TWO-DIMENSIONAL SYSTOLIC COMPLEXES SATISFY PROPERTY A

NIMA HODA

Department of Mathematics and Statistics, McGill University
Burnside Hall, Room 1005
805 Sherbrooke Street West
Montreal, QC, H3A 0B9, Canada

DAMIAN OSAJDA

Instytut Matematyczny, Uniwersytet Wrocławski
pl. Grunwaldzki 2/4, 50–384 Wrocław, Poland

Institute of Mathematics, Polish Academy of Sciences
Śniadeckich 8, 00-656 Warszawa, Poland

Abstract. We show that 2-dimensional systolic complexes are quasi-isometric to quadric complexes with flat intervals. We use this fact along with the weight function of Brodzki, Campbell, Guentner, Niblo and Wright [5] to prove that 2-dimensional systolic complexes satisfy Property A.

1. Introduction

Property A is a quasi-isometry invariant of metric spaces introduced by Guoliang Yu in his study of the Baum-Connes conjecture [16]. It may be thought of as a non-equivariant version of amenability. As is the case for amenability, Property A has plenty of equivalent formulations. In particular, for finitely generated groups, it is equivalent to the exactness of the reduced C^*–algebra of the group and also to the existence of an amenable action on a compact space [10, 14]. Property A implies coarse embeddability into Hilbert space and hence the coarse Baum-Connes Conjecture and the Novikov Conjecture. Classes of groups for which Property A holds include Gromov hyperbolic groups [1], CAT(0) cubical groups [7], and uniform lattices in affine buildings [6], but it is an open question whether it holds for all CAT(0) groups.

In the current article we prove the following.

E-mail addresses: `nima.hoda@mail.mcgill.ca`, `dosaj@math.uni.wroc.pl`.

Date: August 7, 2018.

2010 *Mathematics Subject Classification.* 20F65, 20F69, 57M20.

Key words and phrases. systolic complex, CAT(0) triangle complex, property A, boundary amenability, exact group.

Main Theorem. *Two-dimensional systolic complexes satisfy Property A.*

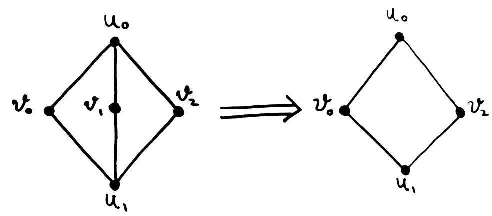
A 2-dimensional *systolic complex* can be defined as a 2-dimensional simplicial complex which is CAT(0) when equipped with a metric in which each triangle is isometric to an equilateral Euclidean triangle. The class of isometry groups of such complexes is vast. It contains many Gromov hyperbolic groups. It also contains lattices in $\tilde{A}_2$ buildings, which were already proved to satisfy Property A by Campbell [6]. Some of these groups satisfy interesting additional properties such as Kazhdan's property (T). Notably, there are numerous well developed techniques for constructing groups acting on 2-dimensional systolic complexes (with various additional features) making them a rich source of examples. For instance, given a finite group F and a generating set $S \subseteq F \setminus \{1\}$ whose Cayley graph $\Gamma(F, S)$ has girth at least 6, Ballmann and Świątkowski [2, Theorem 2 and Section 4] construct a canonical infinite 2-dimensional systolic complex X whose oriented triangles are labeled by elements of $S \cup S^{-1}$ and in which the link of every vertex is isomorphic to $\Gamma(F, S)$ with labels induced from the triangles. The labeled automorphisms of X act simply transitively on the oriented 1-simplices of X and if (F, S) has a relation of the form $(st)^3$ with $s, t \in S$ then X has flat planes and so is not hyperbolic. This construction is a particular case of a development of a complex of groups as described in Bridson and Haefliger [4, Example 4.19(2) of Chapter III.$\mathcal{C}$].

We prove the Main Theorem by showing that 2-dimensional systolic complexes are quasi-isometric to quadric complexes whose intervals with respect to a basepoint are CAT(0) square complexes (Theorem 3.1, Theorem 3.2 and Theorem 3.3). This allows us to apply the weight function and uniform convergence argument of Brodzki, Campbell, Guentner, Niblo and Wright [5] in their proof that finitely dimensional CAT(0) cube complexes satisfy Property A (Theorem 2.6).

