RESIDUALLY FINITE NON-EXACT GROUPS

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Abstract. We construct the first examples of residually finite non-exact groups.

1. Introduction

A finitely generated group is *non-exact* if its reduced C^*–algebra is non-exact. Equivalently, it has no Guoliang Yu's property A (see e.g. [Roe03, Chapter 11.5]). Most classical groups are *exact*, that is, are not non-exact. The first examples of non-exact groups were the so-called *Gromov monsters* [Gro03]. In this paper we rely on author's construction of groups containing isometrically expanders [Osa14]. The isometric embedding of an expanding family of graphs performed in the latter construction is possible thanks to using a graphical small cancellation. That particular construction is crucial for results in the current paper.

Main Theorem. *There exist finitely generated residually finite non-exact groups defined by infinite graphical small cancellation presentations.*

This answers one of few questions from the Open Problems chapter of the Brown-Ozawa book [BO08, Problem 10.4.6]. Some motivations for the question can be found there. Our interest in the problem is twofold: First, we plan to use residually finite non-exact groups constructed here for producing other, essentially new examples of non-exact groups; Second, we believe that our examples might be useful for constructing and studying metric spaces with interesting new coarse geometric features. More precisely, let G be a finitely generated infinite residually finite group, and let $(N_i)_{i=1}^{\infty}$ be a sequence of its finite index normal subgroups with $\bigcap_{i=1}^{\infty} N_i = \{1\}$. The *box space* of G corresponding to (N_i) is the coarse disjoint union $\bigsqcup_{i=1}^{\infty} G/N_i$, with each G/N_i endowed with the word metric coming from a given finite generating set for G. Properties of the group G are often related to coarse geometric properties of its box space. For example, a group is amenable iff its box space has property A [Roe03, Proposition 11.39]. Box spaces provide a powerful method for producing metric spaces with interesting coarse geometric features (see e.g. [Roe03, Chapter 11.3]). The groups constructed

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in the current article open a way to studying box spaces of non-exact groups and make the following questions meaningful.

Questions. What are coarse geometric properties of box spaces of non-exact groups? Can non-exactness of a group be characterized by coarse geometric properties of its box space?[1]

The idea of the construction of groups as in the Main Theorem is as follows. The group is defined by an infinite graphical small cancellation presentation. It is a limit of a direct sequence of groups G_i with surjective bonding maps – each G_i has a graphical small cancellation presentation being a finite chunk of the infinite presentation. Such finite chunks are constructed inductively, using results of [Osa14], so that they satisfy the following conditions. Each group G_i is hyperbolic and acts geometrically on a CAT(0) cubical complex, hence it is residually finite.[2] For every i, there exists a map $\varphi_i \colon G_i \to F_i$ to a finite group such that no nontrivial element of the i–ball around identity is mapped to 1. Every φ_i factors through the quotient maps $G_i \twoheadrightarrow G_j$ so that it induces a map of the limit group G to a finite group injective on a large ball. The residual finiteness of G follows. Finally, G is non-exact since its Cayley graph contains a sequence of graphs (relators) without property A.

In Section 2 we present preliminaries on graphical small cancellation presentations and we recall some results from [Osa14]. In Section 3 we present the inductive construction of the infinite graphical small cancellation presentation proving the Main Theorem.

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2. Preliminaries

We follow closely (up to the notation) [Osa14].

2.1. **Graphs.** All graphs considered in this paper are *simplicial*, that is, they are undirected, have no loops nor multiple edges. In particular, we will consider Cayley graphs of groups, denoted $\mathrm{Cay}(G,S)$ – the Cayley graph

[1]After circulating the first version of the article I was informed that Thibault Pillon introduced a notion of "fibred property A", and proved that a finitely generated residually finite group is exact iff its box space has this property (unpublished).

