TORSION GROUPS DO NOT ACT ON 2-DIMENSIONAL CAT(0) COMPLEXES

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Abstract. We show, under mild hypotheses, that if each element of a finitely generated group acting on a 2-dimensional CAT(0) complex has a fixed point, then there is a global fixed point. In particular all actions of finitely generated torsion groups on such complexes have global fixed points. The proofs rely on Masur's theorem on periodic trajectories in rational billiards, and Ballmann–Brin's methods for finding closed geodesics in 2-dimensional locally CAT(0) complexes. As another ingredient we prove that the image of an immersed loop in a graph of girth 2π with length not commensurable with π has diameter $> \pi$. This is closely related to a theorem of Dehn on tiling rectangles by squares.

1. Introduction

Let H be a group acting properly and cocompactly on a 2-dimensional CAT(0) complex X. In [BB95] Ballmann and Brin proved *Rank Rigidity* for H, saying that if each edge of X is in at least two 2-cells, then either H has an element of rank 1, or X is a Euclidean building. Two other strongly related questions on CAT(0) groups have remained open even in the same 2-dimensional setting. The first one is the *Tits Alternative* stating that subgroups of H are either virtually abelian or contain free subgroups. The second one is even more basic (as it might be seen as a first step for proving the Tits Alternative):

Question. *Can H have an infinite torsion subgroup G?*

(See e.g. [Swe99], [Bes00, Quest 2.11], [Bri07, Quest 8.2], [Kap08, Prob 24], [Cap14, § IV.5] for appearances of the problem.) Surprisingly, prior to our work the answer was not known even in the otherwise very well understood case of lattices in isometry groups of Euclidean buildings of type $\widetilde{A}_2$. Swenson [Swe99] proved that for 1-ended H, negative answer to the Question is equivalent to no cut points in the CAT(0) boundary of X. Moreover, he answered the Question in the negative for $G = H$. (Both independent of the dimension of the CAT(0) space.) Furthermore, if G is finitely presented, then it also acts properly and cocompactly on a 2-dimensional CAT(0) complex [HMP14, §1.1], so the answer stays negative.

Generalising the Question by discarding H one can ask if there is an infinite torsion group G acting nontrivially on a 2-dimensional CAT(0) complex. (We say that an action is *trivial* if it has a global fixed point.) It is natural to ask that for G finitely generated, since infinitely generated torsion groups can act properly on trees (see [BH99, II.7.11]). We answer that, as well as the Question, in the negative in Corollaries 1.3 and 1.4. These are two consequences of our main Theorem 1.1, concerning more general actions of groups on 2-dimensional CAT(0) complexes. Under mild hypotheses, it states that locally elliptic actions are trivial. This result as well as both corollaries are new even in the case of 2-dimensional Euclidean buildings. In particular, in this setting they confirm [Mar15, Conj 1.2], and extend

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results of [Par03]. Note that finite-dimensionality assumption is important, since there are infinite torsion Grigorchuk groups that are amenable [Gri84], and hence they act properly on a Hilbert space, which is CAT(0).

We now pass to discussing the general setup for Theorem 1.1. Let X be a 2-dimensional simplicial complex, which we will call a *triangle complex*. We assume additionally that X has a *piecewise smooth Riemannian metric*, which is a family of smooth Riemannian metrics σ_T, σ_e on the triangles and edges such that σ_T restricts to σ_e for each $e \subset T$. Riemannian metrics σ_T, σ_e induce metrics (i.e. distance functions) d_T, d_e on triangles and edges. We then equip X with the *quotient pseudometric* d (see [BH99, I.5.19]).

We assume that the triangles containing each vertex v of X belong to only finitely many isometry classes of σ_T. We also assume that (X, d) is a complete length space. This holds for example:

- if there are only finitely many isometry classes of σ_T (see [BH99, I.7.13] for X an M_κ *complex*, meaning that each σ_T has Gaussian curvature κ and geodesic sides; the general case follows using a bilipschitz map from X to an M_κ complex), or
- if X is the space $\mathbf{X}$ for the tame automorphism group ([LP18, Prop 5.4, Lem 5.6]). Some cells of $\mathbf{X}$ are polygons instead of triangles, but we can easily transform $\mathbf{X}$ into a triangle complex by subdividing.

We consider an action of a group G on X by *automorphisms*, i.e. simplicial automorphisms that preserve the metrics σ_T. Consequently, they are isometries of (X, d).

See Section 2 for the discussion of the CAT(0) property and the definition of having *rational angles*. For example, if the triangles of X have all angles commensurable with π, then X has rational angles.

Theorem 1.1. *Let (X, d) be a* CAT(0) *triangle complex. Let G be a finitely generated group acting nontrivially on X. Assume that*
 (i) each element of G fixing a point of X has finite order, or
 (ii) X is locally finite, or
 (iii) X has rational angles.
Then G has an element with no fixed point in X.

While we believe that Theorem 1.1 holds also without (i),(ii), and (iii), this does not seem to be tractable with our methods. For example, one of them would require in particular the existence of periodic trajectories in triangular biliards, which is a major open problem (see e.g. [Mas86]).

Applying Theorem 1.1 and then [CL10, Thm 1.1] to the family of fixed point sets for all finitely generated subgroups G of H below gives:

Corollary 1.2. *Let (X, d) be a* CAT(0) *triangle complex. Let H be a group acting on X such that each element of H fixes a point of X. Assume that (i),(ii), or (iii) holds. Then the action of H on $X \cup \partial X$ is trivial.*

Theorem 1.1 under hypothesis (i) can be rephrased in the following way.

Corollary 1.3. *Let (X, d) be a* CAT(0) *triangle complex. Let G be a finitely generated group acting nontrivially on X. Then G has an element of infinite order.*

The following corollary answers the Question. Note that the finite generation assumption is removed here.

Corollary 1.4. *Let (X, d) be a* CAT(0) *triangle complex. Let H be a group acting properly and cocompactly on X. Then any subgroup G of H is finite or has an element of infinite order.*

Proof. Suppose that all elements of G have finite order. By [BH99, II.2.8(2)], there is a finite bound on the size of all finite subgroups of H. Let then F be a maximal finite subgroup of G. We will prove that $F = G$. Otherwise, there is $g \in G - F$. Then $\langle F, g\rangle$ is finitely generated and thus by Corollary 1.3 it is finite. This contradicts the maximality of F. □

In view of the Question and Corollary 1.3 we state the following conjecture, which we could not find elsewhere in the literature. Observe that, as in the proof of Corollary 1.4, the conjecture implies negative answer to the Question in the setting of finite-dimensional CAT(0) complexes.

Conjecture 1.5. *Every finitely generated group acting nontrivially on a finite-dimensional* CAT(0) *complex contains an element of infinite order.*

In the proof of Theorem 1.1 under conditions (i) or (ii) we will need the following graph-theoretic result of independent interest.

Below, by a *graph* we mean a (possibly infinite) metric graph with finitely many possible edge lengths. A closed edge-path embedded in a graph Λ is a *cycle of* Λ. An embedded edge-path P in Λ is a *segment of* Λ if the endpoints of P have degree at least three in Λ, but every internal vertex of P has degree two.

Theorem 1.6. *Let Γ be a graph, and let Γ' be its finite subgraph with all vertices of degree at least two. Assume that*

- *the girth of Γ is $\geq 2\pi$, and*
- *each x, y in Γ' are at distance $\leq \pi$ in Γ.*

Then all cycles and segments of Γ' have length commensurable with π.

Corollary 1.7. *Let Γ be a graph, and let $\gamma\colon C \to \Gamma$ be a closed edge-path immersed in Γ. Assume that*

- *the girth of Γ is $\geq 2\pi$, and*
- *each x, y in $\gamma(C)$ are at distance $\leq \pi$ in Γ.*

Then the length of γ is commensurable with π.

Interestingly, Theorem 1.6 will follow from a generalisation of a classical theorem of Dehn [Deh03] stating that a rectangle tiled by finitely many squares has commensurable side lengths.

