The topology of the punctured Banach-Mazur compacta

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Abstract

The main result of this paper is that the Banach-Mazur compacta are one point compactifications of Q-manifolds. We also show that they are homeomorphic to orbit spaces of some group actions on the Hilbert cube.

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1 Introduction

The Banach-Mazur compactum $BM(n)$, for $n = 2, 3, ...$, is the space of isometry classes of the n-dimensional Banach spaces and was introduced in [B].
A. Pełczyński posed the following two questions concerning the Banach-Mazur compacta (see [We]):
Are $BM(n)$'s absolute retracts ? Are they Hilbert cubes ?
An affirmative answer to the first question was then given in [Fa] for the case $n = 2$ and in [ABF], [An3] for the general case.
The second question remains open in the general case while for $n = 2$ it was solved negatively in [An4] (compare also [AB]). The idea of the proof (following the general idea of [TW]) is to show that the complement of the class of the Euclidean space in $BM(2)$ is not contractible while the complement of one point in the Hilbert cube is (see for example [M]).
In connection with the above results there arises a question about the topology of $BM(n) \setminus \{Eucl.\}$. In this paper we show that $BM(n) \setminus \{Eucl.\}$ is a Q-manifold for every $n \geq 2$. It should be noticed that similar results have been already published ([An5] and [ABR]). In [AR] it is showed that $BM(n) \setminus \{Eucl.\}$

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is a Q-manifold for $n = 2$. However our methods are different, seem to us more elementary (we do not use the Slice Theorem to show the disjoint discs property) and were developed independently (see [O]).
The proof is based on the characterization of Q-manifolds given by H. Toruńczyk in [T]. We use another representations of Banach-Mazur compactum $BM(n)$ (see [An4]) as the quotient of the space $C_0(n)$ (of compact, convex and symmetric with respect to the origin, bodies in $\mathbf{R}^n$, whose minimal volume ellipsoid is the unit ball B^n) with the Hausdorff metric, by a natural $\mathrm{O(n)}$-action.
Then we prove:

Theorem 1 *Let K be a closed subgroup of the group* $\mathrm{O(n)}$, $n \geq 2$. *Then* $(C_0(n) \setminus \{B^n\})/K$ *is a Q-manifold.*

As a corollary for $K = \mathrm{O(n)}$ we get the main result:

Corollary 2 *The complement $BM(n) \setminus \{Eucl.\}$ of the class of the Euclidean space in the Banach-Mazur compactum is a Q-manifold, for $n \geq 2$.*

Referring to the Toruńczyk characterization of the Hilbert cube Q (see [T]) we also get the following

Corollary 3 *For every finite subgroup K of* $\mathrm{O(n)}$, $C_0(n)/K$ *is homeomorphic to the Hilbert cube.*

And hence in the case $K = \{1\}$ the following holds.

Corollary 4 *For every $n \geq 2$ the Banach-Mazur compactum $BM(n)$ is homeomorphic to some orbit space* $Q/\mathrm{O(n)}$.

In Section 2 we give some basic definitions, notions and facts about Banach-Mazur compacta. In the next Section we prove Theorem 1 and Corollary 3. Then, in Section 4 we give some remarks about that proof and as a conclusion we prove Corollary 3. In Section 5 we sketch another proof of Theorem 1 in the case $n = 2$.

2 Preliminaries

In the rest of this paper we will denote by $\| \cdot \|$ the ordinary Euclidean norm on $\mathbf{R}^n$, i.e. $\| x \| = \sqrt{\sum_{i=1}^n x_i^2}$ for any $x = (x_1, ..., x_n) \in \mathbf{R}^n$, by B^n the unit, closed ball: $B^n = \{x \in \mathbf{R}^n : \| x \| \leq 1\}$ and if we not say otherwise, by B_r the closed ball of radius r: $B_r = \{x \in \mathbf{R}^n : \| x \| \leq r\}$ (where the value of n is clear from the context). We will always consider the metric on $S^{n-1} \subset \mathbf{R}^n$ induced by the norm $\| \cdot \|$.

For a given integer $n \geq 2$ the Banach-Mazur compactum $BM(n)$ is defined as the space of isometry classes of n-dimensional Banach spaces with the metric:

$$d([E],[F]) = \ln \inf \left\{ \| T \| \, \| T^{-1} \| : T : E \to F \text{ is a lin. isomorphism} \right\},$$

where $[E]$ denotes the isometry class of a Banach space E.
It is easy to check that this formula defines a metric on $BM(n)$.

Now let $C(n)$ denote the set of all compact, convex, symmetric with respect to the origin subsets of $\mathbf{R}^n$ with non empty interior, (we will call them bodies) with the Hausdorff metric d_{H}, defined by

$$d_{\mathrm{H}}(A,B) = \inf \{ \epsilon > 0 : A \subseteq B_\epsilon(B) \text{ and } B \subseteq B_\epsilon(A) \},$$

where $A, B \in C(n)$ and $B_\epsilon(A) = \{x \in \mathbf{R}^n : \| x - y \| < \epsilon \; for \; some \; y \in A\}$. With the natural action it is a $GL(n)$- space and it is well known (compare [We]) that $BM(n)$ is homeomorphic to the orbit space $C(n)/GL(n)$. Moreover according to [J] for every body $A \in C(n)$ there exists a unique (closed) ellipsoid of minimal volume $j(A)$ containing A (the so called *John ellipsoid* (or *Löwner ellipsoid* - see e.g. [L]). Let $C_0(n)$ be the subspace of $C(n)$ consisting of bodies whose John ellipsoid is the unit ball B^n. From the minimality of the volume and from the uniquness it follows that $C_0(n)$ is O(n) invariant (compare [An4]). One has the following representation of the Banach-Mazur compactum.

Theorem 5 ([An4], Corollary 1) *The Banach-Mazur compactum $BM(n)$ is homeomorphic to the* O(n)*-orbit space* $C_0(n)/\mathrm{O(n)}$.

(Compare also [AR]).

Recall that a space X is a Q-manifold (or a manifold modelled on the Hilbert cube - Q) iff it can be covered by open sets homeomorphic to open subsets of the Hilbert cube.
The following characterization is due to Toruńczyk:

Theorem 6 (see [T]) *A locally compact space X is a Q-manifold iff X is an ANR and for each $\epsilon > 0$ there exist two ϵ-close to the identity maps from X to itself which have disjoint images.*

3 The proof of the main theorem

The aim of this section is to prove Theorem 1 (see Introduction). For $K = \mathrm{O(n)}$ it implies, by Theorem 5, the main result, i.e. Corollary 2.