[2]Note that Pride [Pri89] constructed infinitely presented classical small cancellation groups that are not residually finite. They are limits of hyperbolic CAT(0) cubical (hence residually finite) groups.

of G with respect to the generating set S. For a set S, an *S–labelling* of a graph Θ is the assignment of elements of $S \cup S^{-1}$ (S^{-1} being the set of formal inverses of elements of S) to directed edges (pairs of vertices) of Θ, satisfying the following condition: If s is assigned to (v, w) then s^{-1} is assigned to (w, v). All labellings considered in this paper are *reduced*: If s is assigned to (v, w), and s' is assigned to (v, w') then $s = s'$ iff $w = w'$. For a covering of graphs $p\colon \widehat{\Theta} \to \Theta$, having a labelling (Θ, l) we will always consider the *induced* labelling $(\widehat{\Theta}, \widehat{l})$: the label of an edge e in $\widehat{\Theta}$ is the same as the label of $p(e)$. Speaking about the metric on a connected graph Θ we mean the metric space $(\Theta^{(0)}, d)$, where $\Theta^{(0)}$ is the set of vertices of Θ and d is the path metric. The *ball* of radius i around v in Θ is $B_i(v, \Theta) := \{w \in \Theta^{(0)} \mid d(w, v) \leqslant i\}$. In particular, the metric on $\mathrm{Cay}(G, S)$ coincides with the word metric on G given by S.

2.2. **Graphical small cancellation.** A *graphical presentation* is the following data: $\mathcal{P} = \langle S \mid (\Theta, l)\rangle$, where S is a finite set – the *generating set*, and (Θ, l) is a graph Θ with an S–labelling l. We assume that Θ is a disjoint (possibly infinite) union of finite connected graphs $(\Theta_i)_{i\in I}$, and the labelling l restricted to Θ_i is denoted by l_i. We write $(\Theta, l) = (\Theta_i, l_i)_{i\in I}$. A graphical presentation $\mathcal{P}$ defines a group $G := F(S)/R$, where R is the normal closure in $F(S)$ of the subgroup generated by words in $S \cup S^{-1}$ read along (directed) loops in Θ.

A *piece* is a labelled path occurring in two distinct connected components Θ_i and Θ_j, or occurring in a single Θ_i in two places not differing by an automorphism of (Θ_i, l_i). Observe that if $(\widehat{\Theta}_i, \widehat{l}_i) \to (\Theta_i, l_i)$ is a normal covering then two lifts of a path in Θ_i differ by a covering automorphism. In particular, if the covering corresponds to a characteristic subgroup of $\pi_1(\Theta_i)$ then a lift of a non-piece is a non-piece.

For $\lambda \in (0, 1/6]$, the labelling (Θ, l) or the presentation $\mathcal{P}$ are called *$C'(\lambda)$–small cancellation* if length of every piece appearing in Θ_i is strictly less than $\lambda \mathrm{girth}(\Theta_i)$, where $\mathrm{girth}(\Theta_i)$ is the length of a shortest simple cycle in Θ_i. Such presentations define infinite groups. The introduction of graphical small cancellation is attributed to Gromov [Gro03]. For more details see e.g. [Wis, Osa14]. We will use mostly results proven already in [Wis, Osa14], so we list only the most important features of groups defined by graphical small cancellation presentations.

First, observe that if (Θ, l) is a $C'(\lambda)$–small cancellation labelling, and $\widehat{\Theta}_i \to \Theta_i$ is a covering corresponding to a characteristic subgroup of $\pi_1(\Theta_i)$, for each i, then the induced labelling $(\widehat{\Theta}, \widehat{l})$ is also $C'(\lambda)$–small cancellation. The following result was first stated by Gromov.

Lemma 2.1 ([Gro03])**.** *Let G be the group defined by a graphical $C'(\lambda)$–small cancellation presentation $\mathcal{P}$, for $\lambda \in (0, 1/6]$. Then, for every i, there is an isometric embedding $\Theta_i \to \mathrm{Cay}(G, S)$.*

The isometric embedding above is just an embedding of S–labelled graphs.