Idea of proof of Theorem 1.1. Suppose by contradiction that for each $f \in G$ the set $\mathrm{Fix}(f)$ of fixed points of f is nonempty. We wish to prove inductively that for any finite set of elements $f_1, \ldots, f_n \in G$ (ultimately becoming the generating set) the intersection $\mathrm{Fix}(f_1) \cap \cdots \cap \mathrm{Fix}(f_n)$ is nonempty. Here we illustrate the induction step from $n = 2$ to $n = 3$. Suppose that $\mathrm{Fix}(f_1), \mathrm{Fix}(f_2), \mathrm{Fix}(f_3)$ pairwise intersect but their triple intersection is empty. We then find a simplicial disc Δ with decomposition of its boundary into three paths $\partial\Delta = P \cup Q \cup R$ together with a simplicial map ψ from Δ to X that sends P, Q, R into $\mathrm{Fix}(f), \mathrm{Fix}(g), \mathrm{Fix}(h)$, respectively. We suppose that Δ has minimal possible area. Under hypothesis (iii) of rational angles we find a periodic billiard trajectory ω in Δ outside its 0-skeleton Σ. Then $\psi(\omega)$ develops to an axis in X of a loxodromic element of G and leads to contradiction. If the angle at a vertex $v \in \Sigma$ is not commensurable with π, then developing the image under ψ of the link at v and using hypothesis (i) or (ii) we construct a closed edge-path in the link of $\psi(v)$ in X of length not commensurable with π. Applying Corollary 1.7 to that path we obtain directions at $\psi(v)$ at distance $> \pi$. These directions permit to construct a trajectory through v that develops to an axis of a loxodromic element.

Organisation. In Section 2 we discuss CAT(0) spaces and rational angles property. In Section 3 we outline the proof of Theorem 1.1 for 2-generated G. In Sections 4 and 5 we fill in the details of that outline. In Section 6 we complete the proof of Theorem 1.1. The proof of Theorem 1.6 is postponed till Section 7.

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2. Rational angles

Let v be a vertex of the triangle complex X. By lk_v we denote the *space of directions* at v, which is the set of geodesic rays issuing from v, where we identify rays at Alexandrov angle 0 (see [BH99, II.3.18]). Note that lk_v has an obvious structure of a graph. We treat lk_v as a length metric space (lk_v, d_v) where the length of each edge is the angle in an appropriate triangle of X. Since we assumed that triangles containing v belong to only finitely many isometry classes of σ_T, there are only finite many possible edge lengths in a given lk_v.

For a piecewise smooth simplicial map $\phi\colon X \to X'$ we keep the notation ϕ for all the maps $\mathrm{lk}_v \to \mathrm{lk}_{\phi(v)}$ induced by ϕ.

For X to be CAT(0) it is necessary that

- the girth of each lk_v is $\geq 2\pi$,
- the Gaussian curvature of σ_T at any interior point of T is ≤ 0,
- the sum of geodesic curvatures in any two distinct triangles at any interior point of a common edge is ≤ 0, and
- X is simply connected.

While it seems that these conditions are also sufficient (one would need to couple the ideas from [BB96] and [BH99]), only under the following conditions this has been verified in the literature.

- X is locally finite [BB96, Thm 7.1], or
- X is an M_κ complex with $\kappa \leq 0$ and there are only finitely many isometry classes of σ_T [BH99, II.5.2 and II.5.4], or
- X is the space $\mathbf{X}$ for the tame automorphism group [LP18, Thm A].

Definition 2.1. We say that X has *rational angles* with respect to an action of a group G, if for each vertex v of X we have a discrete set $\Lambda \subset \mathrm{lk}_v$, invariant under the stabiliser G_v, and with the following property. Consider an immersed edge-path γ in lk_v that is disjoint from Λ except possibly at the endpoints. We call γ *flat* if

(i) each edge of γ corresponds to a triangle T of X with Gaussian curvature 0 at any interior point, and
(ii) the sum of geodesic curvatures in any two such consecutive triangles at any interior point of the common edge is 0.

We require that each flat γ is

- finite, and
- if it has endpoints in Λ, then its length is commensurable with π.

For example, if all triangles of X with σ_T of constant Gaussian curvature 0 have only angles commensurable with π, then X has rational angles, since we can take for Λ all the vertices of lk_v. This includes all $\widetilde{A}_2, \widetilde{C}_2$, and $\widetilde{G}_2$ buildings.

Lemma 2.2. *The space* $\mathbf{X}$ *for the tame automorphism group constructed in* [LP18] *has rational angles.*

Proof. The proof is aimed at readers familiar with [LP18]. Let $\rho_+ : \mathbf{X} \to \nabla_+$ be the folding from [LP18, Cor 2.4]. Let $P \subset \mathrm{lk}_{\rho_+(v)}$ be the directions determined by the *principal lines*: through $\rho_+(v)$ and $[1,0,0]$, $[0,1,0]$, or $[0,0,1]$. Let $\Lambda = \rho_+^{-1}(P)$. These are the only possible directions of edges from $\rho_+(v)$ that are geodesics, since all edges lie in admissible lines (see [LP18, §4.A]) and the only admissible lines that are geodesics are principal (equality case in [LP18, Lem 5.1]). Thus by Definition 2.1(ii), the image of γ under ρ_+ in $\mathrm{lk}_{\rho_+(v)}$ might not be immersed only at P. Since consecutive points of P are at distance $\frac{\pi}{3}$ (see [LP18, Rm 5.2]), the length of γ is bounded by $\frac{\pi}{3}$ with equality if and only if γ has endpoints in Λ. □

3. Pairs of generators

In this section we prove Theorem 1.1 for 2-generated G. For an automorphism f of X, $\mathrm{Fix}(f) \subset X$ denotes the set of fixed points of f.

Proposition 3.1. *Let f, g be automorphisms of a* CAT(0) *triangle complex X satisfying condition (i),(ii), or (iii) of Theorem 1.1. Suppose that both* $\mathrm{Fix}(f), \mathrm{Fix}(g)$ *are nonempty. Then* $\mathrm{Fix}(f)$ *intersects* $\mathrm{Fix}(g)$*, or $\langle f, g\rangle$ contains an element with no fixed point in X.*

In the proof we will need the following terminology. We say that an automorphism f of X acts *without inversions*, if whenever f stabilises a triangle T or an edge e, then it fixes T or e pointwise.

Definition 3.2. Let S be a sphere with three marked points p, q, r. Let f_S, g_S and $h_S = g_S^{-1} \circ f_S$ be the elements of the fundamental group of $S - \{p, q, r\}$ corresponding to the punctures p, q, r, respectively. Let $\widetilde{S}$ be the branched cover of S over p, q, r corresponding to $\pi_1(S - \{p, q, r\})$. For a pair (f, g) of automorphisms of X, an *equivariant triangulation* of $\widetilde{S}$ *w.r.t.* (f, g) is the following object.

First of all, it consists of a structure on S of a Δ-*complex* (as defined in [Hat02, §2.1]), which is a generalisation of a simplicial complex allowing the simplices not to be embedded, and allowing several simplices with the same boundary. We require that p, q, and r are vertices. This induces a Δ-complex structure on $\widetilde{S}$.

Secondly, an equivariant triangulation consists of a simplicial map ϕ from $\widetilde{S}$ to X that is equivariant with respect to the homomorphism defined by $f_S \to f, g_S \to g$, which we will denote by ϕ_*.

A simplicial map is *nondegenerate* if it does not collapse an edge to a vertex. A nondegenerate simplicial map is a *near-immersion* if it is a local embedding at edge midpoints.

We begin with listing a lemma and a proposition that piece together to a proof of Proposition 3.1.

Lemma 3.3. *Let $f, g, h = g^{-1} \circ f$ be automorphisms of a simply connected triangle complex X acting without inversions. Suppose that all* $\mathrm{Fix}(f), \mathrm{Fix}(g), \mathrm{Fix}(h)$ *are nonempty and pairwise disjoint. Then there exists an equivariant triangulation of $\widetilde{S}$ w.r.t. (f, g) with ϕ a near-immersion.*

Lemma 3.3 will be proved in Section 4.

If X is a CAT(0) space, an element $g \in G$ is *loxodromic* if there is a geodesic $\omega \subset X$ (called an *axis*) such that g preserves ω and acts on it as a translation. A loxodromic element does not have a fixed point in X.

Proposition 3.4. *Let X be a* CAT(0) *triangle complex satisfying condition (i),(ii), or (iii) of Theorem 1.1. Suppose that for some automorphisms f, g of X we have an equivariant triangulation of $\widetilde{S}$ w.r.t. (f, g) with ϕ a near-immersion. Then $\langle f, g\rangle$ contains a loxodromic element.*

Proposition 3.4 will be proved in Section 5.