By Corollary 2 of [An4], and the 4.5 Corollary 1 of [V] $C_0(n)$ is a compact K-AR. Then according to Theorem 8 of [An2] (see also [A]) the orbit space $C_0(n)/K$ is an AR and, since an open subset of an AR is an ANR (compare [M]), a complement of one point in it, $(C_0(n) \setminus \{B^n\})/K$, is a locally compact ANR. So according to Theorem 6 to prove that it is a Q-manifold it suffices to show that there exist two maps from $(C_0(n) \setminus \{B^n\})/K$ to itself that are ϵ-close to the identity and have disjoint images. To do this we will construct two O(n)-equivariant maps from $C_0(n) \setminus \{B^n\}$ to itself, which are ϵ-close to the identity and have disjoint images. Such maps clearly induce the desired maps on $(C_0(n) \setminus \{B^n\})/K$.

We will construct those two maps in Subsection 3.1 and 3.2. Before that we will introduce the notion of a maximal set that leads to properties that will distinguish the two maps.

Let $C(S^{n-1}, \mathbf{R}^+)$ denote the space of continuous functions from S^{n-1} to $\mathbf{R}^+$ endowed with the supremum metric and $C_0(S^{n-1}, \mathbf{R}^+)$ its subspace consisting of functions bounded by 1 and symmetric with respect to the origin.
By $\tilde{C}(n)$ we denote the space of all subsets of $\mathbf{R}^n$ that are compact, symmetric with respect to the origin and with non-empty interior. We endow this space with the Hausdorff metric d_{H}. By $C(n)$ we will understand, as before, the subspace of $\tilde{C}(n)$ consisting of convex bodies and by $C_0(n)$ a subspace of $C(n)$ consisting of those bodies whose minimal volume elipsoid (John elipsoid) is the unit ball B^n.

Definition 7 *Define a map* $B : C_0(S^{n-1}, \mathbf{R}^+) \to \tilde{C}(n)$ *by* $B(\varphi) = \{0\} \cup \left\{x \in \mathbf{R}^n \setminus \{0\} : \| x \| \leq \varphi(\frac{x}{\|x\|})\right\}$, *where* $\| \cdot \|$ *means the Euclidean norm on* $\mathbf{R}^n$.
Denote also by $\varrho : \mathrm{im}(B) \to C_0(S^{n-1}, \mathbf{R}^+)$ *the radial function of a set given by* $\varrho(A)(x) = \max\{\lambda \in \mathbf{R}^+ : \lambda \cdot x \in A\}$

Remark. Observe that $C_0(n) \subset \mathrm{im}(B)$ and that for $A \in C_0(n)$ we have $\varrho(A)(x) = \frac{1}{\|x\|_A}$, where $\| \cdot \|_A$ is the norm on $\mathbf{R}^n$ whose unit ball is A.

Lemma 8 *a) For given* $\varphi, \psi \in C(S^{n-1}, \mathbf{R}^+)$ *if* $|\varphi - \psi| := \max\{|\varphi(x) - \psi(x)| : x \in S^{n-1}\} < \epsilon$ *then* $d_{\mathrm{H}}(B(\varphi), B(\psi)) < \epsilon$.
b) The map B *is a homeomorphism onto its image with inverse* ϱ.

PROOF. a) If $\lambda \cdot x \in B(\varphi)$ for some $x \in S^{n-1}$, $\lambda > 0$, then $\lambda \leq \varphi(x) < \psi(x) + \epsilon$ hence there exists $0 < \mu \leq \psi(x)$ such that $\lambda = \mu + \epsilon'$ with $0 < \epsilon' < \epsilon$. But $\mu \cdot x \in B(\psi)$ so that since $\| \lambda \cdot x - \mu \cdot x \| < \epsilon$, we get $B(\varphi) \subseteq B_\epsilon(B(\psi))$. Doing the same for points of $B(\psi)$ we get $d_{\mathrm{H}}(B(\varphi), B(\psi)) < \epsilon$.
b) It's clear that B is a bijection on its image with ϱ as its inverse. From a)

we get the continuity of B. The continuity of ϱ follows from the continuity of the map $B \mapsto \| \cdot \|_B$ and the above remark. $\square$

We will need the following

Definition 9 *A maximal point of $\varphi \in C(S^{n-1}, \mathbf{R}^+)$ is a point $x \in S^{n-1}$ satisfying the following condition: for every path $\gamma : [0,1] \to S^{n-1}$ such that $\gamma(0) = x$ and for any $t \in (0,1]$ such that $\varphi(\gamma(t)) > \varphi(x)$ there exists $t_0 \in (0,t)$ such that $\varphi(\gamma(t_0)) < \varphi(x)$. A maximal point of the radial function of a set A, i.e. of $\varrho(A)$ will be called "a maximal point of A". A path connected component of the set of maximal points of φ (or of A) will be called a maximal set of φ (or of A).*

Remark. Observe that a point $x \in S^{n-1}$ with $\varphi(x) = \max\{\varphi(y) : y \in S^{n-1}\}$ is a maximal point of φ . The next two lemmas show some basic technical properties of maximal sets.

Lemma 10 *For $x, y \in S^{n-1}$ belonging to the same maximal set of φ we have $\varphi(x) = \varphi(y)$.*

PROOF. Suppose $\varphi(x) < \varphi(y)$. Let $\gamma : [0,1] \to S^{n-1}$ be a curve with $\gamma(0) = x$, $\gamma(1) = y$ consisting of maximal points. Choose $t_0 \in [0,1)$ such that $\varphi(\gamma(t_0)) = \min\{\varphi(\gamma(t)) : t \in [0,1]\}$. Then for the curve $\gamma'(t) = \gamma(t_0 + t \cdot (1 - t_0))$, $t \in [0,1]$, for which $\gamma'(0) = \gamma(t_0)$ and $\gamma'(1) = \gamma(1) = y$, we have for $t \in [0,1]$, $\varphi(\gamma'(t)) \geq \varphi(\gamma'(0)) < \varphi(\gamma'(1)) = \varphi(y)$ which contradicts the maximality of $\gamma'(0) = \gamma(t_0)$.
Hence $\varphi(x) = \varphi(y)$. $\square$

Lemma 11 *Let $x \in S^{n-1}$ be a maximal point of φ and let $A = \{y \in S^{n-1} : \varphi(y) = \varphi(x)\}$. Then any point in the same (as x) path-connected component of the set A is maximal.*