2.3. **Walls in graphs.** A *wall* in a connected graph is a collection of edges such that removing all interiors of these edges decomposes the graph in exactly two connected components. There are many ways for defining walls in finite graphs Θ_i. We would like however that such walls "extend" to walls in $\mathrm{Cay}(G,S)$. We use a particular construction of walls in finite graphs. For such graph Θ_i, let $\widehat{\Theta}_i$ denote its $\mathbb{Z}_2$–homology cover, that is, a normal cover corresponding to the kernel of the obvious map $\pi_1(\Theta_i,v_0)\to H_1(\Theta_i;\mathbb{Z}_2)$. Wise [Wis] observed that there is a natural structure of walls on $\widehat{\Theta}_i$: for every edge of Θ_i its preimage is a wall in $\widehat{\Theta}_i$. We call these walls *$\mathbb{Z}_2$–homology cover walls*. We will use results of Wise [Wis] to show that such walls, defined for $\widehat{\Theta}_1,\ldots,\widehat{\Theta}_i$ give rise to walls in $\mathrm{Cay}(\widehat{G},S)$, where $\widehat{G}$ is the group with the graphical presentation $\langle S\mid(\widehat{\Theta}_1,\widehat{l}_1),\ldots,(\widehat{\Theta}_i,\widehat{l}_i)\rangle$, for $\widehat{l}_j$ being the labeling induced from l_j. Furthermore, we show that $\widehat{G}$ acts geometrically on the associated CAT(0) cube complex. We begin with a technical lemma. All the walls here are $\mathbb{Z}_2$–homology cover walls.

Lemma 2.2. *Let $\widehat{\gamma}$ be a geodesic in $\widehat{\Theta}_i$ whose first and last edges, $\widehat{e}_1$ and $\widehat{e}_2$, respectively, belong to walls w_1 and w_2. Suppose there is a wall w such that one of its edges $\widehat{e}$ belongs to $\widehat{\gamma}$, and another edge $\widehat{e}'$ is* $(\mathrm{girth}\,(\Theta_i)/24)$*–close to $\widehat{e}_2$. Then there is a wall w' separating $\widehat{e}_1$ and $\widehat{e}_2$ but such that no edge of w' is* $(\mathrm{girth}\,(\Theta_i)/24)$*–close to an edge of w_1 or w_2.*

Proof. By definition w_1, w_2, and w consist of preimages of edges e_1,e_2, and e in Θ_i, respectively. Since $\widehat{e}$ and $\widehat{e}'$ are in the same wall, but distinct, it follows that their distance is at least $\mathrm{girth}\,(\Theta_i)-1$. Hence the length of $\widehat{\gamma}$ is at least $(\mathrm{girth}\,(\Theta_i)-1-\mathrm{girth}\,(\Theta_i)/24)>\mathrm{girth}\,(\Theta_i)/2$. Consider a projection γ of $\widehat{\gamma}$. It is a path (sequence of edges) of length (number of edges) at least $\mathrm{girth}\,(\Theta_i)/2$, without back-tracks. Hence, there exists an edge f in γ that is not $(\mathrm{girth}\,(\Theta_i)/24)$–close to e_1 or e_2 and that is passed by γ an odd number of times. The wall w' defined by f, that is, the wall consisting of preimages of f is the desired wall separating $\widehat{e}_1$ and $\widehat{e}_2$. □

Remark. The above lemma is needed to use [Wis, Theorem 5.40 and Remark 5.41][3] in the sequel. Wise's work concerns the so-called *cubical small cancellation*. It is a far-going generalization of the graphical small cancellation used in the current article (see e.g. [Wis, 3.s. Examples on p. 72 or Section 5.k. p. 124]). For example, in our case Wise's *hyperplanes* reduce just to edges, and there exist only *cone pieces*. Consequently, many assumptions appearing in formulations of results in [Wis] are easily satisfied in the graphical small cancellation case.

