LARGE-TYPE ARTIN GROUPS ARE SYSTOLIC

JINGYIN HUANG AND DAMIAN OSAJDA

Abstract. We prove that Artin groups from a class containing all large-type Artin groups are systolic. This provides a concise yet precise description of their geometry. Immediate consequences are new results concerning large-type Artin groups: biautomaticity; existence of EZ–boundaries; the Novikov conjecture; descriptions of finitely presented subgroups, of virtually solvable subgroups, and of centralizers for infinite order elements; the Burghelea conjecture and the Bass conjecture; existence of low-dimensional models for classifying spaces for some families of subgroups.

Contents

1. Introduction

1.1. **Background and the Main Theorem.** Let Γ be a finite simple graph with its vertex set denoted by V. Each edge of Γ is labeled by a positive integer at least 2. The *Artin group with defining graph* Γ, denoted A_Γ, is the group whose generators are in 1-1 correspondence with elements

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in V, and whose relators are of the form $\underbrace{s_i s_j s_i \cdots}_{m_{ij}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij}}$ whenever s_i and s_j span an edge and m_{ij} is the label of this edge.

It is an open question whether all Artin groups are non-positively curved in the sense that they act geometrically (i.e. properly and cocompactly by isometries) on non-positively curved spaces. One of the earlier motivations for this question comes from the seminal work of Charney and Davis [CD95], where they put a $CAT(0)$ metric on the modified Deligne complex for Artin groups of type FC as well as 2–dimensional Artin groups, and deduce the $K(\pi, 1)$ conjecture for these Artin groups. Though the action of an Artin group on its modified Deligne complex is not proper, this naturally leads to the question whether one can directly construct $CAT(0)$ spaces on which Artin groups act geometrically. This question has its own interests independent of the $K(\pi, 1)$ conjecture, since one can deduce many finer group theoretic and geometric consequences given the existence of such action. Here is a summary of Artin groups which are known to acts geometrically on $CAT(0)$ spaces:

(1) certain classes of 2–dimensional Artin groups [BC02, BM00];
(2) Artin groups of finite type with three generators [Bra00];
(3) 3–dimensional Artin groups of type FC [Bel05];
(4) spherical Artin groups of type A_4 and B_4 [BM10];
(5) 6–strand braid group [HKS16].

In this paper we focus on Artin group of *large type*, i.e. the value of each label of its defining graph is at least 3. Large-type Artin groups were first introduced and studied by Appel-Schupp [AS83]. It is still unknown whether all Artin groups of large type are $CAT(0)$, though some partial results are obtained in [BM00]. Moreover, most of Artin groups of large type can not act geometrically on $CAT(0)$ cube complexes, even up to passing to finite index subgroups [HJP16, Hae15].

Instead of metric non-positive curvature, we turn the attention towards a combinatorial counterpart. Examples of combinatorially non-positively curved spaces and groups are: small cancellation, CAT(0) cubical, and systolic ones. There are many advantages of the combinatorial approach. For example, biautomaticity is proved in various combinatorial settings, while it is still an open problem (with a plausible negative answer) for CAT(0) groups (see the discussion in Subsection 1.3 below).

A suitable setting for our approach are Artin groups in the following class. An Artin group is of *almost large type* if in the defining graph Γ there is no triangle with at least one edge labeled by 2 and no square with at least three edges labeled by 2. Clearly, large-type Artin groups are of almost large type. Right-angled Artin groups with (triangle and square)-free Γ are examples of almost large-type Artin groups that are not of large type. Our main result is the following (see Theorem 5.5 in Section 5).

Main Theorem. *Every Artin group of almost large type is systolic.*

Systolic groups are groups acting geometrically on *systolic complexes*. The latter are simply connected simplicial complexes satisfying some local combinatorial conditions implying many non-positive-curvature-like features (see Subsection 1.3, and Section 2 below for some details). Systolic complexes were first introduced by Chepoi [Che00] under the name *bridged complexes*. However, *bridged graphs*, that is, one-skeleta of systolic complexes were studied before in the frame of metric graph theory. They were introduced by Soltan-Chepoi [SC83] and Farber-Jamison [FJ87]. Later, independently systolic complexes were discovered by Januszkiewicz-Świątkowski [JŚ06] and Haglund [Hag03] in the context of geometric group theory. The combinatorial approach to non-positive curvature allowed them to construct groups and complexes with interesting properties. In particular, the first examples of high-dimensional hyperbolic Coxeter groups are systolic. The theory of systolic complexes and groups have been developed further providing new applications (see e.g. [JŚ07, Wis03, Świ06, OP16] and references therein).

Let us note that there have been other very successful approaches to Artin groups using combinatorial versions of non-positive curvature. Those include: using small cancellation [AS83, Pri86, Pei96], CAT(0) cube complexes (the case of right-angled Artin groups), and Bestvina's approach to Artin groups of finite type [Bes99].

Our complex is a thickening of the presentation complex of the Artin group, and the disk diagrams in our complex is very simple ([Els09a, Lemma 4.2]), so one can study disk diagrams in the presentation complex by using information on the disk diagrams in our systolic complexes. Hence we believe that our complexes can be used to prove finer properties of Artin groups of large type, beyond the ones presented in Subsection 1.3. Also we expect our approach can be adapted for more general classes of Artin groups.

1.2. **Comments on the proof.** First we consider the special case when A_Γ is an Artin group such that the label of each edge in the defining graph is 3. Let X^*_Γ be the universal cover of the presentation complex of A_Γ. Then each 2–cell of X^*_Γ is a hexagon. We put a new vertex (which is called an *interior vertex*) in the interior of each 2–cell and subdivide each 2–cell into 6 triangles around this new vertex. One naturally want to metrize such complex by declaring each triangle is a Euclidean equilateral triangle. However, such metric is not $CAT(0)$, since there exists a pair of 2–cells such that their intersection contains two edges, and this leads to a positively curved point v indicated in Figure 1. We think the configuration around v as a "corner" inside a 3–dimensional Euclidean space, and would like to "fill in" the corner to kill the positive curvature. Combinatorially, we add an edge between the interior vertices of every pair of 2–cells whose intersection contains $\geqslant 2$ edges, and take the flag complex. Though the new complex is still not $CAT(0)$, we find it has enough non-positive curvature to work with (its geometry is actually quite similar to a $CAT(0)$ complex made of equilateral triangles) and realize that the a suitable language to realize this intuition is the theory of systolic complex.

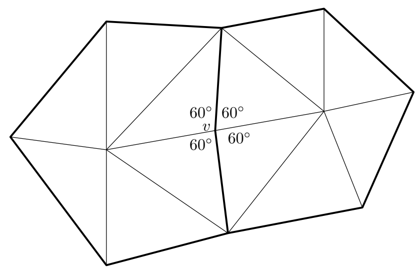

FIGURE 1.

Now we consider more general A_Γ of large type. Again we start with the universal cover of the presentation complex and subdivide each 2–cell (which are $2n$–gons for $n \geqslant 3$) into triangles. Among the many possibilities, we choose the one as in Figure 6 based on the following considerations.

(1) Each triangle is a Euclidean equilateral triangle.
(2) Each 2–cell with the subdivision is flat.

Though (1) gives a strong restriction, the benefits are that the disk diagrams are much simpler and the biautomaticity of the group is easier to obtain (biautomaticity is much harder if the complex has cells of different isometry types, see e.g. [LM12]). The reason for (2) is that Artin groups have a lot of flats inside, if one create some negative curvature at one place, then the complex will be forced to have positive curvature at some other places.

Again a pair of 2–cells with a large piece between them lead to points of positive curvature. We add edges between the interior vertices of these 2–cells to create a "prism-like" configuration as in Figure 7, which resolves these positive curvature points.

The bulk of the paper (Section 4) will actually be devoted to the studying of the above complexes for the dihedral Artin group (i.e. the defining graph Γ of A_Γ is an edge), since these are our building blocks for studying more general Artin groups. A more technical point is that the "prism-like" configuration in the previous paragraph need to be designed carefully so that the complex is still systolic after we glue these building blocks together. The main points of Section 4 are

(1) these building blocks are systolic (Proposition 4.5, Proposition 4.19);
(2) there is no obstruction to systolicity if we glue these blocks together (Lemma 4.16, Remark 4.18).

The process of gluing the building blocks is explained in Section 5.

We end this subsection by noting that dihedral Artin groups are already very well-understood (they are virtually free times $\mathbb{Z}$), they are known to acts on various complexes with features of non-positive curvature [Bes99, BM00, BC02, HJP16, Hae15], and it is known that one can use them as building blocks to obtain $CAT(0)$ complexes for certain family of Artin groups. However, these building blocks are not good enough for constructing $CAT(0)$ complexes for all Artin groups of large type (even the case when the defining graph Γ is a complete graph with $\geqslant 4$ vertices is not known).

1.3. **Immediate consequences of the Main Theorem.** We gather immediate consequences of being systolic for almost large-type Artin groups in the following corollary. To the best of our knowledge all the results listed below are new. Afterwards we provide some details, in particular, commenting on results obtained earlier.

Corollary. *Let G be an Artin group of almost large type. Then:*

(1) G is biautomatic;
(2) G has a boundary in the sense of [OP09]*, which captures the large-scale geometry of G. In particular, G admits an EZ–structure, and hence the Novikov conjecture holds for G;*
(3) finitely presented subgroups of G are systolic, hence they are biautomatic, have solvable word problem, solvable conjugacy problem and all the other properties listed here;
(4) virtually solvable subgroups of G are either virtually cyclic or virtually $\mathbb{Z}^2$;
(5) the Burghelea conjecture and, consequently, the Bass conjecture hold for G;
(6) the centralizer of an infinite order element of G is commensurable with $F_n \times \mathbb{Z}$ or $\mathbb{Z}$;
(7) G admits a finitely dimensional model for $E_{\mathcal{VAB}}G$, the classifying space for the family of virtually abelian subgroups of G.

(1) Biautomaticicty for systolic groups has been established by Januszkiewicz-Świątkowski [JŚ06, Świ06]. Biautomaticity of large-type Artin groups have been a well known open problem. Partial results were obtained by: Pride and Gersten (triangle-free Artin groups) [Pri86, GS90], Charney [Cha92] (finite type), Peifer [Pei96] (extra-large type, i.e., $m_{ij} \geqslant 4$, for $i \neq j$), Brady-McCammond [BM00] (three-generator large-type Artin groups and some generalizations), Holt-Rees [HR12, HR13] (sufficiently large-type Artin groups are shortlex automatic with respect to the standard generating set).

(2) Let G act geometrically on a systolic complex X. In [OP09] a compactification of X by a Z–set $\overline{X} = X \cup \partial X$ has been constructed. This defines the so-called EZ–structure for G [FL05], and ∂X becomes a sort of a boundary of G. Such structures are known only for few classes of groups, most notably, for Gromov hyperbolic groups and CAT(0) groups. Closer relations between algebraic properties of G and the dynamics of its action on ∂X are exhibited in [Pry17]. Existence of an EZ–structure implies, in particular, the Novikov conjecture [FL05]. Ciobanu-Holt-Rees [CHR16] established the (stronger) Baum-Connes conjecture for a subclass of large-type Artin groups, including

Artin groups of extra-large type. They did it by proving the rapid decay property for such groups.

(3) Using towers of complexes Wise [Wis03] showed that finitely presented subgroups of torsion-free systolic groups are systolic. For all systolic groups the result has been shown in [HMP14, Zad14].

(4) By Theorem 2.2 in Section 2 below, virtually solvable subgroups of systolic groups are either virtually cyclic or virtually $\mathbb{Z}^2$. Bestvina [Bes99] showed that solvable subgroups of Artin groups of finite type are abelian. He used there another combinatorial version of non-positive curvature. It is an open question whether virtually solvable subgroups of biautomatic groups are finitely generated virtually abelian groups.

(5) The Burghelea conjecture concerns the periodic cyclic homology of complex group rings; see [EM16] and references therein for details. It is known to be false in general, but has been established for, among others, hyperbolic groups. Engel-Marcinkowski [EM16] showed that the Burghelea conjecture holds for systolic groups. The Burghelea conjecture implies the strong Bass conjecture, that in turns implies the classical Bass conjecture.

(6) Elsner [Els09b] showed that infinite order elements of systolic groups admit a kind of axis, similarly to the hyperbolic and CAT(0) cases. Verifying a conjecture by Wise [Wis03], it is shown in [OP16] that centralizers of infinite order elements in systolic groups are commensurable with a product of $\mathbb{Z}$ and a finitely generated free group (possibly trivial or $\mathbb{Z}$). In the special case of 2–dimensional Artin groups of hyperbolic type, this result was obtained by Crisp [Cri05]. In fact, Crisp computes explicitly the centralizer of a given element up to commensurability.

(7) By a result of Degrijse [Deg17] two-dimensional Artin groups admit finite dimensional models for the family of virtually cyclic subgroups. Similar result has been proved in [OP16] for systolic groups, where it is also shown that systolic groups admit finitely dimensional models for classifying spaces for the family of virtually abelian subgroups.