PROOF. Suppose we have a non-maximal $y \in A$ and a path $\gamma : [0,1] \to A$ with $\gamma(0) = x$ and $\gamma(1) = y$. Then there exists a path $\sigma : [0,1] \to S^{n-1}$ with $\sigma(0) = y$ such that $\forall_{t \in [0,1]} \varphi(\sigma(t)) \geq \varphi(y)$ and $\varphi(y) < \varphi(\sigma(1))$. Hence the existence of the curve

$$\tilde{\gamma}(t) = \begin{cases} \gamma(2 \cdot t) & \text{if } t \in [0, \frac{1}{2}] \\ \sigma(2 \cdot t - 1) & \text{if } t \in [\frac{1}{2}, 1] \end{cases}$$

contradicts the maximality of x. $\square$

We will also need the following technical lemma

Lemma 12 *There exists a constant $C > 1$ such that for each $A \in C_0(n)$ and for any two points $x, y \in S^{n-1}$, $|\varphi(A)(x) - \varphi(A)(y)| \leq C\cdot \parallel x - y \parallel$.*

PROOF. Suppose there is no such constant. By compactness of S^{n-1} we could then find a sequence of pairs of points $\{(x_i, y_i)\}_{i=0}^{\infty} \subset S^{n-1} \times S^{n-1}$ converging to some (x, y) and a sequence $\{A_i\}_{i=0}^{\infty}$ of bodies in $C_0(n)$ such that $|\varrho(A_i)(x_i) - \varrho(A_i)(y_i)| > i\cdot \parallel x_i - y_i \parallel$.
In what follows for a point $x_i \in S^{n-1}$ we will denote by $\tilde{x}_i$ the corresponding point of the boundary of A_i that is $\tilde{x} = x \cdot \varrho(A_i)(x) \in \mathbf{R}^n$.
Consider the triangle $\triangle(0, \tilde{x}_i, \tilde{y}_i)$ (0 - the origin of $\mathbf{R}^n$). As $i \to \infty$, $\parallel x_i - y_i \parallel$ tends to 0 because all $\varrho(A_i)$'s are uniformly bounded by 1. Without loss of generality we can assume that $\varrho(A_i)(x_i) < \varrho(A_i)(y_i)$. Then, for some $a_i \in A_i$,

$$\begin{aligned}\mathrm{dist}(\tilde{x}_i,\ \text{line } 0\tilde{y}_i) = \parallel \tilde{x}_i - a_i \parallel \leq \parallel x_i - y_i \parallel < \\ < \frac{1}{i}(\parallel \tilde{y}_i \parallel - \parallel \tilde{x}_i \parallel) \leq \frac{1}{i}(\parallel \tilde{y}_i \parallel - \parallel a_i \parallel).\end{aligned}$$

It follows that the angle $\angle(0\tilde{y}_i\tilde{x}_i)$ tends to 0 as i tends to infinity , hence there exists $k > 0$ such that for $i > k$ the line $\tilde{x}_i\tilde{y}_i$ intersects the interior of the ball $B_{\sqrt{n}}$ which is contained in every A_i (see [H, Lemma 1.4.3]). Let $\hat{z}_i \in B_{\sqrt{n}}$ be such a point of intersection. Then there exists $\lambda > 1$ such that $z_i = \lambda\cdot\hat{z}_i \in B_{\sqrt{n}}$ and hence z_i belongs to A_i. By convexity the point v_i of intersection of lines $z_i\tilde{y}_i$ and $0x_i$ belongs to A_i and we have $\parallel v_i \parallel > \parallel \tilde{x}_i \parallel$, that contradicts the fact that $\tilde{x}_i = x \cdot \varphi(B_i)(x)$. This finishes the proof. $\square$

In the next two subsections we will construct two maps from $C_0(n) \setminus \{B^n\}$ to itself that were mentioned at the beginning of Section 3. The image of the first map will consist of bodies with finitely many maximal sets and the image of the second one will consist of bodies with infinitely many maximal sets each.

3.1 Construction of the first map

Let $\epsilon > 0$ be given. We shall construct an ϵ-close to the identity map
$G : C_0(n) \setminus \{B^n\} \to C_0(n) \setminus \{B^n\}$ whose image consists of bodies with finitely many maximal sets each.
First for a given body $A \in C_0(n) \setminus \{B^n\}$ define $\delta'(A)$ as
$\frac{1}{2}\max\{\delta > 0 : \exists_{B_\delta \subseteq S^{n-1}} \forall_{x \in B_\delta} \varrho(A)(x) \leq 1 - \delta\}$ where B_δ is the ball of radius δ in S^{n-1} - remember (see the beginning of Section 2) that metric on S^{n-1} is that induced by the Euclidean norm on $\mathbf{R}^n$.

Lemma 13 *The map $A \mapsto \delta'(A)$ is continuous and $\delta'(A) > 0$.*

PROOF. By Lemma 8 the map $A \mapsto \varrho(A)$ is continuous and the continuity of the map $\varrho(A) \mapsto \delta'(A)$ follows easily from the definition. The second assertion is clear. $\square$

Let $\delta(A) = \frac{1}{C} \cdot \min\left\{\frac{1}{4}\delta'(A), \epsilon\right\}$, where C is the constant of Lemma 12
Now define a function $\tilde{G}(A) : S^{n-1} \to \mathbf{R}^+$ by

$$\tilde{G}(A)(x) = \max\left\{\varrho(A)(y) : y \in B_{\delta(A)}(x)\right\},$$

where $B_\delta(x)$ is the δ-ball centered at x.

Lemma 14 *The function $\tilde{G}(A)$ belongs to $C_0(S^{n-1}, \mathbf{R}^+)\backslash\{1\}$ and every maximal point of $\tilde{G}(A)$ belongs to some $\delta(A)$-ball contained in the maximal set of that point.*

PROOF. The first assertion is clear in view of definitions of $\delta(A)$ and $\tilde{G}(A)$. For the second, let $x \in S^{n-1}$ be a maximal point of $\tilde{G}(A)$. Then $\tilde{G}(A)(x) = \varrho(A)(y)$ for some $y \in B_{\delta(A)}(x)$. But then for all $z \in B_{\delta(A)}(y)$, $\tilde{G}(A)(z) = \varrho(A)(y)$, because $\tilde{G}(A)(z) \geq \varrho(A)(y)$ by definition and if $\tilde{G}(A)(z) > \varrho(A)(y)$ then, by the geodesic convexity of $B_{\delta(A)}(y)$ one could find a geodesic segment $\gamma : [0,1] \to S^{n-1}$, with $\gamma(0) = x$, $\gamma(1) = z$, for which we would have $\tilde{G}(A)(\gamma(t)) \geq \tilde{G}(A)(x) < \tilde{G}(A)(z)$ for all $t \in [0,1]$, which contradicts the maximality of x. Since x is maximal we get from Lemma 11 that $B_{\delta(A)}(y)$ is contained in the same maximal set as x. $\square$