Proof of Proposition 3.1. After possibly subdividing the triangles of X, we can assume that f, g act without inversions. Let $h = g^{-1} \circ f$. If h does not have a fixed point in X, then the proof is complete, so we can assume that $\mathrm{Fix}(h)$ is nonempty. Similarly, we can suppose that $\mathrm{Fix}(f)$ is disjoint from $\mathrm{Fix}(g)$. Then $\mathrm{Fix}(h)$ is disjoint from both $\mathrm{Fix}(f)$ and $\mathrm{Fix}(g)$. By Lemma 3.3, there exists an equivariant triangulation of $\widetilde{S}$ w.r.t. (f, g) with ϕ a near-immersion. By Proposition 3.4, $\langle f, g\rangle$ contains a loxodromic element. □

4. Existence of nearly immersed triangulations

The *area* of a finite 2-dimensional Δ-complex is the number of its triangles. We say that an equivariant triangulation of $\widetilde{S}$ w.r.t. (f, g) has *minimal area* if the corresponding Δ-complex structure on S has minimal area among all equivariant triangulations of $\widetilde{S}$ w.r.t. (f, g).

Proof of Lemma 3.3. **Step 1.** *There exists an equivariant triangulation.*

Since f, g act without inversions, $\mathrm{Fix}(f), \mathrm{Fix}(g), \mathrm{Fix}(h)$ are subcomplexes. Consider any vertices $a \in \mathrm{Fix}(f), b \in \mathrm{Fix}(g), c \in \mathrm{Fix}(h)$, and any nontrivial edge-paths α (resp. β) from c to a (resp. b) in X. Let γ be the closed edge path $\alpha^{-1}\beta g(\beta^{-1})f(\alpha)$, which passes through $a, c, b, g(c) = f(c)$. By the relative simplicial approximation [Zee64], there is a disc D with a structure of a simplicial complex and a simplicial map $\psi\colon D \to X$ that restricts to γ on ∂D.

Gluing the parts of ∂D mapping to α and $f(\alpha)$, and the parts mapping to β and $g(\beta)$, and labelling the points mapping to a, b, c by p, q, r, equips S with a structure of a Δ-complex. This defines an equivariant triangulation of $\widetilde{S}$, since there exists an obvious equivariant map ϕ extending ψ viewed as a map from the fundamental domain $D \subset \widetilde{S}$ for the action of $\langle f_S, g_S\rangle$.

Step 2. *A minimal area equivariant triangulation has nondegenerate ϕ.*

Suppose that there is an edge $\widetilde{e} \subset \widetilde{S}$ with $\phi(\widetilde{e})$ a vertex. We will find an equivariant triangulation with strictly smaller area, which contradicts minimality. Let e be the projection of $\widetilde{e}$ to S. Let u, v be the endpoints of e.

Case $u = v$**.** In that case, assume for the moment that u is distinct from p, q, r. Since $u = v$, we have that e is an embedded closed path. Let $B \subset S$ be the open disc bounded by e containing at most one of p, q, r. Let T be the triangle of S adjacent to e outside B, and let t, t' be the edges of T distinct from e. Note that $t \neq t'$ since otherwise T would form the entire outside of B, which would contradict the assumption that there are at least two of p, q, r outside B.

If B does not contain a puncture, then we can remove $T \cup B$ from S and glue along t and t' (independent of whether all the vertices of T coincide or not). This does not change the homeomorphism type of S and decreases its area. If B contains a puncture, say p, then we perform the same operation and additionally relabel u by p. Note that it is clear how to modify ϕ on modified $\widetilde{S}$.

Going back to the possibility that u is one of p, q, r, say, q, we still perform the same operation, except that now B will not contain p (or r), since otherwise $\mathrm{Fix}(f)$ would intersect $\mathrm{Fix}(g)$. Consequently we still have $t \neq t'$ and we can proceed as before.

Case $u \neq v$**.** Let T, T' be the triangles of S adjacent to e, and let y, y' be the vertices opposite to e in T, T'. If $T = T'$, then any lift $\widetilde{T}$ of T to $\widetilde{S}$ is collapsed to a single vertex of X. Let $\widetilde{e}'$ be the edge of $\widetilde{T}$ that does not project to e. Then the projection e' of $\widetilde{e}'$ has coinciding endpoints, which brings us back to Case $u = v$.

Thus we can assume that $T \neq T'$. We want to remove $T \cup T'$ from the triangulation and glue the resulting square in the boundary so that u is identified with v. This amounts to collapsing segments of a foliation in $T \cup T'$, with leaves parallel to e. This does not change the homeomorphism type of S, as long as the leaves do not combine to circles.

Indeed, a single leaf (or a pair of leaves in a common triangle) cannot close up to a circle since the edges yu and yv are distinct, because S does not contain a Möbius band, and analogously the edges uy', vy' are distinct. Furthermore, a pair of leaves in distinct triangles does not form a circle since otherwise S would have only two triangles and consequently both u, v would be in $\{p, q, r\}$, forcing some $\mathrm{Fix}(f), \mathrm{Fix}(g), \mathrm{Fix}(h)$ to intersect.

Removing $T \cup T'$ decreases the area of S. Note that as a result of this operation, vertices p, q, r cannot become identified, since this again would mean that some $\mathrm{Fix}(f), \mathrm{Fix}(g), \mathrm{Fix}(h)$ intersect.

Step 3. *A minimal area equivariant triangulation has ϕ a near immersion.*

By Step 2, ϕ is nondegenerate. Suppose that there is an edge $\widetilde{e} \subset \widetilde{S}$ with midpoint $\widetilde{m}$ where ϕ is not a local embedding. Again, we will reach a contradiction by showing that the area can be decreased. Let e and m be the projections of $\widetilde{e}$ and $\widetilde{m}$ to S. Let $T, T' \subset S$ be the triangles containing e and let y, y' be the vertices opposite to e in T, T'.

To start with, note that $T \neq T'$. Indeed, if $T = T'$, then let τ be the segment in T starting and ending at the distinct copies of m in ∂T. Let $\widetilde{\tau}, w\widetilde{\tau}$ for $w \in \langle f_S, g_S \rangle$ be the two lifts of τ to $\widetilde{S}$ at $\widetilde{m}$, and let $\widetilde{T}$ be the lift of T containing $\widetilde{\tau}$. Then $\phi(\widetilde{T})$ is stabilised by $\phi_*(w)$, but not fixed pointwise, which is a contradiction.

Case $y \neq y'$. In that case removing $T \cup T'$ from the triangulation and gluing the resulting square in the boundary so that y is identified with y' is equivalent to the following. We collapse intervals of a foliation in $T \cup T'$ with leaves parallel to the union δ of segments $ym \subset T, my' \subset T'$. Similarly as in Step 2 these leaves do not form circles, and thus collapsing them does not change the homeomorphism type of S, while it decreases its area. Again as a result of this operation, vertices p, q, r cannot become identified, since this would mean that some $\mathrm{Fix}(f), \mathrm{Fix}(g), \mathrm{Fix}(h)$ intersect.

Case $y = y'$. In that case, assume for the moment that y is distinct from p, q, r. Then δ, defined as in Case $y \neq y'$, is an embedded closed path. Let $B \subset S$ be the open disc bounded by δ containing at most one of p, q, r. Let t, t' be the edges of T, T' outside B. Note that $t \neq t'$ since otherwise halves of T and T' would form the entire outside of B, which would contradict the assumption that there are at least two of p, q, r outside B. We can thus remove $T \cup T' \cup B$ and glue along t, t' to decrease the area of S. If B contained one of p, q, r, say, p, then we relabel y by p. If $y = y'$ is one of p, q, r, say, q, then we define B in the same way and we note that B does not contain p (or r) since otherwise $\mathrm{Fix}(f)$ would intersect $\mathrm{Fix}(g)$. Consequently we still have $t \neq t'$ and we can proceed as before. □

5. Constructing axes

To prove Proposition 3.4 we will need the following famous theorem of Masur establishing the existence of periodic trajectories in rational billiards.

Definition 5.1. A *translation surface* S is a surface obtained from identifying sides of finitely many polygons in $\mathbb{R}^2$ by translations. This equips S with a Riemannian metric of Gauss curvature 0 outside a finite set Σ.