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2. Systolic complexes and systolic groups

All graphs consider in this paper contain no edge-loops nor multiple edges. For vertices $w, v_1, v_1, \ldots$ of a graph, we write $w \sim v_1, v_2, \ldots$ when w is *adjacent* to each v_i, that is, there is an edge containing w and v_i. If w is not adjacent to any of v_i then we write $w \nsim v_1, v_2, \ldots$. A simplicial complex X is *flag* if every set of pairwise adjacent vertices of X spans a simplex in X. In other words, a flag simplicial complex X is determined by its 1–*skeleton* $X^{(1)}$, being a simplicial graph. A subcomplex Y of a simplicial complex X is *full* if any set of vertices of Y spanning a simplex in X spans a simplex in Y as well. A subgraph of a graph is *full* if it is a full subcomplex. The *link* $\mathrm{lk}(v, \Gamma)$ of a vertex v in a graph Γ is the full subgraph of Γ spanned by vertices adjacent to v. A graph is 6–*large* if there are no simple cycles of length 4 and 5 being full subgraphs.

Definition 2.1. A flag simplicial complex is *systolic* if it is connected, simply connected, and links of vertices in its 1–skeleton are 6–large.

In particular, any 2–dimensional piecewise Euclidean $CAT(0)$ complex whose 2–cells are equilateral triangles satisfies the above definition. In general, systolic complexes are not 2–dimensional and they are not necessarily $CAT(0)$ with the most natural metric – the piecewise Euclidean metric with all edges having length 1. Nevertheless, systolic complexes possess many features typical for nonpositively curved spaces. One of them is a version of the Cartan-Hadamard theorem stating that finite dimensional systolic complexes are contractible. Another important feature is that for any embedded simplicial loop in a systolic complex there is a systolic, and hence CAT(0), disc diagram filling it. See e.g. [Che00, Hag03, Wis03, JŚ06, JŚ07, Els09a, Els09b, OP09, Zad14, OP16] for details and further information.

Groups acting geometrically on systolic complexes are called *systolic*. Few consequences of being a systolic group are listed in Corollary above. The following theorem is a consequence of known results on systolic complexes but has been not stated in the literature.[1]

Theorem 2.2 (Solvable Subgroup Theorem)**.** *Solvable subgroups of systolic groups are either virtually cyclic or virtually* $\mathbb{Z}^2$*.*

Proof. Let G be a systolic group. By [OP16, Proposition 5.14] virtually abelian subgroups of G are finitely generated. Hence, the following argument by Gersten and Short [GS91, page 154] shows that solvable subgroups of G are virtually abelian: By a theorem of Mal'cev [Seg83, Theorem 2 on page 25] such subgroups are polycyclic, and thus, by [GS91, Theorem 6.15] they are virtually abelian. By [JŚ07, Corollary 6.5] finitely generated virtually abelian subgroups of G are either virtually cyclic or virtually $\mathbb{Z}^2$. □

3. The complexes for 2–generated groups

3.1. **Precells in the presentation complex.** Let DA_n be the 2–generator Artin group presented by $\langle a, b \mid \underbrace{aba\cdots}_{n} = \underbrace{bab\cdots}_{n} \rangle$. We assume $n \geqslant 2$. Let

[1]It is mistakenly claimed in [JŚ06, JŚ07] that a form of Solvable Subgroup Theorem follows immediately from biautomaticity. In fact it is an open question whether virtually solvable subgroups of biautomatic groups are virtually abelian.

DA_n^+ be the associated Artin monoid presented by the same generators and relations as DA_n. The following is immediate from definition.

Lemma 3.1. *Let w_1 and w_2 be two words in the free monoid generated by a and b. Suppose $w_1 = w_2$ in DA_n^+. Then*

(1) w_1 and w_2 have the same length;
(2) if w_1 has length $\leqslant n$, then either w_1 and w_2 are the same word, or w_1 equals to one of $\underbrace{aba\cdots}_{n}$ and $\underbrace{bab\cdots}_{n}$, and w_2 equals to another.

The following is a special case of [Del72, Theorem 4.14].

Theorem 3.2. *The natural map $DA_n^+ \to DA_n$ is injective.*

Let P_n be the standard presentation complex of DA_n. Namely the 1–skeleton of P_n is the wedge of two oriented circles, one labeled a and one labeled b. Then we attach the boundary of a closed 2–cell C to the 1–skeleton with respect to the relator of DA_n. Let $C \to P_n$ be the attaching map.

Let X^* be the universal cover of the standard presentation complex of DA_n. Edges of X^* are endowed with induced orientations and labellings from P_n. The following is a direct consequence of Theorem 3.2 and Lemma 3.1.

Corollary 3.3. *Any lift of the map $C \to P_n$ to $C \to X^*$ is an embedding.*

These embedded disks in X^* are called *precells*. The following is a picture of a precell Π^*. X^* is a union of copies of Π^*'s.

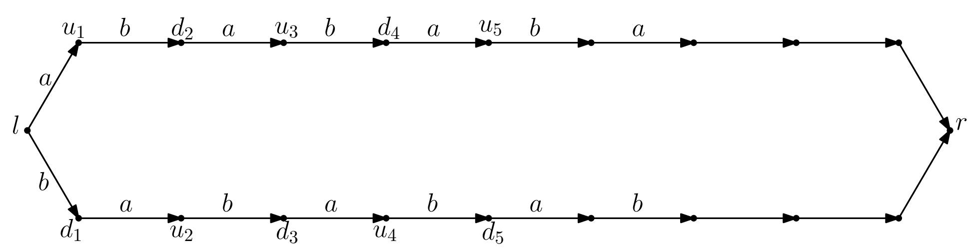

FIGURE 2. Precell Π^*

We label the vertices of Π^* as in Figure 2. The vertices l and r are called the *left tip* and *right tip* of Π^*. The boundary $\partial\Pi^*$ is made of two paths. The one starting at l, going along $\underbrace{aba\cdots}_{n}$ (resp. $\underbrace{bab\cdots}_{n}$), and ending at r is called the *upper half* (resp. *lower half*) of $\partial\Pi^*$. The orientation of edges inside one half is consistent, thus each half has an orientation.

Corollary 3.4. *Let Π_1^* and Π_2^* be two different precells in X^*. Then*

(1) either $\Pi_1^* \cap \Pi_2^* = \emptyset$, or $\Pi_1^* \cap \Pi_2^*$ is connected;

(2) *if* $\Pi_1^* \cap \Pi_2^* \neq \emptyset$, $\Pi_1^* \cap \Pi_2^*$ *is properly contained in the upper half or in the lower half of* Π_1^* *(and* Π_2^**);*
(3) *if* $\Pi_1^* \cap \Pi_2^*$ *contains at least one edge, then one end point of* $\Pi_1^* \cap \Pi_2^*$ *is a tip of* Π_1^**, and another end point of* $\Pi_1^* \cap \Pi_2^*$ *is a tip of* Π_2^**, moreover, among these two tips, one is a left tip and one is a right tip.*

Proof. First we look at the case when $\Pi_1^* \cap \Pi_2^*$ is discrete. Suppose by contradiction that there are two distinct vertices v_1, v_2 in $\Pi_1^* \cap \Pi_2^*$.

If v_1 and v_2 are in the same half of $\partial\Pi_1^*$ and $\partial\Pi_2^*$ then, for $i = 1, 2$, let l_i be the segment joining v_1 and v_2 inside a half of $\partial\Pi_i^*$. If both l_1 and l_2 are oriented from v_1 to v_2, then they give two words in the free monoid that are equal in DA_n. By Theorem 3.2 and Lemma 3.1, these two words have to be in the two situations indicated in Lemma 3.1 (2), however, both situations can be ruled out easily. If l_1 is oriented from v_1 to v_2 and l_2 is oriented from v_2 and v_1, then the concatenation of l_1 and l_2 gives a nontrivial word in the free monoid, which is also nontrivial in DA_n by Theorem 3.2. This contradicts the fact that the concatenation is a loop. Other cases of orientations of l_1 and l_2 can be dealt in a similar way.

If v_1 and v_2 are in different halves of $\partial\Pi_1^*$ and $\partial\Pi_2^*$ then we assume without loss of generality that orientations of halves of Π_1^* and Π_2^* are as in Figure 3 (l_i and r_i are the tips of $\partial\Pi_i$). We also assume without loss of generality that the summation of the length of the path $\overline{l_2v_1r_1}$ and the path $\overline{l_2v_2r_1}$ is $\leqslant 2n$. Let w_1 (resp. w_2) be the word in the free monoid given by $\overline{l_2v_1r_1}$ (resp. $\overline{l_2v_2r_1}$). Then at least one of w_1 and w_2 has length $\leqslant n$. Again, w_1 and w_2 are in the two situations of Lemma 3.1 (2), and both situations can be ruled out easily.

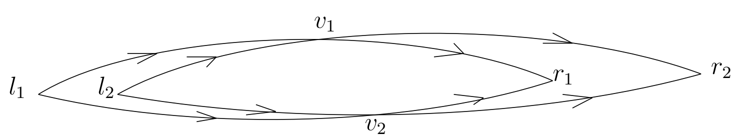

FIGURE 3.

The case where v_1 and v_2 are in different halves of one of $\partial\Pi_1^*$ and $\partial\Pi_2^*$, and are in the same half of the other can be handled in a similar way.

Now we assume $\Pi_1^* \cap \Pi_2^*$ contains an edge f. Let P be the connected component of $\Pi_1^* \cap \Pi_2^*$ that contains this edge. By looking at the labels of edges around $\partial\Pi_1^*$ and $\partial\Pi_2^*$, we deduce that either $P = \partial\Pi_1^* = \partial\Pi_2^*$, or P satisfies conditions (2) and (3) in Corollary 3.4. However, the first case is impossible since that will imply $\Pi_1^* = \Pi_2^*$. Let C be the space obtained by gluing Π_1^* and Π_2^* along P. Then there is a natural map $C \to X^*$. It suffices to show this map is an embedding. We assume without loss of generality that Π_1^* and Π_2^* are positioned as in Figure 4, here l_i and r_i are the tips of Π_i^*, and w_1 (resp. w_2) is an interior vertex of the upper half of Π_1^* (resp. lower half of Π_2^*).

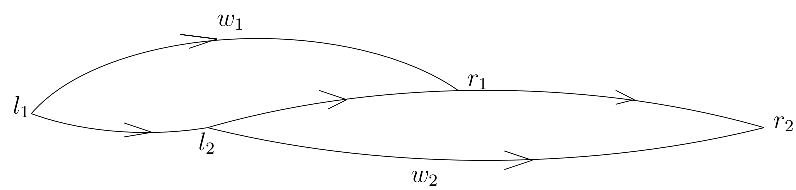

FIGURE 4.

We already know that $\partial\Pi_1^*$ and $\partial\Pi_2^*$ are embedded by Corollary 3.3, and the paths $\overline{l_1l_2r_1r_2}$, $\overline{l_1w_1r_1r_2}$ and $\overline{l_1l_2w_2r_2}$ are embedded because they correspond to words in the free monoid. If w_1 and w_2 are identified in X^*, then $\overline{l_1w_1}$ and $\overline{l_1l_2w_2}$ give two words in the free monoid which are equal in DA_n. By Theorem 3.2 and Lemma 3.1 (2), these two words are identical (note that the length of $\overline{l_1w_1}$ is $< n$), which is a contradiction. $\square$

Corollary 3.5. *Suppose there are three precells Π_1^*, Π_2^* and Π_3^* such that $\Pi_1^* \cap \Pi_2^*$ is a nontrivial path P_1 in the upper half of Π_2^*, and $\Pi_3^* \cap \Pi_2^*$ is a nontrivial path P_3 in the lower half of Π_2^*. Then $\Pi_1^* \cap \Pi_3^*$ is either empty or one point.*

Proof. We glue Π_2^* and Π_1^* along P_1, and glue Π_2^* and Π_3^* along P_3 to obtain space C. There is a natural map $C \to X^*$. By Corollary 3.3 (2) and (3), there are four possibilities of the space C, we only consider the two cases in Figure 5, the other cases are similar.

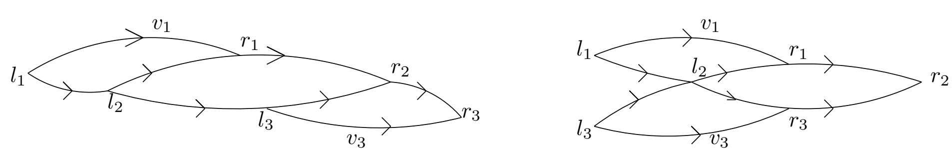

FIGURE 5.

Suppose we are in the case as in Figure 5, on the left. We claim $\partial\Pi_1^* \cap \partial\Pi_3^* = \emptyset$. Since $\Pi_2^* \cup_{P_3} \Pi_3^*$ is embedded in X^* by Corollary 3.3, the path $\overline{l_2r_1}$ is disjoint from $\partial\Pi_3^*$. Similarly, $\overline{l_3r_2}$ is disjoint from $\partial\Pi_1^*$. Moreover, $\overline{l_1l_2}$ is disjoint from $\overline{r_2r_3}$ and $\overline{l_3v_3r_3}$, since $\overline{l_1l_2l_3v_3r_3}$ and $\overline{l_1l_2r_1r_2r_3}$ give words in the free monoid. It remains to show $\overline{l_1v_1r_1} \cap \overline{l_3v_3r_3} = \emptyset$. If this is not true, we assume without loss of generality that v_1 and v_3 are identified. Then $\overline{l_1v_1}$ and $\overline{l_1l_2l_3v_3}$ give two words in the free monoid which are equal in DA_n. Since $\overline{l_1v_1}$ has length $< n$, these words are identical by Theorem 3.2 and Lemma 3.1, which yields a contradiction.