Let $\lambda\in(0,1/24]$, and let $(\Theta_1,l_1),\ldots,(\Theta_i,l_i)$ be a $C'(\lambda)$–small cancellation labelling. Then we call the system of $\mathbb{Z}_2$–homology walls on $(\widehat{\Theta}_1,\widehat{l}_1),\ldots,(\widehat{\Theta}_i,\widehat{l}_i)$ a *proper $\mathbb{Z}_2$–walling*. The following lemma is our main tool for proving residual finiteness of intermediate steps in our construction.

[3]Here and everywhere we refer to the preprint version of [Wis] dated June 19, 2017.

Lemma 2.3. *Let* $(\widehat{\Theta}_1, \widehat{l}_1), \ldots, (\widehat{\Theta}_i, \widehat{l}_i)$ *be equipped with a proper* $\mathbb{Z}_2$*–walling. Then the group* $\widehat{G} = \langle S \mid (\widehat{\Theta}_1, \widehat{l}_1), \ldots, (\widehat{\Theta}_i, \widehat{l}_i)\rangle$ *acts geometrically on a CAT(0) cubical complex.*

Proof. We use [Wis, Theorem 5.40 and Remark 5.41] for proving that $\widehat{G}$ acts properly on a CAT(0) cubical complex. We verify that the graphical presentation $\mathcal{P} := \langle S \mid (\widehat{\Theta}_1, \widehat{l}_1), \ldots, (\widehat{\Theta}_i, \widehat{l}_i)\rangle$ fulfills the conditions (1), (2), and (3) from Theorem 5.40 there.

Condition (1) of [Wis, Theorem 5.40]**.** We have to show that $\mathcal{P}$ satisfies the *generalized B(6) condition* of [Wis, Definition 5.1] and has *short innerpaths.*

[Wis, Lemma 3.67] shows that $\mathcal{P}$ has short innerpaths.

We now turn to [Wis, Definition 5.1]. Points (1), (2), and (5) in this definition are obvious. For (3) and (4) observe that pieces in $\widehat{\Theta}_j$ have length at at most $(\operatorname{girth}(\Theta_j)/24)$ and distinct edges in the same wall in $\widehat{\Theta}_j$ are at distance at least $(\operatorname{girth}(\Theta_j) - 1)$ (compare the proof of Lemma 2.2 above). Hence, (3) and (4) follow as in [Wis, Remark 5.2].

Condition (2) of [Wis, Theorem 5.40]**.** Observe that pieces in each $\widehat{\Theta}_j$ have length at most $(\operatorname{girth}(\Theta_j)/24)$. Hence the condition follows immediately from Lemma 2.2.

Condition (3) of [Wis, Theorem 5.40]**.** This condition is trivially satisfied since every $\widehat{\Theta}_j$ is finite.

Therefore, by [Wis, Theorem 5.40 and Remark 5.41] we conclude that $\widehat{G}$ acts metrically properly on the associated CAT(0) cubical complex.

Since $\widehat{G}$ is hyperbolic, the cocompactness follows e.g. from [HW14, Lemma 7.2]. □