Remark 5.2. Assume that a sphere S has a piecewise smooth Riemannian metric that is smooth of Gauss curvature 0 outside a finite set Σ. Suppose that for each

$v \in \Sigma$ the length of lk_v is commensurable with π. Since $\pi_1(S-\Sigma)$ is generated by the peripheral curves, the image of the holonomy map $\pi_1(S-\Sigma) \to O(2)$ is finite. Consequently, its kernel corresponds to a finite branched cover of S over Σ that has trivial holonomy and is thus a translation surface. Similarly, each finite branched cover of S over Σ has a further cover that is a translation surface.

Theorem 5.3 ([Mas86, Thm 2])**.** *Let S be a translation surface. Then there is a closed local geodesic in $S-\Sigma$.*

We will also need the following results of Ballmann and Brin.

Lemma 5.4 ([BB95, Lem 7.1])**.** *Let S be a compact triangle complex that is locally* CAT(0) *and in which each edge belongs to at least two triangles. Suppose that there is*

- *a point in the interior of a triangle with negative Gauss curvature, or*
- *a point in the interior of an edge with negative sum of geodesic curvatures in a pair of incident triangles.*

Then there is a closed local geodesic in S that is disjoint from the vertices.

Lemma 5.5 (compare [BB95, Lem 7.3])**.** *Let S be a compact triangle complex that is locally* CAT(0) *and in which each edge belongs to at least two triangles. Assume that there is a vertex v and points ξ, η in lk_v with $d_v(\xi,\eta) = \pi$. Then for any $\varepsilon > 0$ there is a closed path $\beta_1\beta_2\beta_3$ in S such that*

- *paths β_i are local geodesics,*
- *the angles at the breakpoints between β_2 and β_1, β_3 are $> \pi - \varepsilon$,*
- *β_i do not pass through vertices except that β_1 starts at v and β_3 ends at v, and*
- *the starting direction ξ' of β_1 and ending direction η' of β_3 satisfy $d_v(\xi,\xi') < \frac{\varepsilon}{2}, d_v(\eta,\eta') < \frac{\varepsilon}{2}$.*

Proof. We refer to the proof of [BB95, Lem 7.3], where the authors work in the universal cover of S (which they call X). Once they construct their geodesic σ, define ω_2 as the subpath of σ between P and $\varphi(P)$, and ω_1, ω_3 as the geodesics joining the endpoints of ω_2 to $v, \varphi v$ in $P, \varphi(P)$. The projection of $\omega_1\omega_2\omega_3$ to S is the required path $\beta_1\beta_2\beta_3$. □

Proof of Proposition 3.4. We equip $\widetilde{S}$ and S with the piecewise smooth Riemannian metric pulled back from X via ϕ. Since ϕ is a near-immersion, $S-\{p,q,r\}$ is locally CAT(0). After replacing S with a finite branched cover over $\{p,q,r\}$, we can assume that S (which is no longer a sphere) is locally CAT(0) at every point. Let Σ be the vertex set of S.

Assume first that a point in the interior of a triangle of S has negative Gauss curvature, or a point in the interior of an edge of S has negative sum of its two geodesic curvatures. Then by Lemma 5.4, there is a closed local geodesic ω in $S-\Sigma$. Let $\widetilde{\omega}$ be an elevation of ω to $\widetilde{S}$, which is an axis for some $w \in \langle f_S, g_S \rangle$. Since ϕ is a near-immersion, $\phi(\widetilde{\omega})$ is a local, hence global, geodesic in X. Thus $\phi_*(w) \in \langle f, g \rangle$ is loxodromic.

It remains to consider the case where $S-\Sigma$ is smooth with Gauss curvature 0. Suppose first that for each $v \in \Sigma$, the length of lk_v is commensurable with π. By Remark 5.2, there is a finite branched cover S' of S over Σ that is a translation surface. By Theorem 5.3, S' has a closed local geodesic ω' outside the vertex set. Projecting ω' to ω in S we can now find a loxodromic element in $\langle f, g \rangle$ as before.

Assume now that for some $v \in \Sigma$ the length of lk_v is not commensurable with π. Let $\widetilde{v}$ be a lift of v to $\widetilde{S}$. If $\mathrm{lk}_{\widetilde{v}}$ is a circle, consider the closed immersed edge-path $\gamma\colon \mathrm{lk}_{\widetilde{v}}) \to \mathrm{lk}_{\phi(\widetilde{v})}$ induced by ϕ. This path fails the conclusion of Corollary 1.7. Thus there are points $\widetilde{\xi}, \widetilde{\eta} \in \mathrm{lk}_{\widetilde{v}}$ such that their images in $\mathrm{lk}_{\phi(\widetilde{v})}$ are at distance $> \pi + \delta$ for some $\delta > 0$.

If $\mathrm{lk}_{\widetilde{v}}$ is a line, we construct $\widetilde{\xi}, \widetilde{\eta}$ in the following way. Assume without loss of generality that $\widetilde{v}$ is fixed by f_S. First, we claim that X does not have rational angles with respect to the action of G. Otherwise, let $\Lambda \subset \mathrm{lk}_{\phi(\widetilde{v})}$ be the discrete set from Definition 2.1. Then the complementary components in $\mathrm{lk}_{\widetilde{v}}$ of $\phi^{-1}(\Lambda)$ are finite, and of length commensurable with π. Moreover, since f preserves Λ, we have that f_S preserves $\phi^{-1}(\Lambda)$. This contradicts the assumption that the length of lk_v is not commensurable with π, justifying the claim.

According to our hypotheses this means that either X is locally finite, or each element of G fixing a point of X has finite order. In both cases there is a directed edge e in $\mathrm{lk}_{\widetilde{v}}$ and $k \geq 1$ such that $\phi(e) = \phi(f_S^k e)$. Thus the path in $\mathrm{lk}_{\widetilde{v}}$ from the endpoint of e to the endpoint of $f_S^k e$ maps under ϕ to a closed edge-path failing the conclusion of Corollary 1.7, and we obtain $\widetilde{\xi}, \widetilde{\eta}$ as before.

Let ξ, η be the projections of $\widetilde{\xi}, \widetilde{\eta}$ to lk_v. Orbits of an irrational rotation on a circle are dense. Thus in lk_v we can find $\xi_1 = \xi, \xi_2, \ldots, \xi_{2n}$ with $d_v(\xi_i, \xi_{i+1}) = \pi$ and $d_v(\xi_{2n}, \eta) < \frac{\delta}{2}$. Inspired by [BB95, Lem 7.4], we construct the following path $\widetilde{\omega} = \omega_1 \cdots \omega_{6n}$ in $\widetilde{S}$. To start with, we put $\varepsilon = \frac{\delta}{12n}$ and apply Lemma 5.5 to ξ_1, ξ_2. We define $\omega_1\omega_2\omega_3$ to be the lift of $\beta_1\beta_2\beta_3$ starting in the direction at distance $< \frac{\varepsilon}{2}$ from $\widetilde{\xi}$. Let $\widetilde{v}_2$ be the endpoint of ω_3 and let $\widetilde{\xi}_2$ in $\mathrm{lk}_{\widetilde{v}_2}$ be the lift of ξ_2 at distance $< \frac{\varepsilon}{2}$ from the ending direction of ω_3. Since $d_{v'}(\xi_2, \xi_3) = \pi$, there is a lift $\widetilde{\xi}_3$ of ξ_3 in $\mathrm{lk}_{\widetilde{v}_2}$ with $d_{\widetilde{v}_2}(\widetilde{\xi}_2, \widetilde{\xi}_3) = \pi$. Apply now Lemma 5.5 to ξ_3, ξ_4 and define $\omega_4\omega_5\omega_6$ to be the lift of the resulting $\beta_1\beta_2\beta_3$ starting in the direction at distance $< \frac{\varepsilon}{2}$ from $\widetilde{\xi}_3$ etc. The endpoint of ω_{6n} has the form $w\widetilde{v}$ for some $w \in \langle f_S, g_S \rangle$, and since $d_v(\xi_{2n}, \eta) \leq \frac{\delta}{2}$, we can choose w so that the ending direction of ω_{6n} is at distance $< \frac{\varepsilon}{2} + \frac{\delta}{2}$ from $w\widetilde{\eta}$.