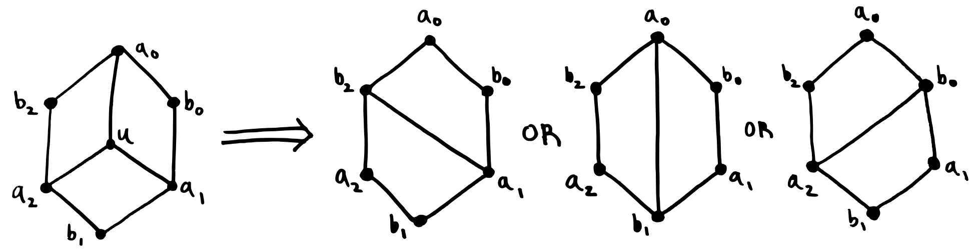
Acknowledgements. The authors would like to thank Jacek Świątkowski for some helpful discussions on the construction described above of 2-dimensional systolic developments. N.H. was partially funded by an NSERC CGS M. D.O. was partially supported by (Polish) Narodowe Centrum Nauki, grant no. UMO-2015/18/M/ST1/00050. Parts of this research were carried out while D.O. was visiting McGill University and while N.H. was visiting the University of Wrocław. The authors would like to thank both institutions for their hospitality.

2. Preliminaries

For basic algebraic topological notions such as those of *CW complexes* and *simple connectedness* we refer the reader to Hatcher [9]. A *combinatorial map* $X \to Y$ between CW complexes is one whose restriction to each cell of X is a homeomorphism onto a cell of Y. All graphs considered in this paper are simplicial. We consider metrics only on the 0-skeleton X^0 of a cell complex X as induced by the shortest path metric on the 1-skeleton X^1. The *metric ball* $B_r(v)$ (respectively, *metric sphere* $S_r(v)$) of radius r centered at a vertex v in a complex is the subgraph induced in the 1-skeleton by the set of vertices of distance at most (respectively, exactly) r from v. The *girth* of a graph is the length of the embedded shortest cycle. The *link* of a vertex v in a 2-dimensional simplicial complex is the graph whose vertices are the neighbours of v and where two vertices are joined by an edge if they span a triangle together with v.



(A)



(B)

FIGURE 1. Replacement rules for quadric complexes.

2.1. **Property A.** Rather than defining Property A in full generality, we give the characterization for graphs of Brodzki, Campbell, Guentner, Niblo and Wright [5, Proposition 1.5]. A graph Γ satisfies *Property A* iff there exists a sequence of constants $(C_n)_{n\in\mathbb{N}}$ and a family of functions $f_{n,v}\colon \Gamma^0 \to \mathbb{N}$ indexed by $\mathbb{N}\times\Gamma^0$ such that the following conditions hold.

(1) $f_{n,v}$ is supported on $B_{C_n}(v)$.
(2) $\frac{||f_{n,v}-f_{n,v'}||_1}{||f_{n,v}||_1} \to 0$ uniformly over all pairs of vertices (v,v') joined by an edge.

2.2. **Two-dimensional systolic complexes.** A 2*-dimensional systolic complex* is a simply connected 2-dimensional simplicial complex in which the girth of the link of every vertex is at least 6.

The following are well known properties of systolic complexes. See Chepoi [8, Theorem 8.1] and Januszkiewicz and Świątkowski [12, Lemma 7.7].

Lemma 2.1. *Let Y be a 2-dimensional systolic complex.*

(1) *Metric spheres in Y are triangle-free.*
(2) *Let $u, v \in Y^0$. Let $w, x \in Y^0$ be neighbours of v that are closer to u than is v. Then w and x are joined by an edge.*
(3) *(Triangle Condition) Let $u, v, w \in Y^0$ such that v and w are joined by an edge and they are equidistant to u. Then there exists $x \in Y^0$ adjacent to v and w and closer to u than are v and w.*

2.3. **Quadric complexes.** A *square complex* is a 2-dimensional CW complex with a simplicial 1-skeleton such that the attaching maps of 2-cells are injective combinatorial maps from 4-cycles. The closed 2-cells of a square complex are called *squares.* We assume that no two squares of a square complex are glued to the same 4-cycle of the 1-skeleton. A *quadric complex* is a simply connected square complex in which, for any subcomplex as on the left-hand side of Figure 1, there exists at

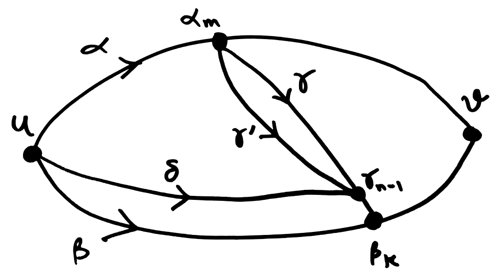


FIGURE 2. Two geodesics α and β between a pair of vertices u and v of a quadric complex along with a geodesic γ joining $\alpha_m \in \alpha$ and $\beta_k \in \beta$. Used in the proof of Theorem 2.4.

least one subcomplex as on its right-hand side having the same boundary path. In other words, quadric complexes are simply connected generalized $(4, 4)$-complexes, as defined by Wise [15], that are built out of squares. They were first studied in depth by Hoda [11]. A quadric complex is a CAT(0) square complex if and only if its 1-skeleton is $K_{2,3}$-free.