Lemma 15 *The map $A \mapsto \tilde{G}(A)$ is continuous.*

PROOF. Let $\tilde{\epsilon} > 0$ be given. By Lemmas 8 and 13 we can choose $\delta_1 > 0$, such that for every $D \in C_0(n)\backslash\{B^n\}$ with $d_{\mathrm{H}}(A, D) < \delta_1$, we have $|\varrho(D) - \varrho(A)| < \frac{\tilde{\epsilon}}{2}$ and $|\delta(D) - \delta(A)| < \frac{\tilde{\epsilon}}{4C}$ (C of Lemma 12). Now, for every $x \in S^{n-1}$ there exist $y \in B_{\delta(A)}(x) \subset S^{n-1}$, such that $\tilde{G}(A)(x) = \varrho(A)(y)$ and $z \in B_{2\cdot|\delta(D)-\delta(A)|}(y)$, such that $\tilde{G}(D)(x) \geq \varrho(D)(z)$ (if $\delta(D) \geq \delta(A)$ take $y = z$; if not one can find such z on the geodesic segment $(x, y) \subset S^{n-1}$) by the definition of $\tilde{G}$. Then $\tilde{G}(A)(x) - \tilde{G}(D)(x) \leq \varrho(A)(y) - \varrho(D)(z) = \varrho(A)(y) - \varrho(D)(y) + \varrho(D)(y) - \varrho(D)(z) \leq \frac{\tilde{\epsilon}}{2} + C\cdot \parallel z - y \parallel \leq \frac{\tilde{\epsilon}}{2} + C \cdot \frac{\tilde{\epsilon}}{2C} \leq \tilde{\epsilon}$. Similarly we have $\tilde{G}(D)(x) - \tilde{G}(A)(x) \leq \tilde{\epsilon}$ which finishes the proof. $\square$

Definition 16 *Define the map $G : C_0(n) \setminus \{B^n\} \to C_0(n) \setminus \{B^n\}$ by $G(A) = conv(B(\tilde{G}(A)))$, where $conv(A)$ means convex hull of A.*

Lemma 17 *Every maximal point of $G(A)$ belongs to some $\delta(A)$-ball contained in the same maximal set as that point.*

PROOF. For $y \in S^{n-1}$ we will denote by $\tilde{y}$ the point $y \cdot \varrho(G(A))(y) \in \mathbf{R}^n$, where $\cdot$ means the standard product by scalar in $\mathbf{R}^n$.
First, we shall show that if $x \in S^{n-1}$ is a maximal point of $\varrho(G(A))$, then $\varrho(G(A))(x) = \tilde{G}(A)(x)$.

Suppose $\varrho(G(A))(x) \neq \tilde{G}(A)(x)$. Then $\tilde{x} \in G(A) \setminus B(\tilde{G}(A))$, so that $\tilde{x} = \lambda_1 \cdot y_1 + ... + \lambda_l \cdot y_l$, for some $l \in N$, $y_1, ..., y_l \in B(\tilde{G}(A))$ and $\lambda_1, ..., \lambda_l \in (0,1)$ with $\lambda_1 + ... + \lambda_l = 1$. But then there exists $i \in \{1, ..., l\}$ such that $\| y_i \| > \| \tilde{x} \|$. If now there is a point y_i' lying on the segment $(\tilde{x}, y_i) \subset \mathbf{R}^n$ such, that $\| y_i' \| \leq \| \tilde{x} \|$ then $\tilde{x} \in conv\{y_1, ..., y_{i-1}, y_i', y_{i+1}, ..., y_l\}$. Then again there exists $j \in \{1, ..., i-1, i+1, ..., l\}$ such that $\| y_j \| > \| \tilde{x} \|$ and if there is $y_j' \in (\tilde{x}, y_j) \subset \mathbf{R}^n$ satisfying $\| y_j' \| \leq \| \tilde{x} \|$ then again
$\tilde{x} \in conv\left\{y_1, ..., y_{j-1}, y_j', y_{j+1}, ..., y_{i-1}, y_i', y_{i+1}, ..., y_l\right\}$. We could repeat this procedure but it is not possible that $\tilde{x} \in conv\{y_1', ..., y_l'\}$ for $y_i' \neq \tilde{x}$ such that $\| y_i' \| \leq \| \tilde{x} \|$, $i = 1, ..., l$. Hence there exists $i \in \{1, ..., l\}$ such that $\| y_i \| > \| \tilde{x} \|$ and for all $z \in (\tilde{x}, y_i]$, $\| z \| > \| \tilde{x} \|$, which contradicts the maximality of x.
Now, since $\varrho(G(A)) \geq \tilde{G}(A)$, x has to be a maximal point of $B(\tilde{G}(A))$ and, if $B_{\delta(A)} \subset S^{n-1}$ is a $\delta(A)$- ball, containing x and contained in the same maximal set of $B(\tilde{G}(A))$ as x(see Lemma 14), then for all $y \in B_{\delta(A)}$ we must have $\varrho(G(A))(y) = \tilde{G}(A)(y)$ and, in view of Lemma 11, every such y is a maximal point of $G(A)$. $\square$

Lemma 18 *The map G is an* O(n)- *equivariant, continuous, ϵ-close to the identity map from $C_0(n) \setminus \{B^n\}$ to itself, whose image consists of bodies with finitely many maximal sets.*

PROOF. It's clear, that $G(A) \in C_0(n) \setminus \{B^n\}$ for $A \in C_0(n) \setminus \{B^n\}$, and, in view of Lemma 17, that $G(A)$ has only finitely many maximal sets.
Since $|\varrho(A) - \tilde{G}(A)| \leq \epsilon$, in view of definitions of the constant C from Lemma 12, of $\delta(A)$ and of $\tilde{G}(A)$, we get $d_{\mathrm{H}}(A, B(\tilde{G}(A))) \leq \epsilon$, by Lemma 8.
Now, to show, that G is ϵ-close to the identity, it's enough to show, that if $d_{\mathrm{H}}(A, D) \leq \epsilon$, for $A, D \subset \mathbf{R}^n$, then $d_{\mathrm{H}}(conv(A), conv(D)) \leq \epsilon$.
We will show only that for each $x \in conv(A)$ there exists $y \in conv(D)$ such that, $\| x - y \| \leq \epsilon$ since analogously one has the same for A and D interchanged.
Let $x = \lambda_1 \cdot x_1 + ... + \lambda_k \cdot x_k$, for some $k \in N$, $x_1, ..., x_k \in A$ and $\lambda_1, ..., \lambda_k \in [0,1]$ with $\lambda_1 + ... + \lambda_k = 1$. Then, by the assumption,we can find $y_1, ..., y_k \in D$ such that $\| x_i - y_i \| \leq \epsilon$, for $i = 1, ..., k$. Hence $y = \lambda_1 \cdot y_1 + ... + \lambda_k \cdot y_k \in conv(D)$ and $\| x - y \| \leq \epsilon$.