Let $x, y = \phi_*(w)x \in X$ be the endpoints of $\phi(\widetilde{\omega})$, and denote by ξ' and η' the starting and ending directions of $\phi(\widetilde{\omega})$. Then $d_x(\phi(\widetilde{\xi}), \xi') < \frac{\varepsilon}{2}$ and $d_y(\phi(w\widetilde{\eta}), \eta') < \frac{\varepsilon}{2} + \frac{\delta}{2}$. Let α be the geodesic from x to y in X, and denote by ξ'' and η'' the starting and ending directions of α. Note that the angles at all the breakpoints of $\phi(\widetilde{\omega})$ are $> \pi - \varepsilon$. By [BB95, Lem 2.5] we have $d_x(\xi', \xi'') + d_y(\eta', \eta'') < (6n-1)\varepsilon$. We thus have:

$$\begin{aligned}
d_y(\phi_*(w)\xi'', \eta'') &\geq d_y(\phi_*(w)\xi', \eta') - d_y(\phi_*(w)\xi'', \phi_*(w)\xi') - d_y(\eta', \eta'') \\
&> d_y(\phi_*(w)\xi', \eta') - (6n-1)\varepsilon \\
&\geq d_y\big(\phi(w\widetilde{\xi}), \phi(w\widetilde{\eta})\big) - d_y\big(\phi_*(w)\xi', \phi(w\widetilde{\xi})\big) - d_y\big(\phi(w\widetilde{\eta}), \eta'\big) - (6n-1)\varepsilon \\
&> (\pi + \delta) - \frac{\varepsilon}{2} - \left(\frac{\varepsilon}{2} + \frac{\delta}{2}\right) - (6n-1)\varepsilon = \pi.
\end{aligned}$$

Consequently the concatenation of $\phi_*^k(w)\alpha$, for $k \in \mathbb{Z}$ is a local (hence global) geodesic, and thus $\phi_*(w)$ is loxodromic. □

6. Triples of generators

In this Section we complete the proof of Theorem 1.1.

Proposition 6.1. *Let f, g, h be automorphisms of a* CAT(0) *triangle complex X satisfying condition (i),(ii), or (iii) of Theorem 1.1. Suppose that all* $\mathrm{Fix}(f) \cap \mathrm{Fix}(g), \mathrm{Fix}(f) \cap \mathrm{Fix}(h)$, *and* $\mathrm{Fix}(g) \cap \mathrm{Fix}(h)$ *are nonempty. Then* $\mathrm{Fix}(f) \cap \mathrm{Fix}(g) \cap \mathrm{Fix}(h)$ *is nonempty or $\langle f, g, h \rangle$ contains a loxodromic element.*

In the proof we will need the following notion. Let Δ be a disc with decomposition of its boundary into three paths $\partial\Delta = P \cup Q \cup R$. An *admissible triangulation* of Δ w.r.t. (f,g,h) is a structure on Δ of a Δ-complex, with $P\cap Q, Q\cap R, P\cap R$ among the vertices, together with a simplicial map ψ from Δ to X that sends P,Q,R into $\mathrm{Fix}(f), \mathrm{Fix}(g), \mathrm{Fix}(h)$, respectively.

Lemma 6.2. *Let f,g,h be automorphisms of a* $\mathrm{CAT}(0)$ *triangle complex X acting without inversions. Suppose that* $\mathrm{Fix}(f), \mathrm{Fix}(g), \mathrm{Fix}(h)$ *pairwise intersect but their triple intersection is empty. Then there exists an admissible triangulation of Δ with ψ a near-immersion.*

Proof. Since f,g,h act without inversions, their fixed point sets are subcomplexes. Thus an admissible triangulation of Δ exists by the relative simplicial approximation [Zee64]. Suppose now that Δ has minimal area among admissible triangulations w.r.t. (f,g,h). We first prove that ψ is nondegenerate. Indeed, suppose that there is an edge $e\subset\Delta$ with $\psi(e)$ a vertex, and let u,v be the endpoints of e. If e lies in $\partial\Delta$, then let T be the triangle containing e. We then remove e and T from Δ and we identify the two remaining sides of T. This decreases the area of Δ, which contradicts minimality. Suppose then that e is contained in two triangles T and T'. If $u=v$, then we modify Δ as in the proof of Lemma 3.3. If $u\neq v$ and $T=T'$ we land back in Case $u=v$. Otherwise, we wish to remove $T\cup T'$, or more precisely to collapse intervals of the foliation in $T\cup T'$ parallel to e. Unless e has both endpoints on $\partial\Delta$, this does not change the homeomorphism type of Δ. If both u and v are on the path P (resp. Q,R), then together with T and T' we remove the entire disc bounded by e and a subpath of P (resp. Q,R), and we identify the two remaining edges of T or T'. If the endpoints of e lie on two distinct paths, say P and Q, then we do the same using a subpath of $P\cup Q$ ($P\cap Q$ is replaced here by u identified with v).

Finally, we prove that ψ is a near-immersion. Suppose that there is an edge $e\subset\Delta$ with midpoint m where ψ is not a local embedding. Let $T,T'\subset\Delta$ be the triangles containing e and let y,y' be the vertices opposite to e in T,T'. As in the proof of Lemma 3.3, we have $T\neq T'$. Consider the path δ as in the proof of Lemma 3.3. If $y=y'$, then we can modify Δ as in the proof of Lemma 3.3. If $y\neq y'$, then we collapse intervals of appropriate foliation in $T\cup T'$. Unless δ has both endpoints on $\partial\Delta$, this does not change the homeomorphism type of Δ. If both endpoints of δ are on the path P (resp. Q,R), then together with T and T' we remove the entire disc bounded by δ and a subpath of P (resp. Q,R), and we identify the two remaining edges of T,T'. If the endpoints of δ lie on two distinct paths, say P and Q, then we do the same using a subpath of $P\cup Q$ ($P\cap Q$ is replaced here by v identified with v'). After finitely many such operations we will arrive at a near-immersion. □

Proof of Proposition 6.1. After possibly subdividing the triangles of X, we can assume that f,g,h act without inversions. Suppose that $\mathrm{Fix}(f)\cap\mathrm{Fix}(g)\cap\mathrm{Fix}(h)=\varnothing$. Then by Lemma 6.2, there is an admissible triangulation of Δ with ψ a near-immersion. Moreover, by possibly passing to a subcomplex of Δ, we can assume that we have the following property.

($\star$): For any edge e of P (resp. Q,R), the triangle containing e is not mapped by ψ to $\mathrm{Fix}(f)$ (resp. $\mathrm{Fix}(g), \mathrm{Fix}(h)$).

Let D be obtained from Δ by attaching its second copy along the side R. Extend ψ to a map from D to X defined as $h\circ\psi$ on the attached copy of Δ. Let $f'=h\circ f, g'=h\circ g$. Denote $\alpha=\psi(P), \beta=\psi(Q)$ to be directed paths starting at $\psi(P\cap Q)$. Then ψ restricts on ∂D to $\alpha^{-1}\beta g'(\beta^{-1})f'(\alpha)$. As in the first paragraph of the proof of Lemma 3.3, we can extend D and ψ to an equivariant triangulation of $\widetilde{S}$ w.r.t. (f',g'), with equivariant map $\phi\colon\widetilde{S}\to X$. Since ψ was a near-immersion and satisfied property $(\star)$, ϕ is a near-immersion. Thus by Proposition 3.4, $\langle f', g' \rangle$ contains a loxodromic element. $\square$

Proof of Theorem 1.1. Suppose by contradiction that for each $f \in G$ the set $\mathrm{Fix}(f)$ is nonempty. We will prove that for any finite set of elements $f_1, \ldots, f_n \in G$ the intersection $\mathrm{Fix}(f_1) \cap \cdots \cap \mathrm{Fix}(f_n)$ is nonempty. Case $n = 2$ follows from Proposition 3.1. Consequently, case $n = 3$ follows from Proposition 6.1. Since the fixed point sets are convex, the cases $n \geq 4$ follow from Helly's Theorem [Iva14, Thm 1.1]. Setting $f_1, \ldots, f_n$ to be the generators of G we obtain that the action of G on X is trivial, which is a contradiction. $\square$

7. Irrational loops have diameter $> \pi$

In this section we prove Theorem 1.6.

The key ingredient in the proof is a reformulation of a theorem of Dehn [Deh03], which states that a rectangle can be tiled by finitely many squares if and only if its sides are commensurable.