Theorem 2.2 (Hoda [11]). *Let X be a quadric complex. Metric balls are isometrically embedded in X^1.*

Lemma 2.3 (Quadrangle Condition). *Let X be a quadric complex. Let $u, v, w, x \in Y^0$ such that v and w are adjacent to x and v and w are closer to u than is x. Then there exists $y \in X^0$ adjacent to v and w and closer to u than are v and w.*

Lemma 2.3 follows from the fact that the 1-skeleta of quadric complexes are precisely the hereditary modular graphs. This characterization is due to the result of Hoda that the 1-skeleta are the bi-bridged graphs [11] and the theorem of Bandelt that a graph is bi-bridged iff it is hereditary modular [3, Theorem 1].

Let X be a quadric complex. The *interval* $I(u, v)$ between a pair of vertices u and v in X^0 is the full subcomplex induced by the union of the geodesics between u and v.

Theorem 2.4. *Let X be a quadric complex and let $u, v \in X^0$. The 1-skeleton of $I(u, v)$ is isometrically embedded in X^1.*

Proof. Suppose not. Then there are geodesics $(\alpha_i)_{i=0}^{\ell}$ and $(\beta_i)_{i=0}^{\ell}$ from u to v and indices m and k such that no geodesic $(\gamma_i)_{i=0}^{n}$ from α_m to β_k is contained in $I(u, v)$. Choose $(\alpha_i)_i$, $(\beta_i)_i$, m and k so as to minimize $n = d(\alpha_m, \beta_k)$. Without loss of generality, $m \le k$.

By Theorem 2.2 we may assume that $(\gamma_i)_i$ is contained in $B_k(u)$. Hence, since X^1 is bipartite, $d(u, \gamma_{n-1}) = k - 1$. Let $(\delta_i)_{i=0}^{k-1}$ be a geodesic from u to γ_{n-1}. Let $(\beta'_i)_{i=0}^{\ell}$ be the concatenation of the sequences $(\delta_i)_{i=0}^{k-1}$ and $(\beta_i)_{i=k}^{\ell}$. Then $(\beta'_i)_i$ is a geodesic from u to v. By the minimality of $\big((\alpha_i)_i, (\beta_i)_i, m, k\big)$, there is a geodesic $(\gamma'_i)_{i=0}^{n-1}$ from α_m to $\beta'_{k-1} = \gamma_{n-1}$ such that $(\gamma'_i)_i$ is contained in $I(u, v)$. But then appending γ_n to $(\gamma'_i)_i$ we obtain a geodesic from α_m to β_k that is contained in $I(u, v)$. This is a contradiction. $\square$

Corollary 2.5. *Intervals in quadric complexes are quadric.*

Proof. Quadric complexes are characterized by metric properties of their 1-skeleta [11], so an isometrically embedded full subcomplex of a quadric complex is quadric. □

Fix a basepoint $* \in X^0$. If, for all $v \in X^0$, the interval $I(*, v)$ is a CAT(0) square complex then $(X, *)$ *has flat intervals*. By Corollary 2.5, $(X, *)$ has flat intervals if and only if every $I(*, v)$ is $K_{2,3}$-free.

We now describe how we apply the results of Brodzki et al. [5] in the special case of 2-dimensional CAT(0) cube complexes to our present situation.