From Lemma 15 we get the continuity of the map $A \mapsto B(\tilde{G}(A))$, so that the continuity of G follows from the continuity of the map $conv(\cdot)$, which is obvious, in view of what was shown above.

The O(n)- equivariance follows easily from the definition of $G(A)$ and the equivariance of Euclidean norm $\|\cdot\|$. $\square$

3.2 Construction of the second map

Let $\epsilon > 0$ be given. We will construct a map $F : C_0(n) \setminus \{B^n\} \to C_0(n) \setminus \{B^n\}$ which is ϵ-close to the identity and such that for $A \in C_0(n) \setminus \{B^n\}$, there are infinitely many maximal sets of $F(A)$.
To get that, given a body $A \in C_0(n) \setminus \{B^n\}$ we will construct an ascending sequence $\{F_k(A)\}_0^\infty$ of bodies in $C_0(n) \setminus \{B^n\}$ such that $F(A)$ will be the closure of the union of $F_k(A)$'s, and sequences $\{F_k(A)_{\min}\}_0^\infty$ of subsets of S^{n-1}, and $\{\varrho_k(A)_{\min}\}_0^\infty$ of positive numbers.

First, we define

$$F_0(A) = A,$$

$$\varrho_0(A)_{\min} = \min\left\{\varrho(A)(x) : x \in S^{n-1}\right\}$$

and

$$F_0(A)_{\min} = \left\{x \in S^{n-1} : \varrho(A)(x) = \varrho_0(A)_{\min}\right\}.$$

Then we shall proceed recursively. Assume, that for $k \geq 1$,
$F_0(A), ..., F_{k-1}(A)$, $F_0(A)_{\min}, ..., F_{k-1}(A)_{\min}$ and $\varrho_0(A)_{\min}, ..., \varrho_{k-1}(A)_{\min}$ are defined.
Let $\varrho_0(A)_{\max} = 1$ and $\varrho_{k-1}(A)_{\max} = \min\{\varrho(F_{k-1}(A))(x) : x \in F_{k-2}(A)_{\min}\}$, for $k \geq 2$.
For $k > 0$ set also $\delta_k = 4^{-k}\min\{\epsilon, \varrho_{k-1}(A)_{\max} - \varrho_{k-1}(A)_{\min}, 1 - \varrho_0(A)_{\min}\}$.
Finally, if we denote by $(*)$ the condition

$$(*) \qquad \varrho(F_{k-1}(A))(x) \geq \varrho_{k-1}(A)_{\min} + \delta_k$$

we can define a function $\tilde{F}_k(A)$ on S^{n-1} by

$$\tilde{F}_k(A)(x) = \begin{cases} \varrho(F_{k-1}(A))(x) & if \quad (*) \\ 2 \cdot (\varrho_{k-1}(A)_{\min} + \delta_k) - \varrho(F_{k-1}(A))(x) & if \quad \neg(*). \end{cases}$$

Then

$$F_k(A) = conv(B(\tilde{F}_k(A))),$$
$$\varrho_k(A)_{\min} = \min\left\{\varrho(F_k(A))(x) : x \in S^{n-1}\right\}$$

and

$$F_k(A)_{\min} = \left\{x \in S^{n-1} : \varrho(F_k(A))(x) = \varrho_k(A)_{\min}\right\}.$$

Remark. Notice that $\tilde{F}_k(A)$ is a function and $F_k(A)$ is a body.
Observe, that by the above definitions we have $\tilde{F}_k(A) \geq \tilde{F}_{k-l}(A)$ and hence $F_{k-l}(A) \subseteq F_k(A)$.

Definition 19 *Let $F(A)$ be the closure (with respect to the Euclidean norm on $\mathbf{R}^n$) of the union of all $F_k(A)$'s:*

$$F(A) = \overline{\bigcup_{k=0}^{\infty} F_k(A)}.$$

It's quite easy to observe that points of $F_k(A)_{\min}$ are "minimal" points of $F_k(A)$ in a sense analogous to the def. 9 of maximal points. Consequently they become maximal points of $\tilde{F}_{k+1}(A)$ which is a kind of "reflection" of $\varphi(F_k(A))$ with respect to the sphere of radius $\varrho_k(A)_{\min} + \delta_{k+1}$. In a few following lemmas we shall see that they are indeed maximal points of all $F_{k+l}(A)$, for $l \geq 1$.

Lemma 20 *For every $k = 0, 1, 2...$, if $x \in F_k(A)_{\min}$, then:*
a) $\tilde{F}_{k+1}(A)(x) = \varrho(F_{k+m}(A))(x) = \varrho(F(A))(x)$, for $m = 1, 2, 3, ...$,
b) there exists $\gamma > 0$ such that if $y \in S^{n-1}$ satisfies $\| x - y \| < \gamma$, then $\varrho(F(A))(x) \geq \varrho(F(A))(y)$, and if additionally $\tilde{F}_{k+1}(A))(x) > \tilde{F}_{k+1}(A))(y)$ then $\varrho(F(A))(x) > \varrho(F(A))(y)$.

PROOF. a) Let $\tilde{x} = \varrho(F_k(A))(x) \cdot x \in \mathbf{R}^n$ be a point of a border of $F_k(A)$ corresponding to x. Let $H \subset \mathbf{R}^n$ be a hyperplane perpendicular to x and containing $\tilde{x}$ (as a point), and H^-, H^+ denote, respectively the two closed half-spaces (into which H separates $\mathbf{R}^n$), the first containing 0. By the convexity of $F_k(A)$ and by definition of $F_k(A)_{\min}$ we have $F_k(A) \subset H^-$. Now, by the definition of $\tilde{F}_{k+1}(A)$ one can see, that $B(\tilde{F}_{k+1}(A)) \subset H^- \cup B_{\varrho_k(A)_{\min} + 2\cdot\delta_{k+1}}$, hence $B(\tilde{F}_{k+1}(A)) \subset (H + 2\delta_{k+1})^-$, where $H + 2\delta_k \subset H^+$ is the hyperplane parallel to H and at distance $2\delta_k$ from it, and $(H + 2\delta_k)^-$ - the corresponding half-space, containing 0. But then $F_{k+1}(A) = conv(B(\tilde{F}_{k+1}(A))) \subset (H + 2\delta_{k+1})^-$, hence $\tilde{F}_{k+1}(A)(x) = \varrho(F_{k+1}(A))(x)$.