Let (X, μ_X), (Y, μ_Y) be measure spaces of finite measure. We will write μ instead of μ_X and μ_Y for brevity. We say that a pair (A, B), such that $A \subseteq X$, $B \subseteq Y$ are measurable, is a *rectangle in* $X \times Y$ with *side lengths* $\mu(A)$ *and* $\mu(B)$. A rectangle is a *square* if its side lengths are equal. We say that a collection $(A_1, B_1), (A_2, B_2), \ldots, (A_k, B_k)$ of rectangles is a *rectangle tiling of* $X \times Y$ if

- $\bigcup_{i=1}^{k} A_i \times B_i = X \times Y$, and
- $\mu(A_i \cap A_j)\mu(B_i \cap B_j) = 0$ for $1 \leq i < j \leq k$.

We say that a rectangle tiling is *a square tiling* if it consists of squares.

Theorem 7.1. *Let (A_1, B_1), $(A_2, B_2), \ldots, (A_k, B_k)$ be a square tiling of $X \times Y$. Then $\mu(X)$, $\mu(Y)$, $\mu(A_1), \ldots, \mu(A_k)$ are all commensurable.*

While it is not hard to see that Theorem 7.1 is equivalent to the theorem of Dehn stated above we include a proof both for completeness, and because we will need a technical modification of the result. The proof we give is essentially the one used by Hadwiger [Had57] to prove a multidimensional generalization of Dehn's theorem, except that we follow the presentation from Aigner and Ziegler [AZ18, Chap 29], which avoids the reliance on the existence of Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$ and thus the dependence on the Axiom of Choice.

Proof of Theorem 7.1. We start by showing that it suffices to consider the case when X and Y are finite. The reduction is fairly straightforward. Define an equivalence relation $\sim$ on X, by setting $x_1 \sim x_2$ if for every $1 \leq i \leq k$ either $\{x_1, x_2\} \in A_i$ or $\{x_1, x_2\} \cap A_i = \emptyset$. Clearly there are finitely many equivalence classes of elements of X with respect to this relation, and we can identify the elements of X lying in the same equivalence class. Thus we can assume that X and, symmetrically, Y are finite. Further, we may assume without loss of generality that every element of X and Y has a positive measure. It follows that for every $x \in X$ and $y \in Y$ there exists a unique index i such that $x \in A_i$ and $y \in B_i$.

Let V be a vector space over $\mathbb{Q}$ spanned by $\{\mu(x)\}_{x \in X} \cup \{\mu(y)\}_{y \in Y}$. Suppose first that $\mu(X)$ and $\mu(Y)$ are not commensurable. Then there exists a linear function $f : V \to \mathbb{Q}$ such that $f(\mu(X)) = 1$ and $f(\mu(Y)) = -1$. By linearity we have

$$-1 = f(\mu(X))f(\mu(Y)) = \sum_{x \in X} f(\mu(x)) \sum_{y \in Y} f(\mu(y))$$

$$= \sum_{i=1}^{k} f(\mu(A_i))f(\mu(B_i)) = \sum_{i=1}^{k} f^2(\mu(A_i)) \geq 0, \tag{1}$$

yielding the desired contradiction.

Suppose finally that $\mu(A_j)$ is not commensurable with $\mu(X)$ for some $0 \leq j \leq k$. Define a linear function $g : V \to \mathbb{Q}$ such that $g(\mu(X)) = 0$ and $g(\mu(A_j)) = 1$. Substituting g instead of f in (1) yields

$$0 = g(\mu(X))g(\mu(Y)) = \sum_{i=1}^{k} g^2(\mu(A_i)) \geq g^2(\mu(A_j)) = 1,$$

a contradiction. □

We will also need a technical variant of Theorem 7.1.

Lemma 7.2. *Let (A_1, B_1), $(A_2, B_2), \ldots, (A_k, B_k)$ be a rectangle tiling of $X \times Y$, such that for some $q, r \in \mathbb{R}_+$ and $a \in \mathbb{Q}$ we have*

- *$\mu(X) = 2q + ar$, $\mu(Y) = q + (a/2 - 1)r$,*
- *(A_1, B_1) and (A_2, B_2) are rectangles with sides r and $q + r$,*
- *$(A_3, B_3), \ldots, (A_k, B_k)$ are squares,*
- *$\mu(A_3), \ldots, \mu(A_j)$ are commensurable with r for some $3 \leq j \leq k$, and $\sum_{i=3}^{j} \mu^2(A_j) > (a - 4)r^2$*

Then q and r are commensurable.

Proof. The proof parallels the proof of Theorem 7.1 above with minor modification to the calculation (1). We define the vector space V as in the proof of Theorem 7.1, and assume for a contradiction that q and r are not commensurable. Then there exists a linear function $f : V \to \mathbb{Q}$ such that $f(q) = a/2 - 1$ and $f(r) = -1$. Thus $f(\mu(X)) = -2$, $f(\mu(Y)) = 0$, and $f(\mu(A_i))f(\mu(B_i)) = 2 - a/2$ for $i = 1, 2$, while

$$\sum_{i=3}^{j} f(\mu(A_j))f(\mu(B_j)) = \sum_{i=3}^{j} \left(\frac{\mu(A_j)}{r}\right)^2 f^2(r) > a - 4.$$

Combining these estimates as in (1) we obtain

$$\begin{aligned} 0 = f(\mu(X))f(\mu(Y)) &= \sum_{i=1}^{k} f(\mu(A_i))f(\mu(B_i)) \\ &= \sum_{i=1}^{2} f(\mu(A_i))f(\mu(B_i)) + \sum_{i=3}^{j} f(\mu(A_i))f(\mu(B_i)) + \sum_{i=j+1}^{k} f^2(\mu(A_i)) \\ &> (4 - a) + (a - 4) = 0, \end{aligned}$$

the desired contradiction. □

Let us now outline the proof of Theorem 1.6. As a first step (Claim 1 below) we prove that every cycle C of Γ' has length commensurable with π. We do so by examining the structure of shortest paths in Γ between the pairs of points in C. The paths joining the pairs of points which lie at distance greater than π in C must take "shortcuts" which we refer to as *chords*. We construct a square tiling of a cylinder, where the squares correspond to the chords and apply Theorem 7.1.

Next we prove in Claim 3 that certain paths of Γ', called bars, also have length commensurable with π. We say that a path B in Γ' with endpoints u and v is a *bar in Γ' joining cycles C_1 and C_2*, if C_1 and C_2 are disjoint cycles of Γ', and, moreover, $u \in C_1$, $v \in C_2$ and C_1 and C_2 are otherwise disjoint from B. Note that for every bar B there exists a closed edge-path immersed in Γ' that traverses B twice and each of C_i once. The existence of such an immersion allows us to adapt the argument we used for cycles to bars, but there are multiple technical hurdles to overcome, making this the most technical part of the proof. It is here, in particular, that we use Lemma 7.2 and we additionally need extra information about the structure of chords of a cycle and between pairs of cycles, which we obtain in Claims 1 and 2.

To unify the argument for cycles and bars we start the proof of Theorem 1.6 in the setting of Corollary 1.7 by considering closed edge-paths immersed in Γ.

Finally, in Claim 4, we show that every segment of Γ' has length commensurable with π by proving that it can be expressed as a rational linear combination of bars and cycles.

Proof of Theorem 1.6. We follow the steps outlined above, starting by considering immersed circles in Γ'.

Let $\gamma : C \to \Gamma'$ be a local isometry mapping a circle C of length l into Γ'. We identify C with $\mathbb{R}/l\mathbb{Z}$. This identification defines a natural $\mathbb{R}$-action on C.

For $x, y \in C$, let $d(x, y)$ denote the distance between points $\gamma(x)$ and $\gamma(y)$ in Γ. As the girth of Γ is at least 2π, if $0 < d(x, y) < \pi$ then there exists a unique path from $\gamma(x)$ to $\gamma(y)$ in Γ of length $d(x, y)$. We denote such a path by P_{xy}.

Let S be the set of points in $s \in C$ such that $\gamma(s)$ has degree at least three in Γ. For $s, t \in S$ we say that (s, t) is a *chord of* γ *of length* d_0 if $0 < d_0 = d(s, t) < \pi$, and P_{st} is disjoint from $\gamma([s-\varepsilon, s) \cup (s, s+\varepsilon] \cup [t-\varepsilon, t) \cup (t, t+\varepsilon])$ for some $\varepsilon > 0$. Note that our chords are directed, that is we distinguish between the chords (s, t) and (t, s).

