24)$–complex X, with given relators r_i, each graph r_i is a graph with walls. Observe that every r_i is in fact an isometrically embedded subgraph of X, by Lemma 3.1. Following [Wis11, Section 5], we define walls in $X^{(1)}$ as follows: Two edges are in the same wall if they are in the same wall in some relator r_i. This relation is then extended transitively for all relators.

In general, the above definition may not result in walls for $X^{(0)}$. We require some further assumptions on walls in relators, which are formulated below.

Definition 4.1 ((β,Φ)–separation)**.** For $\beta \in (0,1/2]$ and a homeomorphism $\Phi\colon [0,+\infty) \to [0,+\infty)$, a graph r with walls satisfies the *(β,Φ)–separation property* if the following two conditions hold:

β–condition: for every two edges e,e' in r belonging to the same wall we have

$$d(e,e')+1 \geqslant \beta\, \mathrm{girth}\ r.$$

Φ–condition: for every geodesic γ in r, the number of edges in γ whose walls have only one edge in common with γ (and thus, in particular, separate the end-points of γ) is at least $\Phi(|\gamma|)$.

A complex X satisfies the *(β,Φ)–separation property* if every its relator does so.

Proposition 4.1 ([AO14, Lemma 3.3])**.** *For every $\beta \in (0,1/2]$ there exists $\lambda \leqslant 1/24$, such that for every $C'(\lambda)$–complex X satisfying the β–condition the following holds. Removing all open edges from a given wall decomposes $X^{(1)}$ into exactly two connected components. The family of the corresponding partitions induced on $X^{(0)}$ defines the structure of the space with walls $(X^{(0)},\mathcal{W})$.*

In what follows we assume that a $C'(\lambda)$–complex X is as in the proposition. We recall further results on $(X^{(0)},\mathcal{W})$ that will be extensively used in Section 5.

For a wall w, its *hypergraph* Γ_w is a graph defined as follows (see [Wis11, Definition 5.18] and [Wis04]). There are two types of vertices in Γ_w (see e.g. Figure 7):

- *edge-vertices* correspond to edges in w,
- *relator-vertices* correspond to relators containing edges in w.

An *edge* in Γ_w connects an edge-vertex to a relator-vertex whenever the corresponding relator contains the given edge.

The *hypercarrier* of a wall w is the 1–skeleton of the subcomplex of X consisting of all relators containing edges in w or of a single edge e if $w=\{e\}$. The following theorem recalls the most important facts concerning walls; see [AO14, Subection 3.3].

Theorem 4.2. *Each hypergraph is a tree. Relators and hypercarriers are convex subcomplexes of* $X^{(1)}$.

5. Proper lacunary walling

In this section we introduce the condition of proper lacunary walling (see Definition 5.1), and we show that for complexes satisfying this condition the wall pseudo-metric is proper; see Theorem 3 in Introduction and Theorem 5.6 below. We follow the notation from Section 3.1 and Section 4. The section is based on and is analogous to [AO14, Section 4]. Note however that whereas the proper lacunary walling condition from the current paper is weaker than the corresponding lacunary walling condition from [AO14], consequences of the former are also weaker: We obtain properness of the wall pseudo-metric, and in [AO14] a linear separation property is established.

For a relator r, let $P(r)$ denote the maximal number of edges in a piece in r (recall that pieces are subgraphs).

Definition 5.1 (Proper lacunary walling)**.** Let $\beta \in (0, 1/2]$, and let D be a natural number larger than 1. Let $0 < \lambda < \beta/2$ be as in Proposition 4.1 (that is, such that $(X^{(0)}, \mathcal{W})$ is a space with walls). Let $\Phi, \Omega, \Delta \colon [0, +\infty) \to [0, +\infty)$ be homeomorphisms. We say that X satisfies the *proper lacunary walling condition* if:

- $X^{(1)}$ has degree bounded by D;
- (Small cancellation) X satisfies the $C'(\lambda)$–condition;
- (Separation) X satisfies the (β, Φ)–separation property;
- (Lacunarity) $\Phi((\beta - \lambda)\operatorname{girth} r_i) - 6\,P(r_i) \geqslant \Omega(\operatorname{girth} r_i)$;
- (Large girth) $\operatorname{girth} r_i \geqslant \Delta(\operatorname{diam} r_i)$.