Let $(X, *)$ be a based quadric complex with flat intervals. Let $Z_v = I(*, v)$ be a CAT(0) square complex interval in a quadric complex. We will describe the weight function $f_{n,v}$ of Brodzki et al. [5] for v in Z_v. For $w \in Z_v^0$, let $\rho(w)$ be the number of neighbours of w in Z_v^1 that lie on geodesics from w to $*$. The *deficiency* of $w \in Z_v^0$ is defined as follows.

$$\delta(w) = 2 - \rho(w)$$

Define $f_{n,v} \colon Z_v^0 \to \mathbb{N}$ as follows.

$$f_{n,v}(w) = \begin{cases} 0 & \text{if } d(w,v) > n \\ 1 & \text{if } d(w,v) \leq n \text{ and } \delta(w) = 0 \\ n - d(w,v) + 1 & \text{if } d(w,v) \leq n \text{ and } \delta(w) = 1 \\ \frac{1}{2}\big(n - d(w,v) + 2\big)\big(n - d(w,v) + 1\big) & \text{if } d(w,v) \leq n \text{ and } \delta(w) = 2 \end{cases}$$

We extend $f_{n,v}$ by zeroes to all of X^0. Note that if v' is a neighbour of v, then $Z_{v'} \subseteq Z_v$ or $Z_v \subseteq Z_{v'}$, say the latter, and that Z_v and $Z_{v'}$ are both intervals of $*$ in $Z_{v'}$, which by flatness is a CAT(0) square complex. So we may apply the results of Brodzki et al. that

$$||f_{n,v}||_1 = \frac{1}{2}(n+2)(n+1)$$

[5, Proposition 3.10] and for a neighbour v' of v,

$$||f_{n,v} - f_{n,v'}||_1 = 2(n+1)$$

[5, Proposition 3.11] and so we have the following.

Theorem 2.6. *Let X be a quadric complex. If there exists $* \in X^0$ such that $(X, *)$ has flat intervals then X satisfies Property A.*

3. The squaring of 2-dimensional systolic complexes

Let $(Y, *)$ be a 2-dimensional systolic complex with a basepoint $* \in Y^0$. Let X_Y^1 be the subgraph of Y^1 given by the union of all edges whose endpoints are not equidistant to u. Note that X_Y^1 is bipartite The *squaring* of $(Y, *)$ is the based square complex $(X_Y, *)$ obtained from X_Y^1 by attaching a unique square along its boundary to each embedded 4-cycle of X_Y^1.

Theorem 3.1. *Let $(Y, *)$ be a based 2-dimensional systolic complex and let $(X_Y, *)$ be the squaring of $(Y, *)$. Then Y^1 is quasi-isometric to X_Y^1.*

Proof. Applying Lemma 2.1(3) to $*$ and an edge e of Y^1 that is not contained in X_Y^1 gives us a triangle, one of whose edges is e and whose remaining edges are contained in X_Y^1. This ensures that distances in X_Y^1 increase by at most a factor of two relative to distances in Y^1. □

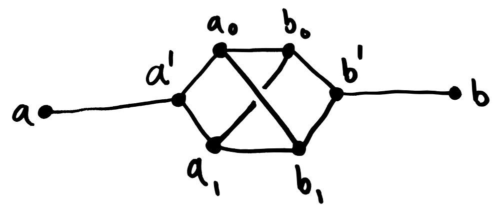


FIGURE 3. A $K_{2,2}$ spanning the vertices a_0, a_1, b_0 and b_1 and embedded in a particular way in the interval $I(a,b)$ of a graph. Such an embedding is not possible in the squaring of a based 2-dimensional systolic complex. Used in the proof of Theorem 3.3.

Theorem 3.2. *Let $(Y,*)$ be a based 2-dimensional systolic complex. The squaring $(X_Y,*)$ of $(Y,*)$ is quadric.*

Proof. We need to show that X_Y is simply connected and that, for every subgraph of X_Y^1 as in the left-hand side of Figure 1B, a pair of antipodal vertices in the outer 6-cycle is joined by an edge.

To show that X_Y is simply connected it suffices to show that, for every embedded cycle α of X_Y^1, there is a 2-dimensional square complex D' homeomorphic to a 2-dimensional disk and a combinatorial map $D' \to X_Y$ whose restriction to the boundary $\partial D'$ of D' is α. By the van Kampen Lemma [13, Proposition 9.2 of Section III.9] since Y is simply connected, there exists a 2-dimensional simplicial complex D homeomorphic to a 2-disk D and a combinatorial map $D \to Y$ which restricts to α on the boundary. Choose such $D \to Y$ so as to minimize the number of triangles of D. By Lemma 2.1(1), each triangle of D has a unique edge e that is not contained in X_Y^1. Then e is contained in the interior of D and, by the minimality of $D \to Y$, the star of e is embedded in Y. Let D' be the result of deleting all such e from D^1 and then spanning a square on each embedded 4-cycle. Since every embedded 4-cycle of X_Y spans a square, we may extend $(D')^1 \to (X_Y)^1$ to $D' \to X_Y$. This proves that X_Y is simply connected.