Suppose we are in the case of Figure 5 right. By looking at label of edges around vertex l_2, we know l_2 is an isolated vertex in $\partial\Pi_1^* \cap \partial\Pi_3^*$. Thus $\partial\Pi_1^* \cap \partial\Pi_3^* = l_2$ by Corollary 3.3. $\square$

3.2. **Subdividing and systolizing the presentation complex.** We subdivide each precell in X^* as in Figure 6 to obtain a simplicial complex $X^{\triangle}$.

In particular, in the case $n = 2$ we do not add any new vertices only an edge $\overline{lr}$. A *cell* of $X^\triangle$ is defined to be a subdivided precell, and we use Π for a cell. The original vertices of X^* in $X^\triangle$ are called the *real vertices*, and the new vertices of $X^\triangle$ after subdivision are called *interior vertices*. Interior vertices in a cell Π are denoted $c_1, c_2, \dots, c_{n-2}$ as in Figure 6. (Here and further we use the convention that the real vertices are drawn as solid points and the interior vertices as circles.)

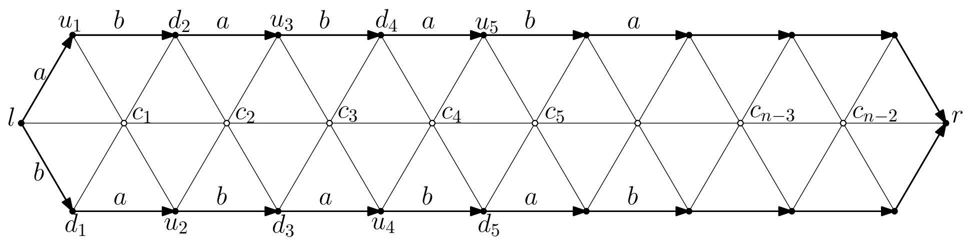

FIGURE 6. Cell Π

If $n = 2$ then $X^\triangle$ is systolic, and we define X to be $X^\triangle$. Form now on we assume $n \geqslant 3$. Note that then $X^\triangle$ is not systolic. Suppose Π_1 and Π_2 are two cells such that $\Pi_1 \cap \Pi_2$ is a path made of $\geqslant 2$ edges. Then they create 4–cycles or 5–cycles in $X^\triangle$ without diagonals, see the thick cycles in Figure 7. In what follows we modify $X^\triangle$ to obtain a systolic complex X. A rough idea is to add appropriate diagonals to these 4–cycles or 5–cycles. We only add new edges between interior vertices.

Example 3.6. For the 5–cycle $lc_{n-6}c_{n-5}d_2c_1$ in the last picture of Figure 7, there is a unique way add diagonals between interior vertices, namely we add edges $\overline{c_{n-6}c_1}$ and $\overline{c_{n-5}c_1}$. Similarly, we add edges $\overline{c_{n-2}c_4}$ and $\overline{c_{n-2}c_5}$. However, adding these edges create new 5–cycles in the space (e.g. $c_1c_{n-5}c_{n-4}u_3c_2$). One can either connect c_1 and c_{n-4}, or c_{n-5} and c_2. We choose the later and add new edges in a zigzag pattern indicated in Figure 7 (see the dashed edges). After adding these edges, we fill in higher dimensional simplexes to obtain a string of 3–dimensional simplexes, starting from $c_{n-6}u_1c_1l$, and ending at $c_{n-2}d_6c_5u_5$.

Now we give a precise description of the new edges added to $X^\triangle$. Let Λ be the collection of all unordered pair of cells of $X^\triangle$ such that their intersection contains at least two edges. Then DA_n acts on Λ. This action is free. To see this, pick a pair $(\Pi_1, \Pi_2) \in \Lambda$ and suppose $\alpha \in DA_n$ stabilize it. In particular, α maps $\partial\Pi_1 \cap \partial\Pi_2$ to itself. However, $\partial\Pi_1 \cap \partial\Pi_2$ is an interval by Corollary 3.4. Thus α fixes a point in $\partial\Pi_1 \cap \partial\Pi_2$. Since $DA_n \curvearrowright X^\triangle$ is free, α is identity.

Pick a base cell Π in $X^\triangle$ such that $l \in \Pi$ coincides with the identity element of DA_n. Let Λ_0 be the collection of pairs of the form $(\Pi, u_i^{-1}\Pi)$, $(\Pi, d_i^{-1}\Pi)$ for $i = 1, \dots, n-2$ (here each vertex of Π can be identified as an

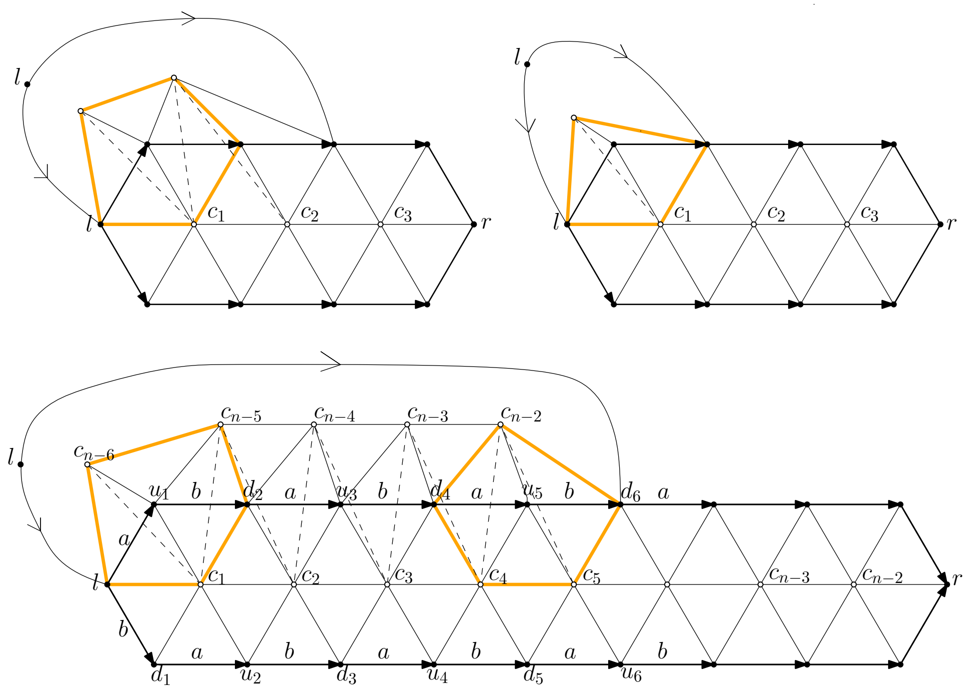

FIGURE 7.

element of DA_n, and $u_i^{-1}\Pi$ means the image of Π under the action of u_i^{-1}). Note that

(1) $\Lambda_0 \subset \Lambda$;
(2) different elements in Λ_0 are in different DA_n–orbits;
(3) every DA_n–orbit in Λ contains an element from Λ_0.

(1) and (2) follow by direct computation. Pick a pair $(\Pi_1, \Pi_2) \in \Lambda$, by Corollary 3.4 (3), one endpoint of $\Pi_1 \cap \Pi_2$ is the left tip of Π_1 or Π_2. Let $\alpha \in DA_n$ be the element represented by such left tip. Then $\alpha^{-1}(\Pi_1, \Pi_2) \in \Lambda_0$.

For the pair $(\Pi, u_i^{-1}\Pi)$, we add an edge between $c_j \in \Pi$ and $u_i^{-1}c_{j+i} \in u_i^{-1}\Pi$ for $j = 1, \dots, n-2-i$, and add an edge between c_j and $u_i^{-1}c_{j+i-1}$ for $j = 1, \dots, n-1-i$, see Figure 8. For the pair $(\Pi, d_i^{-1}\Pi)$, we add an edge between c_j and $d_i^{-1}c_{j+i}$ for $j = 1, \dots, n-2-i$, and add an edge between c_j and $d_i^{-1}c_{j+i-1}$ for $j = 1, \dots, n-1-i$, see Figure 9. Note that the new edges between two cells form a zigzag pattern.

Given a pair of cells $(\Pi_1, \Pi_2) \in \Lambda$, there is a unique element $\alpha \in A$ such that $\alpha^{-1}(\Pi_1, \Pi_2) \in \Lambda_0$. Thus edges between (Π_1, Π_2) are defined to be the α–image of edges between $\alpha^{-1}(\Pi_1, \Pi_2)$. Let X' be the complex obtained by adding all the new edges and let X be the flag completion of X', i.e. X is a flag simplicial complex which has the same 1–skeleton as X'. There is a simplicial action $DA_n \curvearrowright X$.

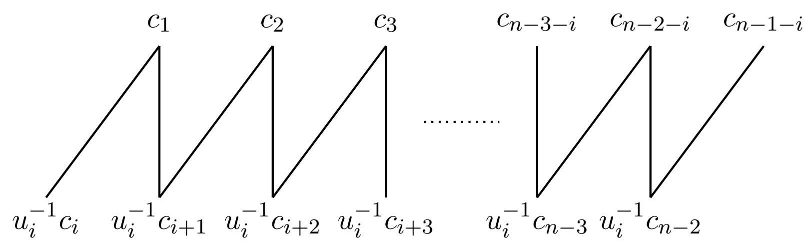

FIGURE 8.

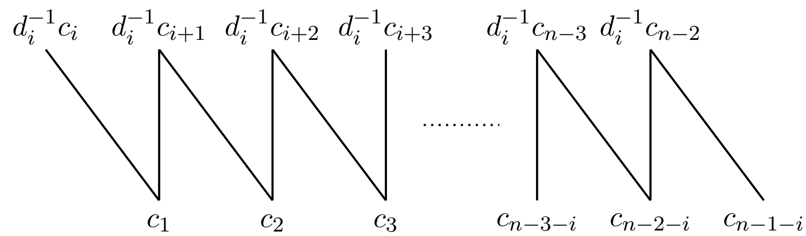

FIGURE 9.

Recall that for two vertices v_1 and v_2 in a simplicial complex, we write $v_1 \sim v_2$ (resp. $v_1 \nsim v_2$) to denote that they are connected by an edge (resp. are not connected by an edge).

For an interior vertex v in a cell, an edge in the boundary of the cell is *facing* v if v and this edge span a triangle in the cell.

Lemma 3.7. *Pick an interior vertex c_k in Π. Then*

(1) c_k is connected to at least one of the interior vertices of $u_i^{-1}\Pi$ (resp. $u_i^{-1}\Pi$) if and only if at least one of the edges in $\partial\Pi$ facing c_k is contained in $u_i^{-1}\Pi \cap \Pi$ (resp. $d_i^{-1}\Pi \cap \Pi$).

(2) if $c_k \in \Pi$ and $u_i^{-1}c_{k'} \in u_i^{-1}\Pi$ (resp. $d_i^{-1}c_{k'} \in d_i^{-1}\Pi$) are adjacent, then there is a vertex in $\Pi \cap \partial(u_i^{-1}\Pi)$ (resp. $\Pi \cap \partial(d_i^{-1}\Pi)$) that is adjacent to both c_k and $u_i^{-1}c_{k'}$ (resp. $d_i^{-1}c_{k'}$).

Proof. We only consider the case of Π and $u_i^{-1}\Pi$. Note that $\Pi \cap \partial(u_i^{-1}\Pi)$ is a path ω in the lower half of Π, starting at l and ending at u_{n-i} or d_{n-i}. Moreover, both c_1 and $u_i^{-1}c_i$ are facing the first edge of ω, both c_k and $u_i^{-1}c_{i+k}$ are facing the $(k+1)$–th edge of ω for $1 \leqslant k \leqslant n-2-i$, and both c_{n-1-i} and $u_i^{-1}c_{n-2}$ are facing the last edge of ω. Now the lemma follows. □

Lemma 3.8.

(1) Suppose $1 \leqslant i < j \leqslant n-2$. Then there are exactly two interior vertices in $u_j^{-1}\Pi$ connected to $u_i^{-1}c_i$, which are $u_j^{-1}c_j$ and $u_j^{-1}c_{j-1}$. There are exactly two interior vertices in $u_i^{-1}\Pi$ connected to $u_j^{-1}c_j$,

which are $u_i^{-1}c_i$ *and* $u_i^{-1}c_{i+1}$ *(see Figure 10 left). Moreover, when* $i \geqslant 2$, $u_j^{-1}c_{j-1} \sim u_i^{-1}c_{i-1}$.

(2) *Suppose* $1 \leqslant i < j = n-1$. *Then there is exactly one interior vertex of* $u_j^{-1}\Pi$ *connected to* $u_i^{-1}c_i$*, which is* $u_j^{-1}c_{j-1}$. *Moreover, when* $i \geqslant 2$, $u_j^{-1}c_{j-1} \sim u_i^{-1}c_{i-1}$.

(3) *Suppose* $1 \leqslant i < j \leqslant n-2$. *Then there are exactly two interior vertices in* $d_j^{-1}\Pi$ *connected to* $d_i^{-1}c_i$*, which are* $d_j^{-1}c_j$ *and* $d_j^{-1}c_{j-1}$. *There are exactly two interior vertices in* $d_i^{-1}\Pi$ *connected to* $d_j^{-1}c_j$*, which are* $d_i^{-1}c_i$ *and* $d_i^{-1}c_{i+1}$ *(see Figure 10 right). Moreover, when* $i \geqslant 2$, $d_j^{-1}c_{j-1} \sim d_i^{-1}c_{i-1}$.