3. The construction

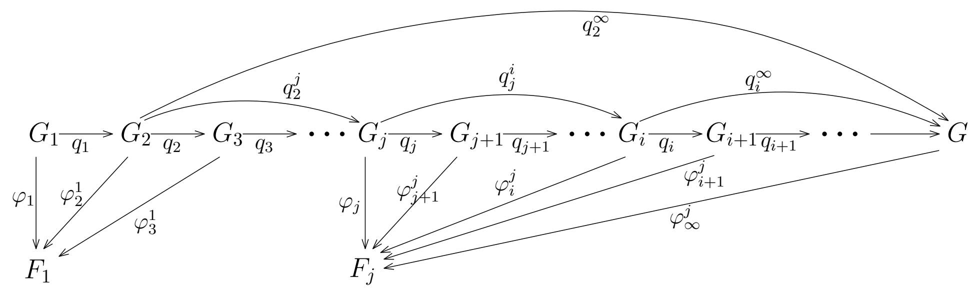
Fix $\lambda \in (0, 1/24]$ and a natural number $D \geqslant 3$. Let $(\Theta, l) = (\Theta_i, l_i)_{i=1}^{\infty}$ be a sequence of D–regular graphs with a labelling l satisfying the following stronger version of $C'(\lambda)$–small cancellation: every path in Θ_i of length greater or equal to $\lambda\operatorname{girth}(\Theta_i)$ has labelling different from any other path. Such sequences are constructed in [Osa14].

We will construct a sequence $(\widehat{\Theta}, \widehat{l}) = (\widehat{\Theta}_i, \widehat{l}_i)_{i=1}^{\infty}$ of normal covers of $(\Theta_i, l_i)_{i=1}^{\infty}$ with the induced labelling $\widehat{l}$. By G_i we will denote the finitely presented group given by the graphical presentation $G_i = \langle S \mid \widehat{\Theta}_1, \widehat{\Theta}_2, \ldots, \widehat{\Theta}_i\rangle$. The associated quotient maps will be $q_i^j \colon G_i \twoheadrightarrow G_j$, with q_i^{i+1} denoted q_i. At the same time we will construct maps to finite groups $\varphi_i^j \colon G_i \to F_j$, for $i \geqslant j$ with φ_i^i denoted φ_i. We will denote $G = \varinjlim(G_i, q_i^j)$, with $q_i^{\infty} \colon G_i \to G$, and $\varphi_{\infty}^i \colon G \to F_i$ being the induced maps.

We require that the labelled graphs $(\widehat{\Theta}_i, \widehat{l}_i)_{i=1}^{\infty}$, and the maps $\varphi_i^j \colon G_i \to F_j$ satisfy the following conditions:

(A) $(\widehat{\Theta}_1, \widehat{l}_1), (\widehat{\Theta}_2, \widehat{l}_2), \ldots$ is a $C'(\lambda)$–small cancellation labelling;
(B) $(\widehat{\Theta}_1, \widehat{l}_1), (\widehat{\Theta}_2, \widehat{l}_2), \ldots, (\widehat{\Theta}_i, \widehat{l}_i)$ admits a proper $\mathbb{Z}_2$–walling for every i;
(C) $\varphi_j(g) \neq 1$, for every j and every $g \in B_j(1, \mathrm{Cay}(G_j, S)) \setminus \{1\}$;
(D) $\varphi_l^j \circ q_k^l = \varphi_k^j$, for all $j \leqslant k \leqslant l$.

In particular, the diagram below is commutative.



We construct the graphs $\widehat{\Theta}_i$, the finite groups F_i, and the maps φ_i^j $(j \leqslant i)$ inductively, with respect to i.

3.1. **Induction basis.** Let $\widehat{\Theta}_1$ be the $\mathbb{Z}_2$–homology cover of Θ_1 (such a cover corresponds to a characteristic subgroup of $\pi_1(\Theta_1)$). Then the following conditions are satisfied:

(A_1) $(\widehat{\Theta}_1, \widehat{l}_1), (\Theta_2, l_2), (\Theta_3, l_3), \ldots$ is a $C'(\lambda)$–small cancellation labelling;
(B_1) $(\widehat{\Theta}_1, \widehat{l}_1)$ admits a proper $\mathbb{Z}_2$–walling $\mathcal{W}^1$.