It is clear that $d_{\mathcal{W}}(p, q) \leqslant d(p, q)$. The rest of this section is devoted to bounding the wall pseudo-metric $d_{\mathcal{W}}$ from below. Let γ be a geodesic in X (that is, in its 1–skeleton $X^{(1)}$) with endpoints p, q. Let $A(\gamma)$ denote the set of edges in γ whose walls meet γ in only one edge (in particular such walls separate p from q). Clearly $d_{\mathcal{W}}(p, q) \geqslant |A(\gamma)|$. We thus estimate $d_{\mathcal{W}}(p, q)$ by closely studying the set $A(\gamma)$. The estimate is first provided locally (in Subsection 5.1 below) and then we use the local bounds to obtain a global one. In what follows, by $E(Y)$ we denote the set of edges of a subcomplex $Y \subseteq X$.

We begin with an auxiliary lemma. Let r be a relator. Since, by Theorem 4.2, r is convex in X, its intersection with γ is an interval $p'q'$, with p' lying closer to p; see Figure 7. Consider the set C of edges e in $p'q'$, whose walls w meet γ at least twice and, moreover, have the following properties. Let $e' \in w$ (considered as an edge-vertex in the hypergraph Γ_w of the wall w) be a closest vertex to e in Γ_w, among edges of w lying on γ. In the hypergraph Γ_w of the wall w, which is a tree by Theorem 4.2, consider the unique geodesic γ_w between vertices e and e'. We assume that there are at least two distinct relator-vertices on γ_w, one of them being r.

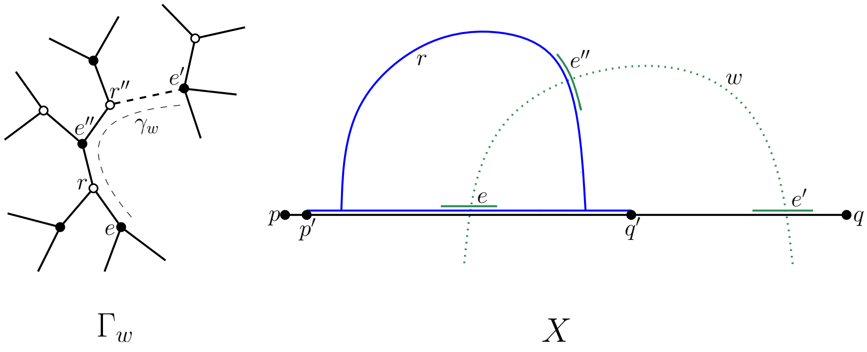

Figure 7. Lemma 5.1.

Lemma 5.1. *In the situation as above we have* $|C| \leqslant 2P(r)$*.*

Proof. The proof is the same as the one of [AO14, Lemma 4.2]. □

5.1. **Local estimate on** $|A(\gamma)|$**.** For a local estimate we need to define neighborhoods N_e – *relator neighborhoods in* γ – one for every edge e in γ, for which the number $|E(N_e) \cap A(\gamma)|$ of edges can be bounded from below.

For a given edge e of γ we define a corresponding relator neighborhood N_e as follows. If $e \in A(\gamma)$ then $N_e = \{e\}$. Otherwise, we proceed in the way described below.

Since e is not in $A(\gamma)$, its wall w crosses γ in at least one more edge. In the wall w, choose an edge $e' \subseteq \gamma$ being a closest edge-vertex to $e \neq e'$ in the hypergraph Γ_w of the wall w. We consider separately the two following cases, see Subsection (5.1.1) and Subsection (5.1.2) below.

5.1.1. **Case I: The edges** e **and** e' **do not lie in common relator.** In the hypergraph Γ_w of the wall w, which is a tree by Theorem 4.2, consider the geodesic γ_w between vertices e and e'. Let r be the relator-vertex in γ_w adjacent to e. Let e'' be an edge-vertex in γ_w adjacent to r. Consequently, let r'' be the other relator-vertex in γ_w adjacent to e''. The intersection of r with γ is an interval $p'q'$. Assume without loss of generality, that q' lies between e and e'; see Figure 7.