For $x, y \in C$ we say that (x, y) *uses a chord* (s, t) if there exists a path P from $\gamma(x)$ to $\gamma(y)$ in Γ of length $d(x, y)$ such that P is obtained by concatenating paths $\gamma([x, s])$, P_{st} and $\gamma([t, y])$, where we denote by $[a, b]$ for $a, b \in C$ the shortest of the two intervals in C with endpoints a and b. Clearly, (x, y) uses a chord (s, t) of length d_0 if and only if there exists $\sigma, \tau \in \mathbb{R}$ such that $x = s + \sigma$, $y = t + \tau$, and $|\sigma| + |\tau| \le \pi - d_0$. Moreover, if the last inequality is strict, then (s, t) is the unique chord used by (x, y).

Note further that for every $(x, y) \in C^2$ if (x, y) uses no chord, then there exist $z, z' \in C$ such that $\gamma(z) = \gamma(z')$, and there exists a path from $\gamma(x)$ to $\gamma(y)$ in Γ of length $d(x, y)$ obtained by concatenating paths $\gamma([x, z])$ and $\gamma([z', y])$. Moreover, if $d(x, y) < \pi$ then the converse also holds. If (x, y) uses no chord and the distance between x and y in C is larger than π then we say that (x, y) is a *spliced pair*. We denote the set of spliced pairs by $Spl(\gamma)$.

We now describe the square tiling to which we apply Theorem 7.1 and Lemma 7.2. Let $R = R(\gamma) \subset C^2$ be the annulus consisting of points (x, y) with x and y at distance at least π in C. Note that R is isometric to the product of a circle of length $l\sqrt{2}$ and an interval of length $l\frac{\sqrt{2}}{2} - \pi\sqrt{2}$. Let $\mathcal{K} = \mathcal{K}(\gamma)$ denote the set of all chords, and let $S(K)$ for $K \in \mathcal{K}$ denote the set of pairs $(x, y) \in C^2$ which use C. For a chord $K = (s, t)$ of length d_0, let $z = \pi - d_0$. As noted above,

$$S(K) = \{(s + \sigma, t + \tau) \mid -z \le \sigma + \tau \le z, -z \le \tau - \sigma \le z\},$$

which is a square in R with side length $z\sqrt{2}$ and one side pair parallel to ∂R. Moreover, for distinct K the squares $S(K)$ have disjoint interiors. Finally note that $Spl(\gamma) = \operatorname{int}(R) - \bigcup_{K \in \mathcal{K}} S(K)$ by definition.

We will also use some of the terminology introduced above when considering a subgraph Λ of Γ, rather than an immersion. For vertices s, t of Λ we say that (s, t) is a *chord of* Λ *of length* d_0 *and area* $2(\pi - d_0)^2$ if $0 < d_0 = d_\Gamma(s, t) < \pi$ and the first and last edge of the unique path P from s to t in Γ of length d_0 do not lie in Λ. Note that if $\gamma : C \to \Gamma$ is an immersion as above, and $s, t \in C$ are such that $(\gamma(s), \gamma(t))$ is a chord of $\gamma(C)$ of length d_0 then (s, t) is a chord of γ of length d_0. If γ is injective then the converse also holds.

We are now ready for the first major claim, which in particular establishes the theorem in the case when Γ' is a cycle.

Claim 1. *Let C be a cycle in Γ' of length l. Then*

a): *l is commensurable with π,*

b): *the length of every chord of C is commensurable with π,*

c): *for every point $p \in C$ the total area of chords of C with both endpoints in $C - \{p\}$ is at least $2\pi(l - 2\pi)$.*

Proof. Let γ be an isometry from a circle of length l onto C. For simplicity of notation we identify the domain of γ with C.

Note that $Spl(\gamma) = \emptyset$. Thus $\mathrm{int}(R(\gamma)) \subseteq \bigcup_{K\in\mathcal{K}(\gamma)} S(K) \subseteq R(\gamma)$. As $\mathcal{K}(\gamma)$ is finite, it follows that $R(\gamma) = \bigcup_{K\in\mathcal{K}(\gamma)} S(K)$, and Theorem 7.1 implies that $l\sqrt{2}$ and $l\frac{\sqrt{2}}{2} - \pi\sqrt{2}$ are commensurable. Thus a) holds, and Theorem 7.1 similarly implies that b) holds.

It remains to establish c). First note that the total area of all chords of C is equal to the area of $R(\gamma)$, which is $l(l - 2\pi)$. It remains to upper bound the area of chords of C with at least one end in p, i.e. chords of γ with at least one end in $s = \gamma^{-1}(p)$. Let $K_1 = (s, t_1), K_2 = (s, t_2), \dots, K_n = (s, t_n)$ be all the chords starting at s. Suppose that K_i has length $\pi - z_i$. Then the area of $S(K_i)$ is equal to $2z_i^2$, and $S(K_i)$ intersects the line $(\{s\} \times C) \cap R$ in an interval of length $2z_i$. As $(\{s\} \times C) \cap R$ has length $l - 2\pi$, we have $\sum_{i=1}^n z_i \leq l/2 - \pi$, and so the sum of the areas of the chords starting at s is at most $2(\sum_{i=1}^n z_i)^2 \leq 2(l/2 - \pi)^2$. It follows that the area of chords of C with at least one end in p, is at most $(l - 2\pi)^2$. As $l(l - 2\pi) - (l - 2\pi)^2 = 2\pi(l - 2\pi)$, c) follows. □

Next we need to extend Claim 1 b) to a pair of disjoint cycles.

Claim 2. *Let C_1, C_2 be disjoint cycles of Γ'. Then every chord of $C_1 \cup C_2$ has length commensurable with π.*

Proof. Let l_i be the length of C_i for $i = 1, 2$. By Claim 1 a) and b) the lengths l_1 and l_2 are commensurable with π and so is the length of every chord of C_1 and C_2. It remains to establish the claim for chords with one endpoint in C_1 and another in C_2.

To do so we parallel the proof of Claim 1 and the preceding construction of the square tiling. Let $\mathcal{K}$ be the set of chords (s, t) of $C_1 \cup C_2$ with $s \in C_1$, $t \in C_2$. We say that a pair $(x, y) \in C_1 \times C_2$ *uses a chord* $(s, t) \in \mathcal{K}$ if there exists a path of length $d_\Gamma(x, y)$ from x to y in Γ obtained by concatenating a path in C_1, the minimal length path from s to t, and a path in C_2. Let $S(K) \subseteq C_1 \times C_2$ be the set of pairs of points using the chord K of length $\pi - z$. Then analogously to the earlier observations we have that $S(K)$ is a square with side length $\sqrt{2}z$, the squares $\{S(K)\}_{K\in\mathcal{K}}$ cover $C_1 \times C_2$ and have disjoint interiors.

To apply Theorem 7.1 we need to transform the tiling of $C_1 \times C_2$ into a tiling of a product set, where the squares inherit the product structure. We do it as follows. Let n_1, n_2 be positive integers such that $n_1 l_1 = n_2 l_2$. By lifting our tiling of $C_1 \times C_2$ to the product of the n_1-fold cover of C_1 and the n_2-fold cover of C_2 we may assume without loss of generality that C_1 and C_2 have the same length l.

Let $\gamma_1 : \mathbb{R}/l\mathbb{Z} \to C_1$ and $\gamma_2 : \mathbb{R}/l\mathbb{Z} \to C_2$ be isometries. Let $\psi : \mathbb{R}/l\mathbb{Z} \times \mathbb{R}/l\mathbb{Z} \to C_1 \times C_2$ be defined by $\psi(x, y) = (\gamma_1(x + y), \gamma_2(x - y))$. Consider a chord $K = (s, t) \in \mathcal{K}$ of length $\pi - z$. Then $\psi^{-1}(S(K))$ consists of two squares of the form $[x - z/2, x + z/2] \times [y - z/2, y + z/2]$, where $(x, y) \in (\mathbb{R}/l\mathbb{Z})^2$ is either one of the two pairs satisfying $\gamma_1(x + y) = s, \gamma_2(x + y) = t$. Thus the preimage of our tiling of $C_1 \times C_2$ is a square tiling of $(\mathbb{R}/l\mathbb{Z})^2$ satisfying the conditions of Theorem 7.1, and by this theorem if $K \in \mathcal{K}$ is a chord of length $\pi - z$, then z is commensurable with l. As l is commensurable with π, the claim follows. □

We are now ready to establish that the conclusion of the theorem holds for bars.