Then $G_1 := \langle S \mid \widehat{\Theta}_1 \rangle$ is hyperbolic and, by Lemma 2.3, acts geometrically on a CAT(0) cubical complex. Therefore, by results of Wise [Wis] and Agol [Ago13] it is residually finite. Let $\varphi_1 \colon G_1 \to F_1$ be a map into a finite group F_1 such that:

(C_1) $\varphi_1(g) \neq 1$ for all $g \in B_1(1, \mathrm{Cay}(G_1, S)) \setminus \{1\}$.

3.2. **Inductive step.** Assume that the graphs $\widehat{\Theta}_1, \widehat{\Theta}_2, \ldots, \widehat{\Theta}_i$, the finite groups $F_1, F_2, \ldots, F_i$, and the maps $\varphi_j^k \colon G_j \to F_k$, for $k \leqslant j \leqslant i$ with the following properties have been constructed.

(A_i) $(\widehat{\Theta}_1, \widehat{l}_1), \ldots, (\widehat{\Theta}_i, \widehat{l}_i), (\Theta_{i+1}, l_{i+1}), (\Theta_{i+2}, l_{i+2}), \ldots$ is a $C'(\lambda)$–small cancellation labelling;
(B_i) $(\widehat{\Theta}_1, \widehat{l}_1), \ldots, (\widehat{\Theta}_i, \widehat{l}_i)$ admits a proper $\mathbb{Z}_2$–walling;
(C_i) $\varphi_j(g) \neq 1$, for every $j \leqslant i$ and every $g \in B_j(1, \mathrm{Cay}(G_j, S)) \setminus \{1\}$;
(D_i) $\varphi_l^j \circ q_k^l = \varphi_k^j$, for all $j \leqslant k \leqslant l \leqslant i$ (that is, the part of the above diagram with all indexes at most i is commutative).

Note that the condition (D_1) is satisfied trivially.

Let H_i be a subgroup of G_i generated by (the images by $F(S) \twoheadrightarrow G_i$ of) all the words read along cycles in (Θ_{i+1}, l_{i+1}). The subgroup $K_i := \bigcap_{j \leqslant i} \ker(\varphi_i^j) \lhd G_i$ is of finite index. Therefore $H_i \cap K_i < H_i$ is of finite index and we can find a finite normal cover $\overline{\Theta}_{i+1}$ of Θ_{i+1} such that the normal closure in G_i of the subgroup generated by words read along $(\overline{\Theta}_{i+1}, \bar{l}_{i+1})$ is contained in K_i, where $\bar{l}_{i+1}$ is the labelling of $\overline{\Theta}_{i+1}$ induced by l_{i+1} via the covering map. Labelled paths of length greater or equal $\lambda\mathrm{girth}(\overline{\Theta}_{i+1})$ in $(\overline{\Theta}_{i+1}, \bar{l}_{i+1})$ do not appear in $(\widehat{\Theta}_j, \widehat{l}_j)$ for $j \leqslant i$, neither in (Θ_j, l_j) for $j \geqslant i+2$. Any two such paths in $(\overline{\Theta}_{i+1}, \bar{l}_{i+1})$ differ by a covering automorphism. Therefore, $(\widehat{\Theta}_1, \widehat{l}_1), \dots, (\widehat{\Theta}_i, \widehat{l}_i), (\overline{\Theta}_{i+1}, \bar{l}_{i+1}), (\Theta_{i+2}, l_{i+2}), (\Theta_{i+3}, l_{i+3}), \dots$ is a $C'(\lambda)$–small cancellation labelling. Let $\widehat{\Theta}_{i+1}$ be the $\mathbb{Z}_2$–homology cover of $\overline{\Theta}_{i+1}$. Then the following properties are satisfied:

(A_{i+1}) $(\widehat{\Theta}_1, \widehat{l}_1), \dots, (\widehat{\Theta}_i, \widehat{l}_i), (\widehat{\Theta}_{i+1}, \widehat{l}_{i+1}), (\Theta_{i+2}, l_{i+2}), (\Theta_{i+3}, l_{i+3}), \dots$ is a $C'(\lambda)$–small cancellation labelling;