Let W be a subgraph of X_Y^1 as in the left-hand side of Figure 1B and with the same vertex labels. By 6-largeness of Y, each of the embedded 4-cycles of W have a diagonal. Since the girth of the link of u is at least 6, these diagonals must join u to each of the b_i. Hence u is adjacent to every vertex in the outer 6-cycle C of W.

Let v be a furthest vertex of C from $*$. By Lemma 2.1(2), the neighbours of v in C are joined by an edge. But then there is a 5-cycle in the link of u which contradicts the 2-dimensional systolicity of Y. □

Theorem 3.3. *Let $(Y,*)$ be a based 2-dimensional systolic complex. The squaring $(X_Y,*)$ of $(Y,*)$ has flat intervals.*

Proof. Suppose there is a $K_{2,3}$ in an interval $I(*,v)$ of X_Y. Let $\{a_0,a_1\}\sqcup\{b_0,b_1,b_2\}$ be the bipartition of the $K_{2,3}$. Some pair of vertices of $\{b_0,b_1,b_2\}$ are equidistant to $*$, say $\{b_0,b_1\}$.

Consider the case where a_0 and a_1 are equidistant to $*$. Let a be the closer of $*$ and v to a_0 and let b be the closer of $*$ and v to b_0. Let a' and b' be obtained by applying Lemma 2.3 to a_0, a_1 and a and to b_0, b_1 and b, as in Figure 3. By 6-largeness of Y, the 4-cycle (a_0,a',a_1,b_0) has a diagonal in Y^1. Since (a',a_0,b_0)

is a geodesic, the diagonal must join a_0 and a_1. Similarly, b_0 and b_1 are joined by edge in Y^1 and hence, by flagness, $\{a_0, a_1, b_0, b_1\}$ spans a 3-simplex in Y. This contradicts the 2-dimensionality of Y.

In the remaining case a_0 and a_1 are not equidistant to $*$. Then the b_i must all be equidistant to $*$ with the (a_0, b_i, a_1) all geodesics. Applying a similar argument as in the previous case to the 4-cycles (a_0, b_i, a_1, b_j) we see that the b_i span a triangle in Y and so, together with a_0, they span a 3-simplex contradicting, again, the 2-dimensionality of Y. $\square$

As an immediate consequence of Theorem 3.1, Theorem 3.2, Theorem 3.3 and Theorem 2.6 we have the Main Theorem.

References

[1] S. Adams. Boundary amenability for word hyperbolic groups and an application to smooth dynamics of simple groups. *Topology*, 33(4):765–783, 1994.

[2] W. Ballmann and J. Świątkowski. On L^2-cohomology and property (T) for automorphism groups of polyhedral cell complexes. *Geom. Funct. Anal.*, 7(4):615–645, 1997.

[3] H.-J. Bandelt. Hereditary modular graphs. *Combinatorica*, 8(2):149–157, 1988.

[4] M. Bridson and A. Haefliger. *Metric spaces of non-positive curvature*, volume 319. Springer, 1999.

[5] J. Brodzki, S. J. Campbell, E. Guentner, G. A. Niblo, and N. J. Wright. Property A and CAT(0) cube complexes. *J. Funct. Anal.*, 256(5):1408–1431, 2009.

[6] S. J. Campbell. Property A and affine buildings. *J. Funct. Anal.*, 256(2):409–431, 2009.

[7] S. J. Campbell and G. A. Niblo. Hilbert space compression and exactness of discrete groups. *J. Funct. Anal.*, 222(2):292–305, 2005.

[8] V. Chepoi. Graphs of some CAT(0) complexes. *Adv. in Appl. Math.*, 24(2):125–179, 2000.

[9] A. Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[10] N. Higson and J. Roe. Amenable group actions and the Novikov conjecture. *J. Reine Angew. Math.*, 519:143–153, 2000.

[11] N. Hoda. Quadric complexes. *Preprint, arXiv:1711.05844*, 2017.

[12] T. Januszkiewicz and J. Świątkowski. Simplicial nonpositive curvature. *Publ. Math. Inst. Hautes Études Sci.*, 104(1):1–85, 2006.

[13] R. C. Lyndon and P. E. Schupp. *Combinatorial group theory.* Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1977 edition.

[14] N. Ozawa. Amenable actions and exactness for discrete groups. *C. R. Acad. Sci. Paris Sér. I Math.*, 330(8):691–695, 2000.

[15] D. T. Wise. Sixtolic complexes and their fundamental groups. Preprint, 2003.

[16] G. Yu. The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space. *Invent. Math.*, 139(1):201–240, 2000.