We define the relator neighborhood N_e as the interval $p'q' = r \cap \gamma$. The following lemma is the same as [AO14, Lemma 4.3].

Lemma 5.2.

$$|E(N_e)| > (\beta - \lambda)\,\mathrm{girth}\; r.$$

We are now ready to state the main result in Case I.

Lemma 5.3 (Local density of $A(\gamma)$ – Case I)**.** *The number of edges in* N_e*, whose walls separate* p *from* q *is estimated as follows:*

$$|E(N_e) \cap A(\gamma)| \geqslant \Phi((\beta - \lambda)\,\mathrm{girth}\, r_e) - 6\,P(r_e).$$

Proof. To estimate $|E(N_e) \cap A(\gamma)|$ we consider first a set B of edges in N_e defined in the following way. An edge f belongs to B if its wall w_f has only one edge in common with N_e. In particular, w_f separates p' from q'.

By the Φ–condition from Definition 4.1, and by Lemma 5.2, we have

$$|B| \geqslant \Phi(|E(N_e)|) \geqslant \Phi((\beta - \lambda)\,\mathrm{girth}\, r_e). \tag{25}$$

We estimate further the number of edges in $A(\gamma) \cap B$. To do this we explore the set of edges f in B outside $A(\gamma)$. We consider separately the two ways in which an edge f of B may fail to belong to $A(\gamma)$ – these are studied in Cases: C and D below.

Since $f \in B \setminus A(\gamma)$, there exists another edge of the same wall w_f in γ outside r_e. Let f' be a closest to f such edge-vertex in the hypergraph Γ_{w_f}. Denote by γ_{w_f} the geodesic in Γ_{w_f} between f and f'. Let r_f be the relator-vertex on γ_{w_f} adjacent to f.

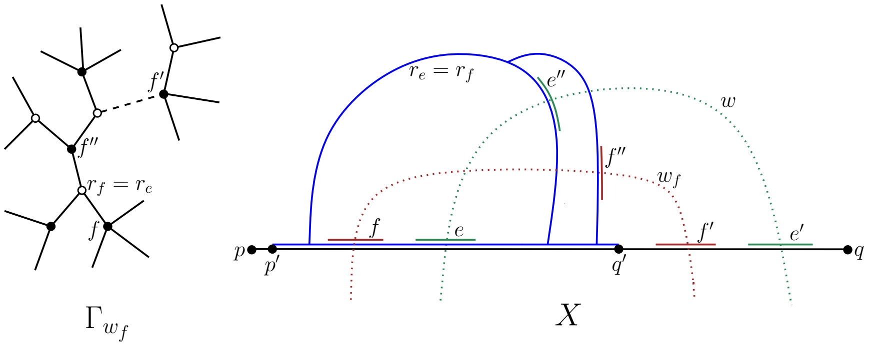

FIGURE 8. Lemma 5.3, Case I(C).

(Case C): $r_f = r_e$. Observe that then there are at least two distinct relator-vertices between f and f' on γ_{w_f}; see Figure 8. The cardinality of the set C of such edges f is bounded, by Lemma 5.1, as follows:

$$|C| \leqslant 2\,P(r_e). \tag{26}$$

(Case D): $r_f \neq r_e$. Let the set of such edges f be denoted by D. In this case we proceed as in the analogous Case D in the proof of [AO14, Lemma 4.4] to obtain the following estimate.

$$|D| \leqslant 4\,P(r_e). \tag{27}$$

Now we combine the cases C, and D, to obtain the following bound in Case I, see estimates (25), (26), and (27) above.

$$\begin{aligned} |E(N_e) \cap A(\gamma)| &\geqslant |B \cap A(\gamma)| \geqslant |B| - |C| - |D| \\ &\geqslant \Phi((\beta - \lambda)\,\mathrm{girth}\, r_e) - 6\,P(r_e). \end{aligned}$$

□

5.1.2. **Case II: The edges e and e' lie in common relator r.** We may assume (exchanging e' if necessary) that e' is closest to e (in X) among edges in w lying in $r_e \cap \gamma$.