(B_{i+1}) $(\widehat{\Theta}_1, \widehat{l}_1), \dots, (\widehat{\Theta}_i, \widehat{l}_i), (\widehat{\Theta}_{i+1}, \widehat{l}_{i+1})$ admits a proper $\mathbb{Z}_2$–walling,

where $q_i \colon G_i \twoheadrightarrow G_{i+1} := \langle S \mid (\widehat{\Theta}_1, \widehat{l}_1), \dots, (\widehat{\Theta}_{i+1}, \widehat{l}_{i+1})\rangle$ is the quotient map. Observe that we have $\ker(q_i) < K_i$.

The group $G_{i+1} = \langle S \mid \widehat{\Theta}_1, \widehat{\Theta}_2, \dots, \widehat{\Theta}_{i+1}\rangle$ is hyperbolic and, by Lemma 2.3, acts geometrically on a CAT(0) cubical complex. Hence, by results of Wise [Wis] and Agol [Ago13], it is residually finite. Therefore, we find a map $\varphi_{i+1} \colon G_{i+1} \to F_{i+1}$ into a finite group F_{i+1} such that $\varphi_{i+1}(g) \neq 1$ for all $g \in B_{i+1}(1, \mathrm{Cay}(G_{i+1}, S))$. Hence, by (C_i), we have

(C_{i+1}) $\varphi_j(g) \neq 1$, for every $j \leqslant i+1$ and every $g \in B_j(1, \mathrm{Cay}(G_j, S)) \setminus \{1\}$.

For $j \leqslant i$ we define $\varphi^j_{i+1} \colon G_{i+1} \to F_j$ as $\varphi^j_{i+1}(q^{i+1}_j(g)) = \varphi_j(g)$. This is a well defined homomorphism: If $q^{i+1}_j(g) = q^{i+1}_j(g')$ then $q^i_j(gg'^{-1}) \in \ker(q_i)$, and hence

$$\begin{aligned}\varphi^j_{i+1}(q^{i+1}_j(g))(\varphi^j_{i+1}(q^{i+1}_j(g')))^{-1} &= \varphi_j(g)\varphi_j(g')^{-1} = \\ &= \varphi_j(gg'^{-1}) = \varphi^j_i(q^i_j(gg'^{-1})) = 1,\end{aligned}$$

by $\ker(q_i) < K_i$. By (D_i) and the definition of φ^j_{i+1} we have:

(D_{i+1}) $\varphi^j_l \circ q^l_k = \varphi^j_k$, for all $j \leqslant k \leqslant l \leqslant i+1$ (that is, the part of the above diagram with all indexes at most $i+1$ is commutative).

This finishes the inductive step.

3.3. **Proof of Main Theorem.** The presentation $\langle S \mid \widehat{\Theta}_1, \widehat{\Theta}_2, \dots\rangle$ is a graphical $C'(\lambda)$–small cancellation presentation, by (A). Thus, the Cayley graph $\mathrm{Cay}(G, S)$ contains isometrically embedded copies of all the graphs $\widehat{\Theta}_i$, by Lemma 2.1. That is, $\mathrm{Cay}(G, S)$ contains a sequence of D–regular graphs of growing girth, and hence G is non-exact, by [Wil11].

We show now that G is residually finite. Take a non trivial element $g \in G$. Let i be such an integer that $g \in B_i(1, \mathrm{Cay}(G, S))$. Then there exists $g' \in G_i$ such that $q^\infty_i(g') = g$, and $g' \in B_i(1, \mathrm{Cay}(G_i, S))$. For the homomorphism $\varphi^i_\infty \colon G \to F_i$ into the finite group F_i we have $\varphi^i_\infty(g) = \varphi^i_\infty \circ q^\infty_i(g') = \varphi_i(g') \neq 1$, by (D) and (C). This shows that G is residually finite.

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