(4) *Suppose* $1 \leqslant i < j = n-1$. *Then there is exactly one interior vertex of* $d_j^{-1}\Pi$ *connected to* $d_i^{-1}c_i$*, which is* $d_j^{-1}c_{j-1}$. *Moreover, when* $i \geqslant 2$, $d_j^{-1}c_{j-1} \sim d_i^{-1}c_{i-1}$.

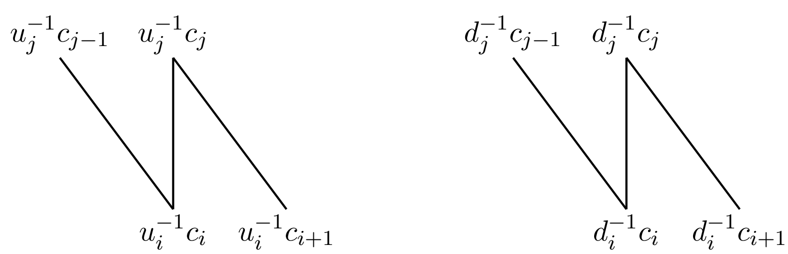

FIGURE 10.

Proof. We prove (1) and (2), the others are similar. Note that $u_iu_j^{-1} = d_{j-i}^{-1}$ when u_i is in the upper half of the cell, and $u_iu_j^{-1} = u_{j-i}^{-1}$ when u_i is in the lower half of the cell. Now we consider the case where u_i is in the upper half of the cell, the other case is similar. It follows from the scheme we add edges between Π and $d_{j-i}^{-1}\Pi$ that

(1) when $1 \leqslant i < j \leqslant n-2$, the only two interior vertices in $d_{j-i}^{-1}\Pi$ connected to c_i are $d_{j-i}^{-1}c_j$ and $d_{j-i}^{-1}c_{j-1}$; and the only two interior vertices in Π connected to $d_{j-i}^{-1}c_j$ are c_i and c_{i+1}; moreover, when $i \geqslant 2$, $c_{i-1} \sim d_{j-i}^{-1}c_{j-1}$;

(2) when $j = n-1$, c_i is only adjacent to $d_{j-i}^{-1}c_{j-1}$ in the interior of $d_{j-i}^{-1}\Pi$, moreover, when $i \geqslant 2$, $c_{i-1} \sim d_{j-i}^{-1}c_{j-1}$.

Now (1) and (2) follows by applying the action of u_i^{-1} (note that $u_i^{-1}d_{j-i}^{-1} = u_j^{-1}$). □

3.3. **Relations to Bestvina's complexes.** We leave a short remark on the relation between X and several complexes defined in Bestvina's paper

[Bes99]. We will not prove the statements in this subsection, since they are not used in the later part of the paper, and their proof follows from the same argument as in Section 4.

The *central segment* of a cell Π in X is the edge path starting at l, traveling through $c_1, c_2, \ldots, c_{n-2}$, and arriving at r (see Figure 6). A *central line* in X is a subset which is homeomorphic to $\mathbb{R}$ and is a concatenation of central segments. Note that for each vertex $v \in X$, there is a unique central line in X that contains v. Two central lines ℓ_1, ℓ_2 are *adjacent* if there exist vertices $v_1 \in \ell_1$ and $v_2 \in \ell_2$ such that v_1 and v_2 are adjacent.

We define a simplicial complex Z as follows. Vertices of Z are in 1-1 correspondence with central lines in X. Two vertices are joined by an edge if the corresponding central lines are adjacent. A collection of vertices spans a simplex if each pair of vertices in the collection are jointed by an edge. Then Z is isomorphic to the simplicial complex $X(\mathcal{G})$ in [Bes99, Definition 2.3]. Moreover, X is homeomorphic to $Z \times \mathbb{R}$. There is another complex in Bestvina's paper [Bes99] which is homeomorphic to $Z \times \mathbb{R}$. It is denoted by $X(\mathcal{A})$ and defined in [Bes99, pp.280]. However, the simplicial structures on X and $X(\mathcal{A})$ are different.

4. Links of vertices in X

4.1. **Prisms.** We recall a standard simplicial subdivision of a prism ([Hat02, Chapter 2.1]). Let Δ^n be the n–dimensional simplex. Let $P = \Delta^n \times [0,1]$ be a prism. We use $[v_0, \ldots, v_n]$ (resp. $[w_0, \ldots . w_n]$) to denote the simplex $\Delta^n \times \{0\}$ (resp. $\Delta^n \times \{1\}$). Then P can be subdivided into $(n+1)$–simplexes, each is of the form $[v_0, \ldots, v_i, w_i, \ldots, w_n]$. The prism P with such simplicial structure is called a *subdivided prism*. Note that in the 1–skeleton $P^{(1)}$, $v_i \sim w_j$ for $j \geqslant i$ and $v_i \nsim w_j$ for $j < i$. This motivates the following definition.

Definition 4.1. Let Γ be a finite simple graph with its vertex set V. Suppose there is a partition $V = W \sqcup W'$. We define $\Gamma = \mathrm{Prism}(W, W')$ if

(1) W span a complete subgraph of Γ, so is W';
(2) W and W' have the same cardinality;
(3) it is possible to order the vertices of W as $\{w_1, \ldots, w_n\}$ such that $W'_1 \supsetneq \cdots \supsetneq W'_n$, where W'_i is the collection of vertices in W' that are adjacent to w_i.

It is clear from the definition that a simple graph Γ is isomorphic to the 1–skeleton of a subdivided prism if and only if its vertex set has a partition such that $\Gamma = \mathrm{Prism}(W, W')$.

Definition 4.1 (2) and (3) imply w_1 is connected to each vertex of W', and w_n is connected to only one vertex of W'. Moreover, we deduce from (2) and (3) that (3) is also true if we switch the role of W and W'. W has a linear order, where $w_i < w_j$ if $W'_i \subsetneq W'_j$. Similarly, W' has a linear order.

Let Γ be a graph. Recall that a subgraph $\Gamma' \subset \Gamma$ is a *full subgraph* if Γ' satisfies that an edge of Γ is inside Γ' if and only if the vertices of this edge are inside Γ'. Let $W \subset \Gamma$ be a collection of vertices. The full subgraph *spanned* by W is the minimal full subgraph that contains W.

Definition 4.2. Let Γ' be a simplicial graph and let W and W' be two disjoint collections of vertices of Γ'. We say that W and W' *span a prism* if the full subgraph spanned by $W \cup W'$ is isomorphic to $\mathrm{Prism}(W, W')$.

Now we discuss a model graph which will appear repeatedly in our computation. The reader can proceed directly to Section 4.2 and come back when needed.

Definition 4.3. A *model graph* is a finite simplicial graph Γ such that its vertex set V admits a partition $V = \{c_l\} \sqcup \{c_r\} \sqcup U_l \sqcup U_r \sqcup D_l \sqcup D_r$ satisfying the following conditions:

(1) the collection of vertices in V that are adjacent to c_l (resp. c_r) is $U_l \cup D_l$ (resp. $U_r \cup D_r$);
(2) there are no edges between a vertex in $U_l \cup U_r$ and a vertex in $D_l \cup D_r$;
(3) U_l and U_r span a prism;
(4) D_l and D_r span a prism.

See Figure 11 for an example of a model graph (edges of the complete subgraphs U_l, U_r, D_l and D_r are not drawn in the picture). Note that if U_l, U_r, D_l and D_r are sets made of a single point, then Γ is a 6–cycle.

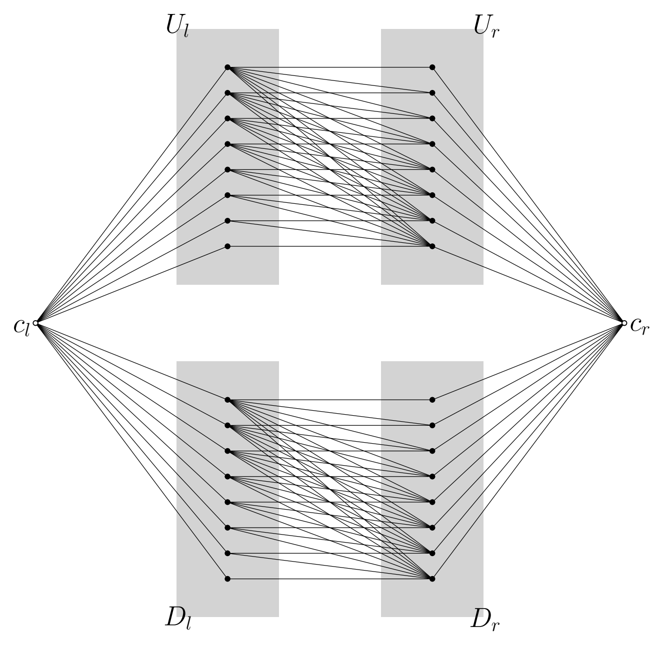

FIGURE 11. A model graph.

Lemma 4.4. *Let Γ be a model graph. Then Γ is 6–large.*

Proof. We use d to denote the combinatorial distance between vertices of Γ. Note that $d(c_l, c_r) = 3$. Let C be a simple 4–cycle or 5–cycle in Γ. We need to show C has a diagonal. First note that it is impossible that both c_l and c_r are inside C, otherwise the length of C is $\geqslant 6$ since it contains two paths from c_l to c_r, each of which has length $\geqslant 3$. Since C is simple, we assume without loss of generality that vertices of C are contained in $c_l \cup U_l \cup U_r$. If $c_l \in C$, then C has a diagonal since each pair of vertices in U_l are connected by an edge. So it remains to consider the case $C \subset \mathrm{Prism}(U_l, U_r)$.

We assume in addition there does not exist a pair of non-consecutive vertices in C such that they are both contained in U_l or U_r, otherwise C has a diagonal since each of U_l and U_r span a complete subgraph. It follows from parity considerations that C has to be a 4–cycle. Let v_1, v_2, v_3, v_4 be the consecutive vertices of C such that $v_1, v_2 \in U_l$, and $v_3, v_4 \in U_r$. If $v_1 > v_2$ with respect to the linear order in the above discussion, then $v_1 \sim v_3$. If $v_1 < v_2$, then $v_2 \sim v_4$. Thus in each case C has a diagonal. □

4.2. **Link of a real vertex.** In this subsection we analyze links of real vertices of the complex X constructed in Section 3.2. Since DA_n acts freely and transitively on the set of such vertices it is enough to describe the link of one of them. We pick the real vertex coinciding with the identity. Following our notation this is the vertex $l \in \Pi$. Otherwise, the same vertex can be described as: $u_i^{-1}u_i \in u_i^{-1}\Pi$, $d_i^{-1}d_i \in d_i^{-1}\Pi$, or $r^{-1}r \in r^{-1}\Pi$. Let V be the set of vertices of X that are adjacent to l, and let Γ_V be the full subgraph of $X^{(1)}$ spanned by V. Our goal in this subsection is the following.

Proposition 4.5. *Γ_V is 6–large.*

The case $n = 2$ is not hard, and we handle it in Remark 4.17 at the end of this subsection. In what follows we assume that $n \geqslant 3$.

Moreover, for reasons that will be explained in Section 5 we need to analyze distances in Γ_V between some particular vertices. The precise statement is in Lemma 4.16 below.

First, we describe vertices adjacent to l in various copies of Π. Observe that $g^{-1}\Pi$ contains a vertex adjacent to l only for $g \in \{l, r\} \cup \{d_i\}_{i=1}^{n-1} \cup \{u_i\}_{i=1}^{n-1}$. We do not provide proofs of the following three lemmas since they are immediate consequences of the form of cells $g^{-1}\Pi$ (see Figure 6).

Lemma 4.6.

1. *There are exactly three vertices in Π adjacent to l, which are u_1, c_1, and d_1.*
2. *There are exactly three vertices in $r^{-1}\Pi$ adjacent to l, which are $r^{-1}u_{n-1}$, $r^{-1}c_{n-2}$, and $r^{-1}d_{n-1}$.*

Lemma 4.7.

1. *There are exactly three vertices in $d_1^{-1}\Pi$ adjacent to l, which are $d_1^{-1}l$, $d_1^{-1}c_1$, and $d_1^{-1}u_2$.*

(2) *Suppose that* $2 \leqslant i \leqslant n-2$. *Then there are exactly four vertices in* $d_i^{-1}\Pi$ *adjacent to* l, *which are* $d_i^{-1}u_{i-1}$, $d_i^{-1}c_{i-1}$, $d_i^{-1}c_i$, *and* $d_i^{-1}u_{i+1}$.
(3) *There are exactly three vertices in* $d_{n-1}^{-1}\Pi$ *adjacent to* l, *which are* $d_{n-1}^{-1}r$, $d_{n-1}^{-1}c_{n-2}$, *and* $d_{n-1}^{-1}u_{n-2}$.

Similarly, Lemma 4.7 still holds with u and d interchanged. That is, we have the following.

Lemma 4.8.