The relator neighborhood N_e is now defined as the interval $p'q' = r \cap \gamma$.

Lemma 5.4 (Local density of $A(\gamma)$ – Case II)**.** *The number of edges in N_e, whose walls separate p from q is estimated as follows:*

$$|E(N_e) \cap A(\gamma)| \geqslant \Phi(\beta \,\mathrm{girth}\, r_e) - 6\, P(r_e).$$

Proof. By the β–separation, we have

$$|E(N_e)| \geqslant \beta \,\mathrm{girth}\, r_e \tag{28}$$

Again, let B be the set of edges f in N_e such that their wall w_f intersects N_e in one edge. Then w_f separates p' and q'. As in Case I (see (25)), by (28), we have the following lower bound:

$$|B| \geqslant \Phi(|E(N_e)|) \geqslant \Phi(\beta \,\mathrm{girth}\, r_e).$$

We estimate again the number of edges f in $B \setminus A(\gamma)$. As in Case I (Lemma 5.3), we consider separately two possibilities: C, D for such an edge f to fail belonging to $A(\gamma)$. The same considerations as in Case I lead to the estimates:

$$|C| \leqslant 2\, P(r_e),$$
$$|D| \leqslant 4\, P(r_e).$$

Combining all the inequalities above we get

$$\begin{aligned} |E(N_e) \cap A(\gamma)| &\geqslant |B \cap A(\gamma)| \geqslant |B| - |C| - |D| \\ &\geqslant \Phi(\beta \,\mathrm{girth}\, r_e) - 6\, P(r_e). \end{aligned}$$

□

5.1.3. *Final local estimate.* We are ready to combine all the previous estimates.

Lemma 5.5 (Local density of $A(\gamma)$)**.** *The number of edges in N_e, whose walls separate p from q is estimated as follows:*

$$|E(N_e) \cap A(\gamma)| \geqslant \Omega(\mathrm{girth}\, r_e).$$

Proof. If $e \in A(\gamma)$ then the assertion is clear. If $e \notin A(\gamma)$ then we use the lacunarity condition from Definition 5.1, and Lemma 5.3 or Lemma 5.4. □

5.2. **Properness of the wall pseudo-metric.** Using the local estimate on the density of $A(\gamma)$ from Lemma 5.5, we now estimate the overall density of edges with walls separating p and q, thus obtaining the properness of the wall pseudo-metric $d_{\mathcal{W}}$.

Theorem 5.6 (Properness)**.** *There exists a homeomorphism $\Psi\colon [0,+\infty) \to [0,+\infty)$ such that*

$$d(p,q) \geqslant d_{\mathcal{W}}(p,q) \geqslant \Psi(d(p,q)).$$

Proof. The left inequality is clear. Now we prove the right one. Define $\Psi\colon [0,+\infty) \to [0,+\infty)$ as a homeomorphism such that $\Psi(d) \leqslant \min\{\sqrt{d}/2, \Omega(\Delta(\sqrt{d}))\}$. For given p,q, we denote $d := d(p,q)$.

We work with the family $\{N_e\}_{e\subseteq\gamma}$ of relator neighborhoods, as defined in Subsection 5.1. We consider separately the following two cases.

(Case 1): There is and edge e in γ with $|E(N_e)| \geqslant \sqrt{d}$. For such an edge e, by the large girth condition from Definition 5.1, we have

$$\operatorname{girth} r_e \geqslant \Delta(\operatorname{diam} r_e) \geqslant \Delta(|E(N_e)|) \geqslant \Delta(\sqrt{d}),$$

and thus, by Lemma 5.5, we obtain

$$|A(\gamma)| \geqslant |A(\gamma) \cap E(N_e)| \geqslant \Omega(\operatorname{girth} r_e) \geqslant \Omega(\Delta(\sqrt{d})). \tag{29}$$