This implies that $\varrho_{k+1}(A)_{\max} = \tilde{F}_{k+1}(A)(x) = \varrho_k(A)_{\min} + 2 \cdot \delta_{k+1}$ and hence, by definition of δ_{k+2} we get $\tilde{F}_{k+1}(A)(x) = \varrho_k(A)_{\min} + 2 \cdot \delta_{k+1} \geq \varrho_{k+1}(A)_{\min} + 2 \cdot \delta_{k+2}$ for all $k > 0$ and hence, by induction:

$$\tilde{F}_{k+1}(A)(x) = \varrho_k(A)_{\min} + 2 \cdot \delta_{k+1} \geq \varrho_l(A)_{\min} + 2 \cdot \delta_{l+1}, \tag{1}$$

for $l \geq k$. This implies that for $l \geq k$, $F_{l+1}(A) \subset conv(F_l(A) \cup B_{\tilde{F}_{k+1}(A)(x)})$ and hence, by induction $F_{l+1}(A) \subset (H+2\delta_{k+1})^-$, for all $l \geq k+1$. This shows that $\tilde{F}_{k+1}(A)(x) = \varrho(F_{k+m}(A))(x)$ for $m \geq 1$ and that $\tilde{F}_{k+1}(A)(x) = \varrho(F(A))(x)$.

b) Following the notations from a) we can similarily observe, that $B(\tilde{F}_{k+1}(A)) \subset (H^- \cap B^n) \cup B_{\varrho_k(A)_{\min}+2\cdot\delta_{k+1}}$ and, that $H \cap ((H^- \cap B^n) \cup B_{\varrho_k(A)_{\min}+2\cdot\delta_{k+1}}) = \{(\varrho_k(A)_{\min} + 2 \cdot \delta_{k+1}) \cdot x\}$. One can now find a positive ν such, that for $y \in S^{n-1}$ with $\| x-y \| \leq \nu$ one has $H_y \cap ((H^- \cap B^n) \cup B_{\varrho_k(A)_{\min}+2\cdot\delta_{k+1}}) = \{(\varrho_k(A)_{\min} + 2 \cdot \delta_{k+1}) \cdot y\}$, where H_y is the hyperplane perpendicular to y and containing $(\varrho_k(A)_{\min}+2\cdot\delta_{k+1})\cdot y$ (it is enough to choose ν such that then $H_y \cap (H^- \cap B^n) = \emptyset$). As in a) we have $F_l(A) \subset H_y^-$ (where H_y^- is the corresponding closed half-space containing 0) for all $l \geq k$ and hence $F(A) \subset H_y^-$ which means, that $\varrho(F(A))(x) \geq \varrho(F(A))(y)$.

Now, if $\tilde{F}_{k+1}(A)(x) > \tilde{F}_{k+1}(A)(y)$, then $H_y \cap B(\tilde{F}_{k+1}(A)) = \emptyset$ and, hence there exists $\alpha > 0$ such, that $B(\tilde{F}_{k+1}(A)) \subset (H_y - \alpha)^-$, where $H_y - \alpha \subset H_y^-$ is the hyperplane parallel to H_y lying at distance α from it, and $(H_y - \alpha)^-$ is the corresponding halfspace containing 0. This means that $\varrho(F_{k+1}(A))(x) > \varrho(F_{k+1}(A))(y)$.

Now observe that if $\delta_{i+1} > 0$ then for any $\alpha > 0$ one can find $x \in F_i(A)_{\min}$ and $y \in S^{n-1}$ satisfying $\| x-y \| \leq \alpha$ and $\tilde{F}_{i+1}(A)(x) > \tilde{F}_{i+1}(A)(y)$. Hence as above $\varrho(F_{i+1}(A))(x) > \varrho(F_{i+1}(A))(y)$ and consequently $\delta_{i+2} > 0$. Since $\delta_1 > 0$ we get by induction $\delta_i > 0$ for all $i > 0$.

By definition of δ_i it follows that we have strict inequality in formula (1), i.e.:

$$\tilde{F}_{k+1}(A)(x) = \varrho_k(B)_{\min} + 2 \cdot \delta_{k+1} > \varrho_l(B)_{\min} + 2 \cdot \delta_{l+1}, \tag{2}$$

for $l \geq k$.
Thus there exists $\beta > 0$ such that $F_{k+1}(A) \cup B_{\varrho_{k+1}(A)_{\min}+2\cdot\delta_{k+2}} \subset (H_y - \beta)^-$. By the same ineqality for $k+1$ instead of k we have $F_l(A) \subset conv(F_{l-1}(A) \cup B_{\varrho_{k+1}(A)_{\min}+2\cdot\delta_{k+2}})$, for $l > k+1$ hence, by induction $F_l(A) \subset (H_y - \beta)^-$, for all $l > k+1$. It follows that $F(A) \subset (H_y - \beta)^-$, which implies $\varrho(F(A))(x) > \varrho(F(A))(y)$. $\square$

Lemma 21 *The function $A \mapsto F(A)$ is an ϵ-close to the identity, continuous function from $C_0(n) \setminus \{B^n\}$ into itself.*

PROOF. First realize, that $|\tilde{F}_k(A) - \tilde{F}_{k+1}(A)| \leq \delta_k$ and hence $d_{\mathrm{H}}(F_k(A), F_{k+1}(A)) \leq \delta_k$ by arguments of the proof of Lemma 18. It follows that $d_{\mathrm{H}}(F_k(A), F_{k+l}(A)) \leq \frac{\epsilon}{2^k}$ for every $k, l \geq 0$ and hence, that F is ϵ-close to the identity.
Similarly we have $|\tilde{F}_k(A) - 1| \geq \frac{1-\varrho_0(A)_{\min}}{2}$, for every $k \geq 0$ so, that $F(A) \in$

$C_0(n) \setminus \{B^n\}$.
Now we shall show the continuity of F. Because of the above observations and since $\bigcup_{i=0}^{k} F_i(A) = F_k(A)$, it's enough to prove that maps $A \mapsto F_k(A)$ are continuous for all $k \in N$.
Let's proceed inductively.
For $k = 0$ the assertion is clear.
Now if we know that $A \mapsto F_i(A)$ is continuous for $i = 0, 1, ..., k-1$ then $\varrho_{k-1}(A)_{\min}$ depends continuously on A, because of its definition. By the above Lemma 20 we have $\varrho_{k-1}(A)_{\max} = \varrho_{k-2}(A)_{\min} + 2 \cdot \delta_{k-1}$ for $k > 1$, so that it is a continuous function of A (in the case $k = 1$ it is obvious and further we assume by induction that δ_{k-1} is a continous function of A - for $k = 2$ it is clear) and hence δ_k depends continuously on A. It follows that $A \mapsto \tilde{F}_k(A) \mapsto F_k(A)$ is continuous. $\square$