(1) *There are exactly three vertices in* $u_1^{-1}\Pi$ *adjacent to* l, *which are* $u_1^{-1}l$, $u_1^{-1}c_1$, *and* $u_1^{-1}d_2$.
(2) *Suppose that* $2 \leqslant i \leqslant n-2$. *Then there are exactly four vertices in* $u_i^{-1}\Pi$ *adjacent to* l, *which are* $u_i^{-1}d_{i-1}$, $u_i^{-1}c_{i-1}$, $u_i^{-1}c_i$, *and* $u_i^{-1}d_{i+1}$.
(3) *There are exactly three vertices in* $d_{n-1}^{-1}\Pi$ *adjacent to* l, *which are* $u_{n-1}^{-1}r$, $u_{n-1}^{-1}c_{n-2}$, *and* $u_{n-1}^{-1}d_{n-2}$.

Let us make few other immediate observations.

Lemma 4.9. *There are exactly four real vertices adjacent to* l, *which are* d_1, u_1, $r^{-1}d_{n-1}$, *and* $r^{-1}u_{n-1}$. *The following identifications hold:*

(1) $u_1^{-1}l = r^{-1}d_{n-1}$, $u_1^{-1}d_2 = d_1$, $d_1^{-1}l = r^{-1}u_{n-1}$, *and* $d_1^{-1}u_2 = u_1$.
(2) *Suppose that* $2 \leqslant i \leqslant n-2$. *Then* $d_i^{-1}u_{i-1} = r^{-1}u_{n-1}$, $d_i^{-1}u_{i+1} = u_1$, $u_i^{-1}d_{i-1} = r^{-1}d_{n-1}$, *and* $u_i^{-1}d_{i+1} = d_1$.
(3) $u_{n-1}^{-1}r = d_1$, $u_{n-1}^{-1}d_{n-2} = r^{-1}d_{n-1}$, $d_{n-1}^{-1}r = u_1$, *and* $d_{n-1}^{-1}u_{n-2} = r^{-1}u_{n-1}$.

We define the following mutually disjoint collections of vertices:

- $D_l = \{r^{-1}d_{n-1}\} \cup \{u_j^{-1}c_{j-1}\}_{j=2}^{n-1}$;
- $D_r = \{d_1\} \cup \{u_j^{-1}c_j\}_{j=1}^{n-2}$;
- $U_l = \{r^{-1}u_{n-1}\} \cup \{d_j^{-1}c_{j-1}\}_{j=2}^{n-1}$;
- $U_r = \{u_1\} \cup \{d_j^{-1}c_j\}_{j=1}^{n-2}$.

By Lemma 4.6, Lemma 4.7, and Lemma 4.8, $V = \{c_1, r^{-1}c_{n-2}\} \sqcup U_l \sqcup U_r \sqcup D_l \sqcup D_r$. Then Proposition 4.5 is a consequence of the following result and Lemma 4.4.

Proposition 4.10. *With the above definition of* U_l, U_r, D_l *and* D_r, Γ_V *satisfies each condition of Definition 4.3 with* c_l *replaced by* $r^{-1}c_{n-2}$ *and* c_r *replaced by* c_1 *(see Figure 12).*

To prove Proposition 4.10 we need few preparatory lemmas. We check consecutive properties in Definition 4.3.

Lemma 4.11.

(1) $c_1 \nsim r^{-1}c_{n-2}$, *and* $c_1 \nsim r^{-1}d_{n-1}, r^{-1}u_{n-1}$, *and* $r^{-1}c_{n-2} \nsim u_1, d_1$.
(2) $c_1 \sim d_1, u_1$, *and* $r^{-1}c_{n-2} \sim r^{-1}d_{n-1}, r^{-1}u_{n-1}$.
(3) *For* $1 \leqslant j \leqslant n-2$, *we have* $c_1 \sim u_j^{-1}c_j, d_j^{-1}c_j$, *and* $c_1 \nsim u_{j+1}^{-1}c_j, d_{j+1}^{-1}c_j$.

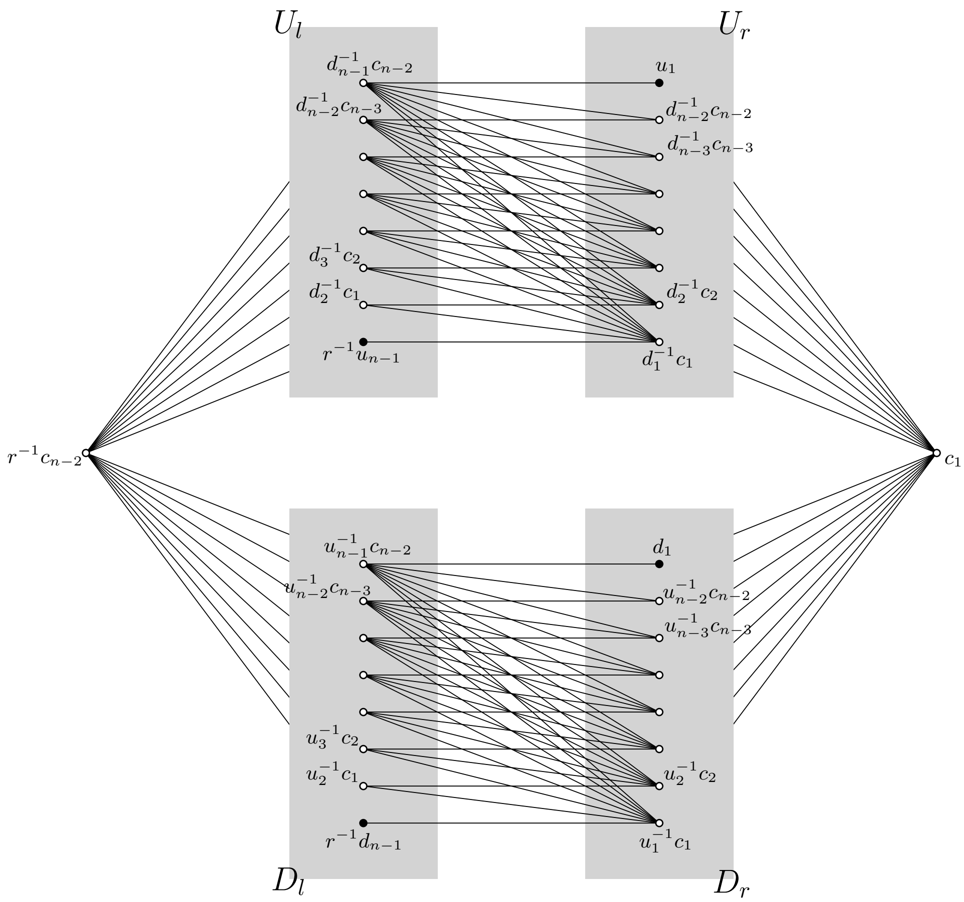

FIGURE 12. The structure of the link Γ_V of a real vertex, the case $n = 9$. (Edges in the complete graphs U_r, U_l, D_r, and D_l are not shown.)

(4) For $1 \leqslant j \leqslant n-2$, *we have* $r^{-1}c_{n-2} \sim u_{j+1}^{-1}c_j, d_{j+1}^{-1}c_j$, *and* $r^{-1}c_{n-2} \nsim u_j^{-1}c_j, d_j^{-1}c_j$.

Consequently, the collection of vertices in V *that are adjacent to* c_1 *(resp.* $r^{-1}c_{n-2}$*) is* $U_r \sqcup D_r$ *(resp.* $U_l \sqcup D_l$*).*

Proof. Observe that $\Pi \cap r^{-1}\Pi = \{l\}$, hence there is no edge between c_1 and $r^{-1}c_{n-2}$. Furthermore, it follows that $r^{-1}d_{n-1}, r^{-1}u_{n-1}$ are real vertices not belonging to Π, hence $c_1 \nsim r^{-1}d_{n-1}, r^{-1}u_{n-1}$. Similarly, $r^{-1}c_{n-2} \nsim u_1, d_1$. This shows (1). Edges in (2) are edges in cells Π and $r^{-1}\Pi$. Recall that the only vertices in $u_j^{-1}\Pi$ adjacent to $c_1 \in \Pi$ are $u_j^{-1}c_j$, and $u_j^{-1}c_{j+1}$ (see Figure 8), and the only vertices in $d_j^{-1}\Pi$ adjacent to c_1 are $d_j^{-1}c_j$, and $d_j^{-1}c_{j+1}$ (see Figure 9), hence (3). For (4) the argument is similar: the only vertices in $u_j\Pi$ adjacent to c_{n-2} are u_jc_{n-2-j}, and u_jc_{n-1-j}, and the only vertices in $d_j\Pi$ adjacent to c_{n-2} are d_jc_{n-2-j}, and d_jc_{n-1-j}. Observe that $r^{-1}u_j$ is equal to u_{n-j}^{-1} (for $(n-j)$ even) or d_{n-j}^{-1} (for $(n-j)$ odd), and that $r^{-1}d_j$ is equal to d_{n-j}^{-1} (for $(n-j)$ even) or u_{n-j}^{-1} (for $(n-j)$ odd). Thus, the only vertices adjacent to $r^{-1}c_{n-2}$ are of the form $u_{n-j}^{-1}c_{(n-j)-1}$, and $d_{n-j}^{-1}c_{(n-j)-1}$. Thus (4) holds. □

Lemma 4.12.

(1) No two of vertices $d_1, r^{-1}d_{n-1}, u_1, r^{-1}u_{n-1}$ are adjacent.
(2) For $1 \leqslant j \leqslant n-2$, we have $u_1 \nsim u_j^{-1}c_j, u_{j+1}^{-1}c_j$, and $r^{-1}u_{n-1} \nsim u_j^{-1}c_j, u_{j+1}^{-1}c_j$.
(3) For $1 \leqslant j \leqslant n-2$, we have $d_1 \nsim d_j^{-1}c_j, d_{j+1}^{-1}c_j$, and $r^{-1}d_{n-1} \nsim d_j^{-1}c_j, d_{j+1}^{-1}c_j$.
(4) For $1 \leqslant j, k \leqslant n-2$, we have $u_j^{-1}c_j \nsim d_k^{-1}c_k, d_{k+1}^{-1}c_k$, and $u_{j+1}^{-1}c_j \nsim d_k^{-1}c_k, d_{k+1}^{-1}c_k$.

Consequently, no vertex in $U_l \cup U_r$ is adjacent to a vertex in $D_l \cup D_r$.

Proof. All the vertices in (1) are real, and there are no edges between them in $X^\triangle$. Therefore, (1) follows from the fact that passing from $X^\triangle$ to X we do not add edges between real vertices. For (2), observe that $u_1 \notin u_j^{-1}\Pi$ so that $u_1 \nsim u_j^{-1}c_j, u_{j+1}^{-1}c_j$. By Lemma 4.9 (1), we have $u_1^{-1}l = r^{-1}d_{n-1}$. For $j \geqslant 2$, by Lemma 4.9 (2), we have $u_j^{-1}d_{j-1} = r^{-1}d_{n-1}$. Therefore, for $1 \leqslant j \leqslant n-2$, we have $r^{-1}d_{n-1} \in u_j^{-1}\Pi$. Since, by Corollary 3.4 (2), the intersection of two cells is contained either in the upper half or in the lower half, we have that $r^{-1}u_{n-1} \notin u_j^{-1}\Pi$. (3) can be proven the same way as (2), interchanging u and d. For (4), observe that $u_j^{-1}\Pi \cap \Pi$ is contained in the lower half of $\partial\Pi$, and $d_j^{-1}\Pi \cap \Pi$ is contained in the upper half of $\partial\Pi$. Then Corollary 3.5 implies that $d_j^{-1}\Pi \cap u_k^{-1}\Pi$ is at most one point, hence no internal vertices of the two cells are adjacent. □

Lemma 4.13. *Every two vertices in D_r are adjacent. The same is true for U_l, U_r, and D_l.*

Proof. We prove the statement for D_r and D_l. The other cases are proven similarly.

By Lemma 4.9(1), we have $d_1 = u_1^{-1}d_2$. Hence, by the form of the cell $u_1^{-1}\Pi$ we have $d_1 \sim u_1^{-1}c_1$ (see Figure 6). Similarly, by Lemma 4.9(2), we have $d_1 = u_j^{-1}d_{j+1}$, hence $d_1 \sim u_j^{-1}c_j$, for $2 \leqslant j \leqslant n-2$. By Lemma 3.8(1), we have that $u_j^{-1}c_j \sim u_k^{-1}c_k$, for $1 \leqslant j < k \leqslant n-2$. Therefore, every two vertices in D_r are adjacent.

By Lemma 4.9 (2)(3), we have $r^{-1}d_{n-1} = u_j^{-1}dj-1$, for $2 \leqslant j \leqslant n-1$. Hence, by the form of the cell $u_j^{-1}\Pi$ we have $u_j^{-1}d_{j-1} \sim u_j^{-1}c_{j-1}$ (see Figure 6). By Lemma 3.8(1)(2), we have $u_j^{-1}c_{j-1} \sim u_k^{-1}c_{k-1}$, for $2 \leqslant j < k \leqslant n-1$. Therefore, every two vertices in D_l are adjacent. □

Now we show that D_l and D_r span a prism. This relies on the following lemmas.