(Case 2): For every edge e in γ we have $|E(N_e)| < \sqrt{d}$. Then, as in the proof of [AO15, Lemma 2.1] there is a family $\{e_1, e_2, \dots, e_k\}$ of edges in γ, such that $N_{e_i} \cap N_{e_j} = \emptyset$, for $i \neq j$ and $k \geqslant \sqrt{d}/2$. Therefore, by Lemma 5.5, we have

$$|A(\gamma)| \geqslant \sum_{i=1}^{k} |A(\gamma) \cap E(N_{e_i})| \geqslant \sum_{i=1}^{k} 1 \geqslant \sqrt{d}/2. \tag{30}$$

Combining formulas (29) and (30), we obtain

$$d_{\mathcal{W}}(p,q) \geqslant |A(\gamma)| \geqslant \Psi(d(p,q)).$$

□

6. PW non-A groups

In this section we prove Theorem 2 from Introduction; see Theorem 6.2 below. For the whole section we assume that Θ consists of D–regular graphs, for some $D \geqslant 3$. (This assumption could be 'coarsely weakened'; see [Wil11].) First we derive from (Θ, m) an appropriate sequence of labeled graphs $(\widehat{\Theta}, \widehat{m})$.

6.1. From (Θ, m) to $(\widetilde{\Theta}, \widetilde{m})$ and $(\widehat{\Theta}, \widehat{m})$.

In what follows, by $P(\Theta_n)$ we denote the maximal number of edges in a *piece* in (Θ_n, m_n), that is, a labeled subgraph of Θ_n that appears in Θ in two different places. Labeled graphs $(\widetilde{\Theta}, \widetilde{m})$ and $(\widehat{\Theta}, \widehat{m})$ will be defined below as appropriate *coverings of labeled graph* (Θ, m), that is, graph coverings with labellings induced from m by the covering map. By a *piece* in $(\widetilde{\Theta}, \widetilde{m})$ (respectively, $(\widehat{\Theta}, \widehat{m})$) we mean a labeled subgraph appearing in two *essentially different* (that is, not differing by a covering automorphisms) places in $(\widetilde{\Theta}, \widetilde{m})$ (respectively, $(\widehat{\Theta}, \widehat{m})$). By $P(\widetilde{\Theta}_n)$ and $P(\widehat{\Theta}_n)$ we denote the maximum number of edges in pieces in, respectively, $(\widetilde{\Theta}_n, \widetilde{m}_n)$ and $(\widehat{\Theta}_n, \widehat{m}_n)$. By Lemma 2.7, simple paths in pieces in Θ_n are shorter than $\lambda \operatorname{girth} \Theta_n$, and we have $P(\Theta_n) = P(\widetilde{\Theta}_n) = P(\widehat{\Theta}_n)$, for every n.

The labeled graph covering $(\widetilde{\Theta},\widetilde{m})$ is chosen so that $\mathrm{girth}\,\widetilde{\Theta}_n$ is large compared to $P(\Theta_n)$; see Lemma 6.1. The labeled graph $(\widehat{\Theta},\widehat{m})$ is the labeled graph covering defined as follows: For every n, $\widehat{\Theta}_n$ is the $\mathbb{Z}_2$–homology cover of $\widetilde{\Theta}_n$. As observed by Wise (see [Wis11, Section 9] and [Wis12, Section 10.3]), every $\widehat{\Theta}_n$ is then equipped with a structure of graph with walls – a wall corresponds to edges in $\widehat{\Theta}_n$ being preimages of a given edge in $\widetilde{\Theta}_n$ (see also [AGŠ12, Section 3] and [Ost12, Lemma 6]). With this system of walls we obtain the following.