Claim 3. *The length of every bar of* Γ' *is commensurable with* π.

Proof. Let B be a bar in Γ' with endpoints u and v, of length b, joining cycles C_1 and C_2 of length l_1 and l_2 respectively such that $u \in C_1$, $v \in C_2$. Let $\gamma : C \to \Gamma'$ be a local isometry from a circle C of length $l = l_1 + l_2 + 2b$, traversing B twice, and each of C_1 and C_2 once. Our goal is to apply Lemma 7.2 to a tiling of $R(\gamma)$ defined in the beginning of the proof.

Unlike in Claim 1, the set $Spl(\gamma)$ of spliced pairs is not empty, but it is not hard to analyze. Let $\gamma^{-1}(B) = [s_1, s_1 + b] \cup [s_2 - b, s_2]$ for some $s_1, s_2 \in C$. Then $Spl(\gamma)$ consists of two rectangles: one with corners $(s_1 - \pi, s_2), (s_1, s_2 + \pi), (s_1 + b + \pi, s_2 - b), (s_1 + b, s_2 - b - \pi)$, and the other obtained from it by permuting the coordinates.

Let $r = \pi\sqrt{2}$, and let $q = b\sqrt{2}$. Let $a = (l_1 + l_2)/\pi$. Then $l\sqrt{2} = 2q + ar$, and thus $R(\gamma)$ can be considered as a product of a circle ∂R of length $2q + ar$ and an interval of length $q + (a/2 - 1)r$. Each of rectangles in $Spl(\gamma)$ has a side of length r parallel to $\partial R(\gamma)$, and a side of length $q + r$ orthogonal to it. The cylinder $R(\gamma)$ is tiled by the rectangles in $Spl(\gamma)$ and squares in $\{S(K)\}_{K\in\mathcal{K}(\gamma)}$, and the first three conditions of Lemma 7.2 are satisfied for this tiling. Note that Lemma 7.2 would immediately imply the claim.

Thus to prove the claim it remains to verify the last condition of Lemma 7.2, i.e. to show that the total area of the squares corresponding to the chords of γ with length commensurable with π is strictly greater than $(a-4)r^2 = 2\pi(l_1 + l_2) - 8\pi^2$.

By Claims 1 b) and 2, every chord of $C_1 \cup C_2$ has length commensurable with π, and if such a chord has no endpoint in $\{u, v\}$ then it corresponds to a chord of γ of the same length. By Claim 1 c) the total area of the chords of C_i with no endpoint in $\{u, v\}$ is at least $2\pi l_i - 4\pi^2$. It remains to find at least one chord of $C_1 \cup C_2$ with one endpoint in $C_1 - \{u\}$ and another in $C_2 - \{v\}$. Indeed, consider a point $x \in C_1$ at distance at least π from u in C_1, and a point $y \in C_2$ at distance at least π from u in C_2. Then the chord used by (x, y) cannot have an endpoint in $\{u, v\}$ and so is as required. □

Let $V(\Gamma'), E(\Gamma')$ denote the vertex and edge set of the graph Γ'. We finish the proof of the theorem by reducing it to Claims 1 a) and 3. For the reduction it will be convenient for us to think of subgraphs of Γ' as elements of $\mathbb{Q}^{E(\Gamma')}$, the vector space of formal linear combinations of edges of Γ' with rational coefficients. Thus we identify every subgraph Λ of Γ' with $\sum_{e\in E(\Lambda)} e$. The theorem immediately follows from Claims 1 a) and 3 and the next claim, which uses the above convention.

Claim 4. *Every segment of* Γ' *is a rational linear combination of cycles and bars of* Γ'.

Proof. Let P be a segment of Γ'. We assume without loss of generality that Γ' is chosen minimal subject to the conditions that the minimum degree of Γ' is at least two, and P is a segment of Γ'. We further assume by suppressing vertices of degree two in Γ' that every vertex of Γ' has degree at least three. In particular, every segment of Γ' is an edge. Subject to these assumptions there are only a few isomorphism types of graphs to consider, and the proof proceeds by exhaustive case analysis.

Note first that every non-loop edge $e \in E(\Gamma')$ shares at least one endpoint with P, as otherwise $\Gamma' - e$ contradicts the minimality of Γ'. Similarly, if $w \in V(\Gamma') - V(P)$ is incident to a loop then $\deg(w) = 3$. (We use $\deg(x)$ to denote the degree of a vertex x in Γ'.) By the previous observation the non-loop edge incident to w shares an endpoint with P. We call such a vertex w a *pseudoleaf* of Γ'.

Let u and v be the endpoints of P. Suppose first that $\deg(u) \geq 4$. Let $e \in E(\Gamma') - E(P)$ be any edge incident to u. By minimality of Γ', either u or v has degree at most two in $\Gamma' - e$. Thus either e is a loop and $\deg(u) = 4$, or e joins u and v and $\deg(v) = 3$. As one of these outcomes holds for every edge in $E(\Gamma') - E(P)$ incident to u, we conclude that Γ' consists of P, a loop incident to u, an edge parallel to P, a pseudoleaf adjacent to v, which we denote by w, and loop incident to w.

It is not hard to verify that the claim holds in this case, and we do so using the following notation, which will also be used in all the remaining cases. We denote the edge with endpoints x and y by xy, and in the case when there are several such edges in Γ' we denote them by xy_1, xy_2,$\dots$ In particular, let $P = uv_1$.

Returning to our case, note that wv is a bar joining a loop at w and a cycle $P + uv_2$, while $wv + P$ is a bar joining loops at w and u. Thus $P = (wv + P) - wv$ is a difference of two bars, and the claim holds in this case. The case $\deg(v) \geq 4$ is symmetric.

It remains to consider the case $\deg(u) = \deg(v) = 3$. Suppose next that there exists $w \in V(\Gamma') - \{u, v\}$ such that w is not a pseudoleaf. As every non pseudoleaf vertex in $V(\Gamma') - \{u, v\}$ is incident to at least three edges which have u or v as their second endpoint, every vertex in $V(\Gamma')$ except u, v and w is a pseudoleaf.

If $\deg(w) \geq 4$, then w is joined to each of u and v by a pair of parallel edges. In this case each of $wv_1 + wv_2$, $wu_1 + wu_2$, $wv_1 + wu_1 + P$, $wv_2 + wu_2 + P$ is a cycle, and P is a rational linear combination of these cycles. Thus the claim holds.

If $\deg(w) = 3$, we suppose without loss of generality that w is joined to u by a pair of parallel edges, and to v by an edge. Then the remaining edge incident to v must have a pseudoleaf, which we denote by x, as its second endpoint. In this case, xv is a bar joining the loop at x to the cycle $P + uw_1 + wv$, and $xv + P$ is a bar joining the loop at x and the cycle $uw_1 + uw_2$. Therefore, P is the difference of two bars, and the claim holds.

We reduced our analysis to the case when $\deg(u) = \deg(v) = 3$ and every vertex in $V(\Gamma') - \{u, v\}$ is a pseudoleaf. We now consider subcases depending on the number of edges joining u to v.

If there are three such edges, then $P + uv_2$, $P + uv_3$ and $uv_2 + uv_3$ are each cycles, and P is their rational linear combination. If exactly two edges join u to v, then there is a pseudoleaf adjacent to each of u and v. Denote these pseudoleafs by u' and v' respectively. Then uu' is a bar joining the loop at u' and $P + uv_2$, and similarly vv' is a bar. The path $uu' + P + vv'$ is also a bar, joining the loops at u' and v', and P is a rational linear combination of the above three bars.

It remains to consider the case when P is the unique edge between u and v. In this case each of u and v is either incident to a loop, or adjacent to two pseudoleafs. Up to symmetry there are three final cases to consider.

If both u and v are incident to a loop then P is a bar. If u is incident to a loop and v is adjacent to pseudoleafs v' and v'', then each of the paths $v'v + P$, $v''v + P$ and $v'v + v''v$ is a bar joining the loops at its endpoints, and P is a rational linear combination of these bars. Finally, suppose that v is adjacent to pseudoleafs v' and v'', and u is adjacent to pseudoleafs u' and u''. Then P is a rational combination of bars $v'v + v''v$, $u'u + u''u$, $v'v + P + u'u$, $v''v + P + u''u$. Thus the claim holds in this last case. □

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