Lemma 4.14. *The only vertex in D_l (resp. D_r) adjacent to d_1 (resp. $r^{-1}d_{n-1}$) is $u_{n-1}^{-1}c_{n-2}$ (resp. $u_1^{-1}c_1$).*

Proof. By Lemma 4.12(1), we have $d_1 \nsim r^{-1}d_{n-1}$. For $2 \leqslant j \leqslant n-2$, we have $d_1 = u_j^{-1}d_{j+1}$ (by Lemma 4.9(2)), hence $d_1 \nsim u_j^{-1}c_{j-1}$. Since $d_1 = u_{n-1}^{-1}r$ (by Lemma 4.9(3)), we have $d_1 \sim u_{n-2}^{-1}c_{n-2}$. It follows that $u_{n-2}^{-1}c_{n-2}$ is the only vertex in D_l adjacent to d_1. The statement about $r^{-1}d_{n-1}$ and D_r has an analogous proof. □

Lemma 4.15. *Let $1 \leqslant j, k \leqslant n-2$. Then $u_j^{-1}c_j \sim u_{k+1}^{-1}c_k$ iff $j \leqslant k+1$.*

Proof. If $j = k+1$ then $u_j^{-1}c_j \sim u_{k+1}^{-1}c_k$ since $c_{j-1} \sim c_j$ in Π. If $j < k+1$ and $k+1 \leqslant n+2$ then $u_j^{-1}c_j \sim u_{k+1}^{-1}c_k$, by Lemma 3.8 (1). If $j < k+1$ and $k+1 = n+1$ then $u_j^{-1}c_j \sim u_{k+1}^{-1}c_k$, by Lemma 3.8 (2). If $j > k+1$ then $u_j^{-1}c_j \nsim u_{k+1}^{-1}c_k$, by Lemma 3.8 (1). □

Proof of Proposition 4.10. By Lemma 4.13, graphs spanned by, respectively, U_r, U_l, D_r, and D_l are complete, hence the condition (3) in Definition 4.3 is satisfied for Γ_V. Now, we prove that the condition (4) from Definition 4.3 holds, that is, D_r and D_l span a prism. Clearly, the cardinalities of the two sets are equal $n-1$. We have to order appropriately (see Definition 4.1 (3)) vertices of D_r as $\{w_1, \ldots, w_{n-1}\}$. We set $w_{n-1} = d_1$, and $w_j = u_j^{-1}c_j$, for $1 \leqslant j \leqslant n-2$. Define $W_1' := U_l$, and $W_j' := \{u_k^{-1}c_{k-1}\}_{k=2}^{j}$, for $2 \leqslant j \leqslant n-1$. Then, by Lemma 4.14 and Lemma 4.15, the set W_j' is exactly the collection of vertices in D_l that are adjacent to w_j. It follows that $D_r \cup D_l$ spans a prism $\mathrm{Prism}(D_l, D_r)$. Analogously one proves that $U_r \cup U_l$ spans a prism $\mathrm{Prism}(U_l, U_r)$, hence the condition (5) from Definition 4.3 holds. The properties (1) and (2) from the same definition hold by, respectively, Lemma 4.11 and Lemma 4.12. □

Lemma 4.16. $d_{\Gamma_V}(r^{-1}u_{n-1}, r^{-1}d_{n-1}) = d_{\Gamma_V}(u_1, d_1) = d_{\Gamma_V}(r^{-1}u_{n-1}, u_1) = d_{\Gamma_V}(r^{-1}u_{n-1}, d_1) = 2$*;* $d_{\Gamma_V}(r^{-1}d_{n-1}, u_1) = d_{\Gamma_V}(r^{-1}u_{n-1}, d_1) = 3$.

Proof. The first statement follows by Lemma 4.11 (2) and Lemma 4.12 (1). It is clear that $d_{\Gamma_V}(r^{-1}d_{n-1}, u_1), d_{\Gamma_V}(r^{-1}u_{n-1}, d_1) \leqslant 3$. By Lemma 4.12 (1), we have $d_{\Gamma_V}(r^{-1}d_{n-1}, u_1) \geqslant 2$, and $d_{\Gamma_V}(r^{-1}u_{n-1}, d_1) \geqslant 2$. If we had $d_{\Gamma_V}(r^{-1}d_{n-1}, u_1) = 2$ then there would exist a vertex adjacent to both $r^{-1}d_{n-1}$, and u_1. This would contradict Proposition 4.10. Similarly one shows that $d_{\Gamma_V}(r^{-1}u_{n-1}, d_1) = 3$. □

Remark 4.17. The group DA_2 is isomorphic to $\mathbb{Z}^2$. To obtain the complex X we add the diagonal $\overline{lr}$ in every precell being the square (we do not add interior vertices c_i). As a result, X is the equilateral triangulation of the plane, and links of vertices are 6–cycles; see Figure 13. Note that it is a

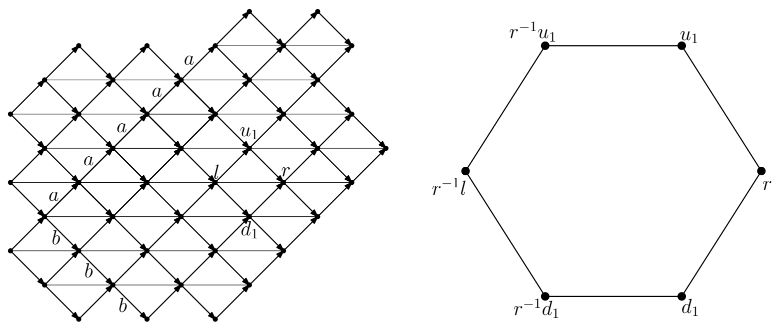

FIGURE 13. The complex X in the case $n = 2$ (a fragment on the left), and the link of its vertex (right).

special case of a model graph (see Definition 4.3), where all U_l, U_r, D_l, and D_r reduce to single vertices, and hence the prisms $\text{Prism}(U_l, U_r)$ etc. reduce to edges.

Remark 4.18. Let X^* be as in Section 3.1 with its edges labeled by the two generators a and b of DA_n. Recall that Γ_V is the link of the identity vertex l in $X^{(1)}$, and the real vertices of $X^{(1)}$ comes from the vertices of X^*. Thus there are exactly four real vertices in Γ_V, two of them arise from incoming and outgoing a–edges at l, which we denoted by a^+ and a^-; and two of them arise from incoming and outgoing b–edges at l, which we denoted by b^+ and b^-. On the other hand, by Lemma 4.9, the four real vertices in Γ_V are described as $d_1, u_1, r^{-1}d_{n-1}$ and $r^{-1}u_{n-1}$. One readily verifies the identification $u_1 = a^-$, $d_1 = b^-$, $r^{-1}u_{n-1} = b^+$ and $r^{-1}d_{n-1} = a^+$ (see Figure 7). Thus the following statements hold:

(1) if $n \geqslant 3$, then by Lemma 4.16, $d_{\Gamma_V}(a^+, b^+) = d_{\Gamma_V}(a^+, b^-) = d_{\Gamma_V}(a^-, b^+) = d_{\Gamma_V}(a^-, b^-) = 2$ and $d_{\Gamma_V}(a^+, a^-) = d_{\Gamma_V}(b^+, b^-) = 3$;
(2) $n = 2$, then by Remark 4.17, $d_{\Gamma_V}(a^+, b^+) = d_{\Gamma_V}(a^-, b^-) = 2$, $d_{\Gamma_V}(a^+, b^-) = d_{\Gamma_V}(a^-, b^+) = 1$ and $d_{\Gamma_V}(a^+, a^-) = d_{\Gamma_V}(b^+, b^-) = 3$;

4.3. **Link of an interior vertex.** Pick an interior vertex $c_i \in \Pi$. Let V be the set of vertices of X that are adjacent to c_i, and let Γ_V be the full subgraph of $X^{(1)}$ spanned by V. Our goal in this subsection is the following.

Proposition 4.19. *Γ_V is 6–large.*

First we characterize elements of V. They fall into two disjoint classes:

A vertices in Π that are adjacent to c_i;
B vertices outside Π that are adjacent to c_i, they must be interior vertices of some cells other than Π.

Class A consists of 6 vertices around c_i. There are two cases.

(1) The number i is odd. Then c_i is connected to u_i and d_{i+1} in the upper half of $\partial\Pi$, and d_i and u_{i+1} in the lower half of $\partial\Pi$. In such case c_i is facing a b–edge in the upper half and an a–edge in the

lower half. Moreover, c_i is connected to c_{i-1} or l (when $i = 1$) on the left, and c_{i+1} or r (when $i = n-2$) on the right.

(2) The number i is even. Then c_i is connected to d_i and u_{i+1} in the upper half of $\partial\Pi$, and u_i and d_{i+1} in the lower half of $\partial\Pi$. In such case c_i is facing an a–edge in the upper half and a b–edge in the lower half. Moreover, c_i is connected to c_{i-1} or l (when $i = 1$) on the left, and c_{i+1} or r (when $i = n-2$) on the right.

Now we assume that i is odd. The case of even i is similar. Next we study vertices of class B. Let $w\Pi$ be a cell such that it contains interior vertices that are adjacent to c_i. By Lemma 3.7 (1), $w\Pi \cap \Pi$ is a path of length $\geqslant 2$ such that it contains either $\overline{u_i d_{i+1}}$ or $\overline{d_i u_{i+1}}$. There are two cases.

(1) If $\overline{u_i d_{i+1}} \subset w\Pi \cap \Pi$, then $w = u_i u_j^{-1}$ for $0 \leqslant j \leqslant n-1$ and $j \neq i$ (we set $u_0 = l$). We define $p_j = u_i u_j^{-1}$.
(2) If $\overline{d_i u_{i+1}} \subset w\Pi \cap \Pi$, then $w = d_i d_j^{-1}$ for $0 \leqslant j \leqslant n-1$ and $j \neq i$ (we set $d_0 = l$). We define $q_j = d_i d_j^{-1}$.

Lemma 4.20.

(1) There is only one vertex in $p_0\Pi$ adjacent to c_i, which is $p_0 c_1$.
(2) Suppose $1 \leqslant j < i \leqslant n-2$. Then there are exactly two vertices in $p_j\Pi$ adjacent to c_i, which are $p_j c_j$ and $p_j c_{j+1}$.
(3) Suppose $1 \leqslant i < j \leqslant n-2$. Then there are exactly two vertices in $p_j\Pi$ adjacent to c_i, which are $p_j c_{j-1}$ and $p_j c_j$.
(4) There is only one vertex in $p_{n-1}\Pi$ adjacent to c_i, which is $p_{n-1} c_{n-2}$.

Proof. Recall that the only vertex in Π adjacent to $u_i^{-1} c_i \in u_i^{-1}\Pi$ is c_1. Thus (1) follows by applying the action of u_i. For (2), we apply Lemma 3.8 (1) with i and j interchanged to deduce that $u_i^{-1} c_i$ is adjacent to $u_j^{-1} c_j$ and $u_j^{-1} c_{j+1}$. Then (2) follows by applying the action of u_i (recall that $p_j = u_i u_j^{-1}$). (3) follows from Lemma 3.8 (1) in a similar way, and (4) follows from Lemma 3.8 (2). □

The following lemma can be proved in a similar way to Lemma 4.20, using Lemma 3.8 (3) and (4).

Lemma 4.21. *Lemma 4.20 still holds with p replaced by q.*

We define the following mutually disjoint collections of vertices:

- $U_l = \{u_i\} \cup \{p_j c_j\}_{j=1}^{i-1} \cup \{p_j c_{j-1}\}_{j=i+1}^{n-1}$;
- $U_r = \{d_{i+1}\} \cup \{p_j c_{j+1}\}_{j=0}^{i-1} \cup \{p_j c_j\}_{j=i+1}^{n-2}$;
- $D_l = \{d_i\} \cup \{q_j c_j\}_{j=1}^{i-1} \cup \{q_j c_{j-1}\}_{j=i+1}^{n-1}$;
- $D_r = \{u_{i+1}\} \cup \{q_j c_{j+1}\}_{j=0}^{i-1} \cup \{q_j c_j\}_{j=i+1}^{n-2}$;

By Lemma 4.20 and Lemma 4.21, $V = \{c_{i-1}, c_{i+1}\} \cup U_l \cup U_r \cup D_l \cup D_r$ (when $i = 1$, let $c_{i-1} = l$, when $i = n-2$, let $c_{i+1} = r$). Then Proposition 4.19 is a consequence of the following result and Lemma 4.4.