Lemma 6.1. *There exist coverings* $(\widetilde{\Theta}_n,\widetilde{m}_n)\to(\Theta_n,m_n)$ *of appropriately large girth such that the following holds. There exist:* $\beta\in(0,1/2]$*,* $\lambda\in(0,1/24]$*, and homeomorphisms* $\Phi,\Omega,\Delta\colon[0,+\infty)\to[0,+\infty)$ *such that, for every* $n\in\mathbb{N}$ *we have:*

1. *the degree of* $\widehat{\Theta}_n$ *is bounded by* D*;*
2. *the diameter of each piece in* $\widehat{\Theta}_n$ *is at most* $\lambda\,\mathrm{girth}\,\widehat{\Theta}_n$*;*
3. $\widehat{\Theta}_n$ *satisfies the* (β,Φ)*–separation property;*
4. $\Phi((\beta-\lambda)\mathrm{girth}\,\widehat{\Theta}_n)-6\,P(\widehat{\Theta}_n)\geqslant\Omega(\mathrm{girth}\,\widehat{\Theta}_n)$*;*
5. $\mathrm{girth}\,\widehat{\Theta}_n\geqslant\Delta(\mathrm{diam}\,\widehat{\Theta}_n)$*.*

Proof. (1) is immediate. (2) follows from Theorem 2.7 and the fact that sizes of pieces do not change when passing to a covering. For (3), by [AO14, Lemma 7.1] one may take $\beta=1/2$, and the existence of Φ follows from [AGŠ12, Lemma 7.2]. Since $\beta=1/2$ and $\lambda\leqslant1/24$, we may choose the girth of each $\widetilde{\Theta}_n$ so large that (4) holds for some Ω. (5) follows from the fact that $\mathrm{girth}\,\widehat{\Theta}_n\to\infty$ as $n\to\infty$. □

6.2. **The group.** Now we construct a coarsely non-amenable group acting properly on a CAT(0) cubical complex announced in Theorem 2. The group is defined by a graphical small cancellation presentation over the sequence $\widehat{\Theta}$; see Section 3 for notations. Again, Γ is a bouquet of loops, and the local isometries $\varphi_n\colon\Theta_n\to\Gamma$ are defined by the labellings $\widehat{m}_n$.

Theorem 6.2 (PW non-A group)**.** *Let* G *be the group defined by the graphical presentation* $\langle\Gamma\,|\,\widehat{\Theta}\rangle$*, where the local isometries* $\widehat{\Theta}_n\to\Gamma$ *are defined by labellings* $\widehat{m}_n$*. Then* G *acts properly on a CAT(0) cubical complex and* G *does not have property A.*

Proof. As in the proof of Theorem 3.2, we construct a small cancellation complex X, on which G acts, by gluing cones over graphs $\widehat{\Theta}_n$ to the Cayley graph corresponding to the presentation $\langle\Gamma\,|\,\widehat{\Theta}\rangle$. We do not distinguish here cones differing by a covering automorphisms. (This does not happen in the proof of Theorem 3.2 but does happen here due to the existence of covering maps for nontrivial coverings $(\widehat{\Theta}_n,\widehat{m}_n)$.) By Lemma 6.1(2), X is a $C'(\lambda)$–complex, where relators (r_i) are copies of graphs $(\widehat{\Theta}_n)$.

Therefore, by Lemma 3.1, the graphs $\widehat{\Theta}_n$ embed isometrically into the Cayley graph $X^{(1)}$ of G. Since $\widehat{\Theta}_n$ are regular of degree $D\geqslant3$ and with girths tending to infinity, by a result of Willett [Wil11], the graph $X^{(1)}$ and, consequently, G have no property A.

To show that G acts properly on a CAT(0) cubical complex it is enough [Nic04, CN05] to show that G acts properly (with respect to the wall pseudometric) on a space with walls. Clearly G acts properly on $X^{(0)}$ and thus it remains to show that X satisfies the proper lacunary walling condition to conclude, from Theorem 5.6, that G acts properly on $(X^{(0)}, \mathcal{W})$. The proper lacunary walling condition follows from Lemma 6.1: separation follows from (3), lacunarity from (4), and the large girth condition follows from (5). □

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Instytut Matematyczny, Uniwersytet Wrocławski, pl. Grunwaldzki 2/4, 50–384 Wrocław, Poland

Universität Wien, Fakultät für Mathematik, Oskar-Morgenstern-Platz 1, 1090 Wien, Austria.
E-mail address: `dosaj@math.uni.wroc.pl`