Lemma 22 *If $x \in F_k(A)_{\min}$ for some $k = 0, 1, 2...$, then x is a maximal point of $F(A)$.*

PROOF. Let $\gamma : [0, 1] \to S^{n-1}$ be a path with $\gamma(0) = x$. Assume, there exists such a $t_0 \in (0, 1]$, that $\varrho(F(A))(\gamma(t_0)) > \varrho(F(A))(x)$. Then we have to show, that there exists $t_1 \in (0, t_0)$ satisfying $\varrho(F(A))(\gamma(t_1)) < \varrho(F(A))(x)$. According to Lemma 20 a) $\varrho(F(A))(x) = \tilde{F}_{k+1}(A)(x)$. As is easy to observe and as it has been already pointed out x is a maximal point of $\tilde{F}_{k+1}(A)$, hence either for all $t \in [0, 1]$ $\tilde{F}_{k+1}(A)(\gamma(t)) = \tilde{F}_{k+1}(A)(x)$, which is impossible in view of Lemma 20 a) and of our assumption for t_0, or there is $t_2 \in (0, t_0)$ such, that for all $t \in [0, t_2]$ $\tilde{F}_{k+1}(A)(\gamma(t)) = \tilde{F}_{k+1}(A)(x)$ and for every $t > t_2$ there exists $s \in (t_2, t)$ such, that $\tilde{F}_{k+1}(A)(\gamma(s)) < \tilde{F}_{k+1}(A)(x)$. But then, by Lemma 20 b) we can find $s \in (0, t_0)$ such, that $\varrho(F(A))(\gamma(s)) < \varrho(F(A))(x)$ which ends the proof by setting $s = t_1$. $\square$

Lemma 23 *The map F is an* O(n)*- equivariant, continuous, ϵ-close to the identity map from $C_0(n) \setminus B^n$ to itself, whose image consists of bodies with infinitely many maximal sets.*

PROOF. By Lemma 21, F is continuous and ϵ-close to the identity and it follows easily from construction of $F(A)$ that it's O(n)-equivariant.
Now, by Lemma 20, for each $k > 0$ if $x \in F_k(A)_{\min}$ then x is a maximal point of $F(A)$ with $\varrho(F(A))(x) = \tilde{F}_{k+1}(A)(x)$. If $l > k$ and $y \in F_l(A)_{\min}$, then $\tilde{F}_l(A)(y) < \tilde{F}_k(A)(x)$, as in the proof of Lemma 20. Hence, by Lemma 10, x and y belong to distinct maximal sets of $F(A)$. In conclusion we get infinitely many maximal sets of $F(A)$ - at least one contained in each $F_k(A)_{\min}$ (in fact at least two, by symmetry of $F(A)$). $\square$

4 Remarks and corollaries

In this section we give some remarks and corollaries concerning the proof from the previous section. In particular we show that for any finite subgroup K of O(n) the space $C_0(n)/K$ is homeomorphic to the Hilbert cube Q (Corollary 3).

4.1 Remarks

There is also another characterization of Q-manifolds (compare Theorem 6), which states that *a locally compact ANR X is a Q-manifold iff for each $n \in \mathbf{N}$ any two maps $f, g : B^n \to X$ may be approximated by maps with disjoint images* (see [T], Remark 3).
Hence in order to prove that $(C_0(n) \setminus \{B^n\})/K$ is a Q-manifold for each closed subgroup K of O(n) it suffices to show that for every compact subset $Y \subseteq C_0(n) \setminus \{B^n\}$ and every $\epsilon > 0$ there exist two O(n)-equivariant, continuous, ϵ-close to the identity maps $F, G : Y \to C_0(n) \setminus \{B^n\}$ with disjoint images (because then the two compositions $F \circ f$ and $G \circ g$ would approximate f and g and have disjoint images).
If now one reminds the constructions of the maps G and F in Sections 3.1 and 3.2 one can notice that if we assume they both are defined on a compact set Y then the function $A \mapsto \delta(A)$ (see the definition in Section 3.1) can be chosen to be constant $\delta(A) \equiv \delta$ so that there exist $m \in \mathbf{N}$ such that, for every $A \in Y$, $G(A)$ consists of bodies with less than m maximal sets . Then, according to what was shown in Section 3.2 one could redefine the map F as follows: $F(A) = F_m(A)$, getting (see the proof of Lemma 23) that $\operatorname{im}(F) \cap \operatorname{im}(G) = \emptyset$.

4.2 The proof of Corollary 3

Let K be a finite subgroup of O(n). Choose $z_0 \in S^{n-1}$.
Let now $\epsilon > 0$ be given. As in the previous section we shall construct two K-equivariant, ϵ-close to the identity maps from $C_0(n)$ to itself, with disjoint images. To do that we use the maps F and G from the previous section as follows.
For $\alpha \in [0, 1)$ and $z \in S^{n-1}$ denote by $H_z - \alpha$ the hyperplane perpendicular to z (in a sense of an euclidean structure on $\mathbf{R}^n$) and at a distance $1 - \alpha$ from the origin. Let $\alpha_0 \in (0, 1)$ be such, that for all $y \neq z$, both belonging to $\{kz_0 : k \in K\} \cup \{k(-z_0) : k \in K\}$, one has $(H_z - \alpha) \cap (H_y - \alpha) \cap B^n = \emptyset$, for every $\alpha \leq \alpha_0$.
For $\alpha \in [0, 1)$ and $A \in C_0(n)$ define
$\tilde{J}_\alpha(A) = A \cap \bigcap_{k \in K}(H_{kz_0} - \alpha)^- \cap \bigcap_{k \in K}(H_{k(-z_0)} - \alpha)^-$, where $(H_z - \alpha)^-$ is one

of the closed half-spaces , the one containing the origin, into which $(H_z - \alpha)$ divides $\mathbf{R}^n$.
Clearly $\tilde{J}_\alpha(A)$ is a continuous function of A and α.
Unfortunately $\tilde{J}_\alpha(A)$ does not have to belong to $C_0(n)$. But according the John ellipsoid $j(\tilde{J}_\alpha(A))$ depends continuously on A and α (compare [ABF]). Hence if $k \in \mathrm{O(n)}$ maps the axes of $j(\tilde{J}_\alpha(A))$ onto the coordinate axes of $\mathbf{R}^n$ so that $k(j(\tilde{J}_\alpha(A))) = \left\{(x_1, ..., x_n) \in \mathbf{R}^n : \sum_{i=1}^n \frac{x_i^2}{a_i^2}\right\}$ then the map (stretching of A along axes of it's John ellipsoid):