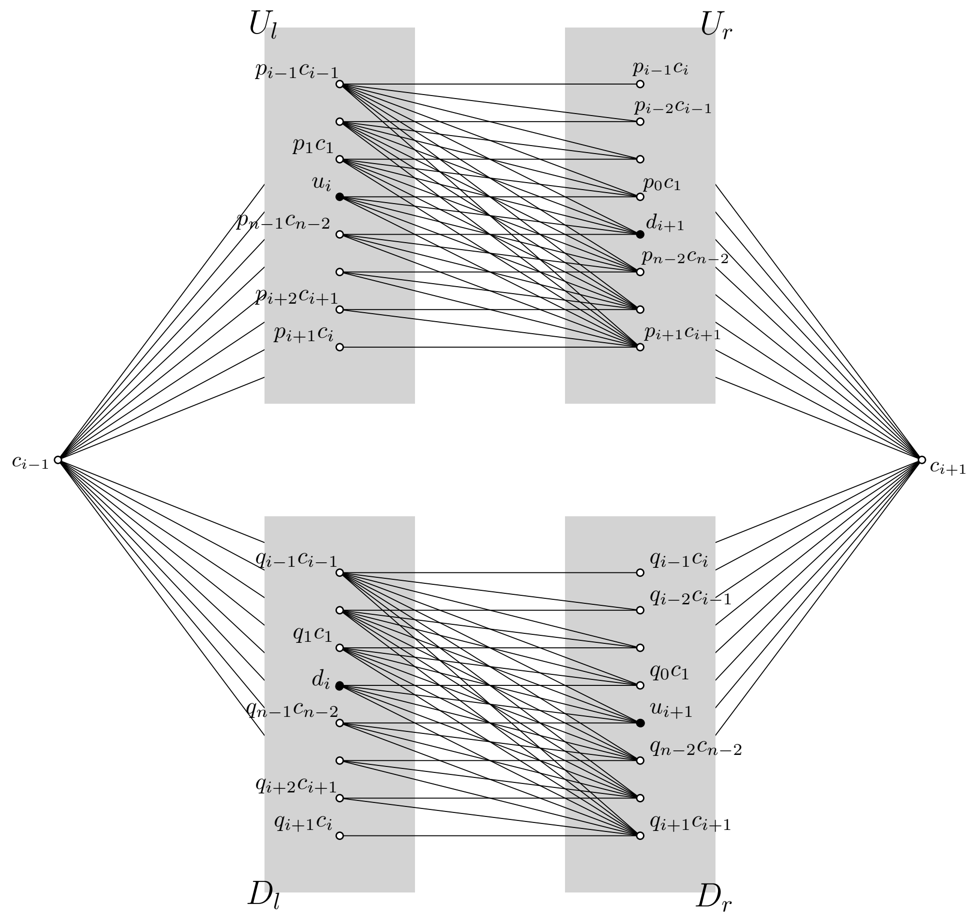

FIGURE 14. The structure of the link Γ_V of the interior vertex c_i, the case $n = 9$. (Edges in the complete graphs U_r, U_l, D_r, and D_l are not shown.)

Proposition 4.22. *With the above definition of U_l, U_r, D_l and D_r, the graph Γ_V satisfies each condition of Definition 4.3 with c_l replaced by c_{i-1} and c_r replaced by c_{i+1}.*

The rest of this section is devoted to the proof of Proposition 4.22.

Lemma 4.23. *We set $c_{n-1} = r$ and $c_0 = l$.*

(1) $c_{i-1} \nsim p_0c_1$ and $c_{i+1} \sim p_0c_1$.
(2) For $1 \leqslant j < i \leqslant n-2$, $c_{i-1} \sim p_jc_j$, $c_{i-1} \nsim p_jc_{j+1}$, $c_{i+1} \nsim p_jc_j$, and $c_{i+1} \sim p_jc_{j+1}$.
(3) For $1 \leqslant i < j \leqslant n-2$, $c_{i-1} \sim p_jc_{j-1}$, $c_{i-1} \nsim p_jc_j$, $c_{i+1} \nsim p_jc_{j-1}$, and $c_{i+1} \sim p_jc_j$.
(4) $c_{i-1} \sim p_{n-1}c_{n-2}$ and $c_{i+1} \nsim p_{n-1}c_{n-2}$.

Moreover, all the statements still hold with p replaced by q. As a consequence, the collection of vertices in V that are adjacent to c_{i-1} (resp. c_{i+1}) is $U_l \cup D_l$ (resp. $U_r \cup D_r$).

Proof. For (1), it follows from the fact that the zigzag pattern between $u_i^{-1}\Pi$ and Π is as in Figure 8 that $c_1 \sim u_i^{-1}c_{i+1}$ and $c_1 \nsim u_i^{-1}c_{i-1}$. (1) follows by applying the action of u_i. Now we prove (2). If $i < n-2$, then we apply Lemma 3.8 (1) with i and j interchanged to deduce that $u_i^{-1}c_i \sim u_j^{-1}c_j$ and $u_i^{-1}c_i \sim u_j^{-1}c_{j+1}$. By the zigzag pattern (see Figure 15 left), we know $u_i^{-1}c_{i-1} \sim u_j^{-1}c_j$, $u_i^{-1}c_{i-1} \nsim u_j^{-1}c_{j+1}$, $u_i^{-1}c_{i+1} \nsim u_j^{-1}c_j$ and $u_i^{-1}c_{i+1} \sim u_j^{-1}c_{j+1}$. Thus (2) follows by applying the action of u_i. If $i = n-2$, then $u_j u_{n-2}^{-1} r = u_{j+2}$. Thus $u_{n-2}^{-1} r = u_j^{-1} u_{j+2}$ is connected to $u_j^{-1}c_{j+1}$. Then we have a similar zigzag pattern as in Figure 15 right. (3) is similar to (2) (since $i < j$, we have a zigzag pattern as in Figure 16), and (4) is similar to (1). □

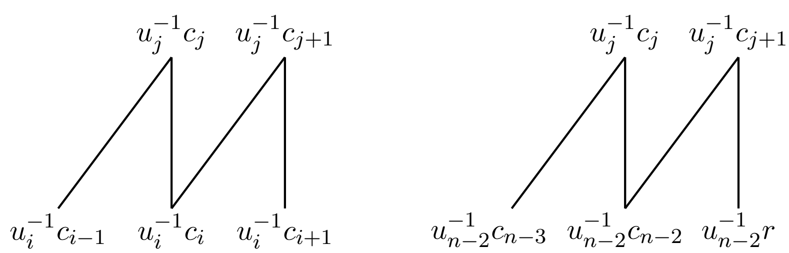

FIGURE 15.

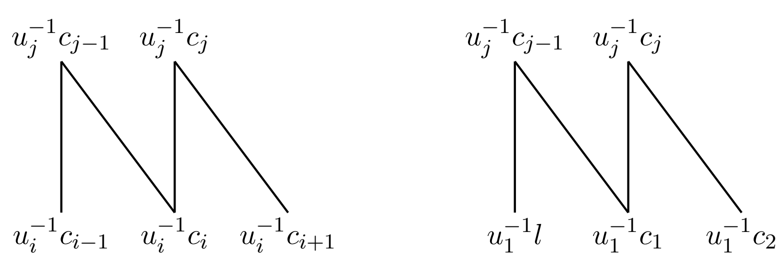

FIGURE 16.

Lemma 4.24. *Let $v \in U_l \cup U_r$ and $v' \in D_l \cup D_r$. Then v and v' are not adjacent.*

Proof. We first look at the case when v and v' are interior vertices. Suppose $v \in w\Pi$ and $v' \in w'\Pi$. Recall that $\overline{u_i d_{i+1}} \subset w\Pi$ and $\overline{d_i u_{i+1}} \subset w'\Pi$. Thus $w\Pi \cap \Pi$ (resp. $w'\Pi \cap \Pi$) is contained in the upper (resp. lower) half of $\partial\Pi$

by Corollary 3.4 (2). Then Corollary 3.5 implies $w\Pi \cap w'\Pi$ is at most one point, thus v and v' are not adjacent.

The case where neither of v and v' is interior is clear. It remains to consider the case when only one of v and v', say v', is interior. Each real vertex adjacent to v' is inside $\partial(w'\Pi)$. However, $w'\Pi \cap \Pi$ is contained in the lower half of $\partial\Pi$, thus v and v' are not adjacent. □

Lemma 4.25. *Every two vertices in U_l are connected by an edge. The same is true for U_r, D_l and D_r.*

Proof. We only prove U_l spans a complete subgraph, and U_r spans a complete subgraph. The cases for D_l and D_r are similar. First we prove u_i is connected to other vertices in U_l. Note that $p_j u_j = u_i$. However, $u_j \sim c_{j-1}$ and $u_j \sim c_j$. By applying the action of p_j, we have $u_i \sim p_j c_{j-1}$ and $u_i \sim p_j c_j$. Similarly, by using $p_j d_{j+1} = d_{i+1}$, $d_{j+1} \sim c_j$ and $d_{j+1} \sim c_{j+1}$, we know that d_{i+1} is connected to other vertices in U_r.

By Lemma 3.8 (1), $u_j^{-1} c_j \sim u_{j'}^{-1} c_{j'}$ for $1 \leqslant j, j' \leqslant n-2$. By applying the action of u_i, each of $\{p_j c_j\}_{j=1}^{i-1}$ and $\{p_j c_j\}_{j=i+1}^{n-2}$ spans a complete subgraph. Moreover, by the zigzag pattern between $u_j^{-1}\Pi$ and $u_{j'}^{-1}\Pi$, $u_j^{-1} c_{j-1} \sim u_{j'}^{-1} c_{j'-1}$ for $2 \leqslant j, j' \leqslant n-2$ and $u_j^{-1} c_{j+1} \sim u_{j'}^{-1} c_{j'+1}$ for $1 \leqslant j, j' \leqslant n-3$. By Lemma 3.8 (2), we know actually $u_j^{-1} c_{j-1} \sim u_{j'}^{-1} c_{j'-1}$ for $2 \leqslant j, j' \leqslant n-1$. By Figure 8, $c_1 \sim u_{j'}^{-1} c_{j'+1}$ for $1 \leqslant j' \leqslant n-3$. Applying the action of u_i, each of $\{p_j c_{j-1}\}_{j=i+1}^{n-1}$ and $\{p_j c_{j+1}\}_{j=0}^{i-1}$ spans a complete subgraph.

By Lemma 3.8 (1) and (2), $u_j^{-1} c_j \sim u_{j'}^{-1} c_{j'-1}$ for $1 \leqslant j < j' \leqslant n-1$ and $u_j^{-1} c_{j+1} \sim u_{j'}^{-1} c_{j'}$ for $1 \leqslant j < j' \leqslant n-2$. Moreover, $c_1 \sim u_{j'}^{-1} c_{j'}$ for $1 \leqslant j' \leqslant n-2$ (see Figure 8). By applying the action of u_i, we know that a vertex from $\{p_j c_j\}_{j=1}^{i-1}$ (resp. $\{p_j c_{j+1}\}_{j=0}^{i-1}$) and a vertex from $\{p_j c_{j-1}\}_{j=i+1}^{n-1}$ (resp. $\{p_j c_j\}_{j=i+1}^{n-2}$) are adjacent. Now it follows that each of U_l and U_r spans a complete subgraph. □

Now we show U_l and U_r span a prism. This relies on the following 4 lemmas.

Lemma 4.26. *Suppose $i < j \leqslant n-2$. Then a vertex $v \in U_l$ is adjacent to $p_j c_j$ if and only if $v \in \{u_i\} \cup \{p_k c_k\}_{k=1}^{i-1} \cup \{p_k c_{k-1}\}_{k=j}^{n-1}$.*

Proof. Note that $p_j u_j = u_i$. Since $c_j \sim u_j$, $u_i = p_i u_j \sim p_j c_j$. Suppose $1 \leqslant k \leqslant i-1$. Thus $1 \leqslant k < j \leqslant n-2$. By Lemma 3.8 (1), $u_k^{-1} c_k \sim u_j^{-1} c_j$, thus $p_k c_k \sim p_j c_j$. Suppose $i+1 \leqslant k < j$. Then $u_k^{-1} c_{k-1} \nsim u_j^{-1} c_j$ by Lemma 3.8 (1), thus $p_k c_{k-1} \nsim p_j c_j$. It is clear that $p_j c_{j-1} \sim p_i c_j$. Suppose $j < k \leqslant n-1$. Then $u_j^{-1} c_j \sim u_k^{-1} c_{k-1}$ by Lemma 3.8 (1) and (2), hence $p_j c_j \sim p_k c_{k-1}$. □

Lemma 4.27. *A vertex $v \in U_l$ is adjacent to $d_{i+1} \in U_r$ if and only if $v \in \{u_i\} \cup \{p_k c_k\}_{k=1}^{i-1} \cup \{p_{n-1} c_{n-2}\}$.*

Proof. It is clear that $u_i \sim d_{i+1}$. Since $u_i = p_k u_k$, $d_{i+1} = p_k d_{k+1}$. Suppose $1 \leqslant k \leqslant n-2$. Since $d_{k+1} \sim c_k$ and $d_{k+1} \nsim c_{k-1}$, $d_{i+1} = p_k d_{k+1} \sim p_k c_k$ and $d_{i+1} \nsim p_k c_{k-1}$. Suppose $k = n-1$. Then $d_{i+1} = p_{n-1} r$. Since $r \sim c_{n-2}$, $p_{n-1} c_{n-2} \sim p_{n-1} r = d_{i+1}$. □

Lemma 4.28. *A vertex $v \in U_l$ is adjacent to $p_0 c_1 \in U_r$ if and only if $v \in \{u_i\} \cup \{p_k c_k\}_{k=1}^{i-1}$.*

Proof. Recall that $p_0 = u_i$. Since $c_1 \sim l$, $p_0 c_1 \sim u_i l = u_i$. Suppose $1 \leqslant k \leqslant n-2$. Then $c_1 \sim u_k^{-1} c_k$, and $c_1 \nsim u_k^{-1} c_{k-1}$ when $k > 1$. By applying the action of $p_0 = u_i$, $p_0 c_1$ is adjacent to each vertex in $\{p_k c_k\}_{k=1}^{i-1}$ when $\{p_k c_k\}_{k=1}^{i-1}$ is nonempty, and $p_0 c_1$ is adjacent to non of $\{p_k c_{k-1}\}_{k=i+1}^{n-2}$. When $k = n-1$, Π and $u_{n-1}^{-1}\Pi$ only intersect along one edge, so is $p_0\Pi$ and $p_{n-1}\Pi$. Thus interior vertices of $p_0\Pi$ and interior vertices of $p_{n-1}\Pi$ are not adjacent, in particular $p_0 c_1 \nsim p_{n-1} c_{n-2}$. □

Lemma 4.29. *Suppose $1 \leqslant j \leqslant i-1$. A vertex $v \in U_l$ is adjacent to $p_j c_{j+1} \in U_r$ if and only if $v \in \{p_k c_k\}_{k=j}^{i-1}$.*

Proof. Since $c_{j+1} \nsim u_j$. Thus $u_j^{-1} c_{j+1} \nsim u_j^{-1} u_j = l$. Hence $p_j c_{j+1} \nsim u_i l = u_i$. Suppose $1 \leqslant k < j$. Then $u_k^{-1} c_k \nsim u_j^{-1} c_{j+1}$ by Lemma 3.8 (1). Thus $p_k c_k \nsim p_j c_{j+1}$. Suppose $k = j$. Then it is clear that $p_j c_j \sim p_j c_{j+1}$. Suppose $j < k \leqslant n-2$. Then $u_k^{-1} c_k \sim u_j^{-1} c_{j+1}$ and $u_k^{-1} c_{k-1} \nsim u_j^{-1} c_{j+1}$ by Lemma 3.8 (1). Thus $p_k c_k \sim p_j c_{j+1}$ and $p_k c_{k-1} \nsim p_j c_{j+1}$. It follows that $p_j c_{j+1}$ is adjacent to each of $\{p_k c_k\}_{k=j+1}^{i-1}$ and is adjacent to none of $\{p_k c_{k-1}\}_{k=i+1}^{n-2}$. Suppose $k = n-1$. By Lemma 3.8 (2), $u_{n-1}^{-1} c_{n-2} \sim u_j^{-1} c_j$ and $u_{n-1}^{-1} c_{n-2} \sim u_j^{-1} c_{j-1}$. Thus $u_{n-1}^{-1} c_{n-2} \nsim u_j^{-1} c_{j+1}$ and $p_{n-1} c_{n-2} \nsim p_j c_{j+1}$. □

It follows from Lemma 4.26, Lemma 4.27, Lemma 4.28 and Lemma 4.29 that U_l and U_r span a prism with the linear order on U_r given by $p_{i+1} c_{i+1} > \ldots > p_{n-2} c_{n-2} > d_{i+1} > p_0 c_1 > \ldots > p_{i-1} c_i$. Similarly, we can prove D_l and D_r span a prism. Thus all the conditions in Definition 4.3 are satisfied and we have finished the proof of Proposition 4.22.

5. The complexes for Artin groups of almost large type

Let A_Γ be an Artin group with defining graph Γ. Let $\Gamma' \subset \Gamma$ be a full subgraph with induced edge labeling and let $A_{\Gamma'}$ be the Artin group with defining graph Γ'. Then there is a natural homomorphism $A_{\Gamma'} \to A_\Gamma$. By [vdL83], this homomorphism is injective. Subgroups of A_Γ of the form $A_{\Gamma'}$ are called *standard subgroups*.

Let P_Γ be the standard presentation complex of A_Γ, and let X_Γ^* be the universal cover of P_Γ. We orient each edge in P_Γ and label each edge in P_Γ by a generator of A_Γ. Thus edges of X_Γ^* have induced orientation and labeling. There is a natural embedding $P_{\Gamma'} \hookrightarrow P_\Gamma$. Since $A_{\Gamma'} \to A_\Gamma$ is injective, $P_{\Gamma'} \hookrightarrow P_\Gamma$ lifts to various embeddings $X_{\Gamma'}^* \to X_\Gamma^*$. Subcomplexes of X_Γ^* arising in such way are called *standard subcomplexes*.

Now we assume A_Γ is of almost large type. A *block* of X^*_Γ is a standard subcomplex which comes from an edge in Γ. This edge is called the *defining edge* of the block. The block is *large* (resp. *small*) if its defining edge is labeled by an integer $\geqslant 3$ (resp. $= 2$).

We define precells of X^*_Γ as in Section 3.1, and subdivide each precell as in Figure 6 to obtain a simplicial complex $X^\triangle_\Gamma$. Interior vertices and real vertices of $X^\triangle_\Gamma$ are defined in a similar way.

Within each block of $X^\triangle_\Gamma$, we add edges between interior vertices as in Section 3.2. Since each element of A_Γ maps one block to another block with the same defining edge, and the stabilizer of each block is a conjugate of a standard subgroup of A_Γ, one readily verifies that the newly added edges are compatible with the action of deck transformations $A_\Gamma \curvearrowright X^\triangle_\Gamma$. Let X'_Γ be the complex obtained by adding all the new edges, and let X_Γ be the flag completion of X'_Γ. The action $A_\Gamma \curvearrowright X^\triangle_\Gamma$ extends to a simplicial action $A_\Gamma \curvearrowright X_\Gamma$, which is proper and cocompact. A *block* in X_Γ is defined to be the full subcomplex spanned by vertices in a block of $X^\triangle_\Gamma$.

Remark 5.1. It is not hard to see that if two cells are in different blocks of $X^\triangle_\Gamma$, then their intersection is at most one edge. This is why we did not add edges between interior vertices from different blocks.

Lemma 5.2. *The isomorphism between a block in X^*_Γ and the space X^* in Section 3.1 naturally extends to an isomorphism between a block in X_Γ and the space X in Section 3.2.*

Proof. By our construction, it suffices to show that if two vertices v_1 and v_2 in a block $B \subset X^*_\Gamma$ are not adjacent in this block, then they are not adjacent in X^*_Γ. However, this follows from the fact that $B^{(1)}$ is convex with respect to the path metric on the 1–skeleton of X^*_Γ ([CP14]). □

Lemma 5.3. *X_Γ is simply connected.*

Proof. Let f be an edge of X_Γ not in $X^\triangle_\Gamma$. Since ∂f is inside one block, we assume without loss of generality that f connects an interior point of Π and an interior point of $u_i^{-1}\Pi$. Lemma 3.7 (2) implies that f and a vertex in $\Pi \cap u_i^{-1}\Pi$ span a triangle. By flagness of X_Γ, f is homotopic rel its end points to the concatenation of other two sides of this triangle, which is inside $X^\triangle_\Gamma$.

Now we show that each loop in X_Γ is null-homotopic. It suffices to consider the case where this loop is a concatenation of edges of X_Γ. If some edges of this loop are not in $X^\triangle_\Gamma$, then we can homotopy these edges rel their end points to paths in $X^\triangle_\Gamma$ by the previous discussion. Thus this loop is homotopic to a loop in $X^\triangle_\Gamma$, which must be null-homotopic since $X^\triangle_\Gamma$ is simply connected. □

Lemma 5.4. *The link of each vertex in the 1–skeleton $X^{(1)}_\Gamma$ is a 6–large graph.*

Proof. Let $x \in X_{\Gamma}^{(1)}$ be a vertex. If x is an interior vertex, then there is a unique block $B \subset X_{\Gamma}$ containing this vertex, and any other vertex in $X_{\Gamma}^{(1)}$ adjacent to x is contained in this block. Since B is a full subcomplex of X_{Γ}, $\mathrm{lk}(x, X_{\Gamma}^{(1)}) = \mathrm{lk}(x, B^{(1)})$, which is 6–large by Lemma 5.2 and Proposition 4.19.

Let x be a real vertex and let ω be a simple 4–cycle or 5–cycle in $\mathrm{lk}(x, X_{\Gamma}^{(1)})$. We need to show that ω has a diagonal. If the number of real vertices in ω is $\leqslant 1$, then ω is inside a block B. However, $\mathrm{lk}(x, B)$ is 6–large by Proposition 4.5. Thus ω has a diagonal. Now we assume ω has $\geqslant 2$ real vertices. Let $\{v_i\}_{i\in\mathbb{Z}/n\mathbb{Z}}$ be consecutive real vertices on ω (then n is the number of real vertices on ω). Note that the segment $\overline{v_i v_{i+1}}$ of ω is contained in one block, which we denote by B_i. Then $\overline{v_i v_{i+1}}$ is an edge-path in $\mathrm{lk}(x, B_i)$ traveling between two real different vertices (since $n \geqslant 2$). Thus if B_i is large, then $\overline{v_i v_{i+1}}$ has length $\geqslant 2$ by Remark 4.18. Thus the number of large blocks among $\{B_i\}_{i=1}^{n}$ is $\leqslant 2$.

Each v_i arises from an edge between x and v_i. This edge is inside X_{Γ}^{*}, hence it is labeled by a generator of A_{Γ}, corresponding to a vertex $z_i \in \Gamma$. Since v_i corresponds to either an incoming, or an outgoing edge labeled by z_i, we will also write $v_i = z_i^{+}$ or $v_i = z_i^{-}$. Let e_i be the defining edge of B_i. Then

$$z_{i+1} \in e_i \cap e_{i+1}. \tag{1}$$

Case 1: There are two large blocks. Note that there is at most one small block, so the two large blocks must be consecutive. We assume without loss of generality that B_0 and B_1 are large. We claim $B_0 = B_1$, and there are no other blocks. Then ω is inside one block and by Proposition 4.5, it has a diagonal. Now we prove the claim.

First we show there are no other blocks. By contradiction we assume there is another small block B_2. If $B_0 \neq B_1$, then e_0, e_1 and e_2 are pairwise distinct. We deduce from (1) that z_1, z_2 and z_0 form a triangle in Γ which contains an edge labeled by 2, contradiction. If $B_0 = B_1$, then (1) implies $z_2 = z_0$. Thus $\overline{v_2 v_0}$ is an edge path in B_2 from z_2^{+} to z_2^{-}, which has length $\geqslant 3$ by Remark 4.18; and the concatenation of $\overline{v_0 v_1}$ and $\overline{v_1 v_2}$ is an edge path in B_1 from z_2^{+} to z_2^{-}, which also has length $\geqslant 3$ by Remark 4.18. This implies ω has length $\geqslant 6$, which is a contradiction.

Now we show $B_0 = B_1$. If $B_0 \neq B_1$, since there are no other blocks, we must have $z_0 = z_1$ by (1). Now we argue as before to show that the length of ω is $\geqslant 6$, which is a contradiction.

Case 2: There is only one large block. We denote this block by B_0 and claim $n = 1$. If there are other small blocks, then $n \leqslant 4$.

We first show that $n = 4$ is impossible. We argue by contradiction Let B_1, B_2 and B_3 be small blocks. If these small blocks are pairwise distinct, then (1) implies that z_0, z_1, z_2 and z_3 are consecutive vertices in a 4–cycle of Γ with three edges labeled by 2, which is a contradiction. If $B_1 = B_2$ and $B_1 \neq B_3$, then there is a 3–cycle of Γ with two edges labeled by 2, contradiction. Then case $B_3 = B_2$ and $B_1 \neq B_3$ can be ruled out similarly. If $B_1 = B_3$, then (1) implies that $z_0 = z_1$. Thus the segment $\overline{v_0 v_1}$ is an edge path in B_0 from z_0^+ to z_0^-, which has length $\geqslant 3$ by Remark 4.18. On the other hand, the concatenation of $\overline{v_1 v_2}$, $\overline{v_2 v_3}$ and $\overline{v_3 v_0}$ has length $\geqslant 3$, hence ω has length $\geqslant 6$, which is a contradiction.

The case $n = 3$ and $n = 2$ can be ruled out in a similar way, since they either give rise to a 3–cycle of Γ with at least one edge labeled 2, or imply ω has length $\geqslant 6$.

Case 3: there are no large blocks. Then $n \leqslant 5$. If $n \leqslant 4$, or $n = 5$ but the blocks are not pairwise distinct, then by the argument as before, we can either deduce that there is a 3–cycle or 4–cycle in Γ such that all of its edges are labeled by 2, or ω has length $\geqslant 6$. It remains to consider the case $n = 5$ and the blocks are pairwise distinct. Then (1) implies that the defining edges of these blocks form a 5–cycle in Γ. Since the length of ω is $\leqslant 5$, v_i and v_{i+1} are adjacent for all i. Suppose without loss of generality that $v_0 = z_0^+$, then we must have $v_1 = z_1^-$ by Remark 4.18 (2). Similarly, $v_2 = z_2^+$, $v_3 = z_3^-$, $v_4 = z_4^+$ and $v_0 = z_0^-$, which is a contradiction. □

The following is a direct consequence of Lemma 5.3 and Lemma 5.4.

Theorem 5.5. *The complex X_Γ is systolic. Hence if A_Γ is an Artin group of almost large type, then it acts properly and cocompactly by automorphisms on a systolic complex X_Γ.*

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Dept. of Math. & Stats., McGill University, Montreal, Quebec, Canada H3A 0B9

E-mail address: jingyin.huang@mcgill.ca

Instytut Matematyczny, Uniwersytet Wrocławski, pl. Grunwaldzki 2/4, 50–384 Wrocław, Poland

Institute of Mathematics, Polish Academy of Sciences, Śniadeckich 8, 00-656 Warszawa, Poland

Dept. of Math. & Stats., McGill University, Montreal, Quebec, Canada H3A 0B9

E-mail address: dosaj@math.uni.wroc.pl