$$(0,1) \times C_0(n) \ni (\alpha, A) \mapsto J_\alpha(A) \in C_0(n),$$

where

$$J_\alpha(A) = k^{-1} \begin{bmatrix} a_1^{-1} & 0 & \dots & 0 \\ 0 & a_2^{-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n^{-1} \end{bmatrix} k(\tilde{J}_\alpha(A))$$

is well defined (does not depend on a choice of k), continuous, because of the continuity of the map j (compare [ABF]) and K-equivariant (as K acts on A). Morover $J_\alpha(A) \neq B^n$ if $\alpha \in (0, \alpha_0)$ since if it were equal then $\tilde{J}_\alpha(A)$ would be an ellipsoid and hence $\tilde{J}_\alpha(A) \cap (\bigcup_{k \in K}(H_{kz_0} - \alpha) \cup \bigcup_{k \in K}(H_{k(-z_0)} - \alpha))$ would be finite. But if $A \neq \tilde{J}_\alpha(A)$ this intersection can't, clearly, be finite and if $A = \tilde{J}_\alpha(A)$ then A would not belong to $C_0(n)$ which would contradict our assumptions.
Now since for every $A \in C_0(n)$, $J_0(A) = A$ by continuity and the compactness of $C_0(n)$ one can choose $\delta \in (0, \alpha_0)$ such that $d_{\mathrm{H}}(A, J_\delta(A)) \leq \frac{\epsilon}{2}$ for every $A \in C_0(n)$.
Now let F and G be the $\frac{\epsilon}{2}$-close to the identity maps from $C_0(n) \setminus \{B^n\}$ to itself constructed in subsections 3.1 and 3.2. Then by Lemmas 18 and 23 and because of what was shown above we get

Lemma 24 *The two maps $A \mapsto F(J_\delta(A))$ and $A \mapsto G(J_\delta(A))$ are two continuous, ϵ-close to the identity, K-equivariant maps from $C_0(n)$ to itself, having disjoint images.*

These two maps induce two continuous, ϵ-close to the identity, maps from $C_0(n)/K$ into itself, having disjoint images. Since $C_0(n)$ is compact the orbit space $C_0(n)/K$ is compact too. Since $C_0(n)/K$ is an AR using the Torunczyk characterization of the Hilbert cube ([T]) we proved Corollary 3 and in particular, for $K = \{1\}$ we get Corollary 4.

5 Another proof in the case $n = 2$

In this section we will give a sketch of another proof of Theorem 1 in the case $n = 2$, which probably can be extended to an arbitrary $n \geq 2$.

As previously it suffices to construct two maps from $C_0(2) \setminus \{B^2\}$ to itself that are $O(2)$-equivariant, ϵ-close to the identity, one of them having an image consisting of bodies with differentiable boundaries and the other with a disjoint image. Those maps induce two maps from $BM(2) \setminus \{B^2\}$ to itself which satisfy the assumptions of Theorem 6. Since, as was shown in Section 3, $BM(2) \setminus \{B^2\}$ is a locally compact ANR, this will finish the proof of Theorem 1 in the case $n = 2$.

In general we follow the notation of Section 3.
Let, in addition $C^1(S^1, \mathbf{R}^+)$ denote the subspace of $C(S^1, \mathbf{R}^+)$ consisting of differentiable functions from S^1 to $\mathbf{R}^+$.

Let now $\epsilon > 0$ be given.

5.1 Construction of the first map

Definition 25 *Define the map $F : C_0(2) \setminus \{B^2\} \to C_0(2) \setminus \{B^2\}$ by:*

$$F(A) = a(A)(\overline{B_{\delta(A)}(A)}),$$

where the map

$$a(A) = k^{-1} \begin{bmatrix} a_1^{-1} & 0 \\ 0 & a_2^{-1} \end{bmatrix} k \in GL(2, \mathbf{R}),$$

$k \in O(2)$, is a right "stretching" as defined in Section 4.2;

i.e. $F(A)$ is the "stretching" of the closure of the $\delta(A)$-neighbourhood of A in $\mathbf{R}^2$; here $\delta(A) = \min\{\delta'(A), \epsilon\}$ for $\delta'(A)$ given in Section 3.1.
By fact 13 and the properties of δ-neighbourhood we obtain

Lemma 26 *The map F is an $O(2)$-equivariant, continuous, ϵ-close to the identity map from $C_0(2) \setminus \{B^2\}$ into $(C_0(2) \setminus \{B^2\}) \cap B(C^1(S^1, \mathbf{R}^+))$, i.e. into the subset of $C_0(2) \setminus \{B^2\}$ consisting of bodies with differentiable boundaries.*

First define a function $p : \mathbf{R} \to S^1 \subset \mathbf{R}^2$ by $p(t) = (\cos t, \sin t)$.
Let again $A \in C_0(2)$ and $G : C_0(2)\setminus\{B^2\} \to C_0(2)\setminus\{B^2\}$ be the $\frac{\epsilon}{4}$-close to the identity map of subsection 3.1. Then, by Lemma 18, $G(A)$ has only finitely many, let us say k maximal sets. Let $m_1 < n_1 < m_2 < n_2 < .... < m_k < n_k$ be numbers such that the images $p([m_1, n_1]), ..., p([m_k, n_k])$ are pairwise distinct (and hence all) maximal sets of $G(A)$. Define now $p^*(A) : \mathbf{R} \to \mathbf{R}^+$ by $p^*(A)(t) = \varrho(G(A))(p(t))$, where ϱ is the map defined in Section 3.
Now let us define:

$$R(A, x, y) = \int_x^y (\max\{p^*(A)(x) - p^*(A)(s) : x \le s \le t\})dt$$

$$L(A, x, y) = \int_x^y (\max\{p^*(A)(y) - p^*(A)(s) : t \le s \le y\})dt$$

$$r(A, x, y) = \int_x^y (\max\{p^*(A)(s) - p^*(A)(x) : x \le s \le t\})dt$$

$$l(A, x, y) = \int_x^y (\max\{p^*(A)(s) - p^*(A)(y) : t \le s \le y\})dt$$

and for $i = 1, 2, ..., k$ let $x_i \in \mathbf{R}$ be such, that $L(A, x_i, m_i) = 7$, $y_i \in \mathbf{R}$ such that $R(A, n_i, y_i) = 7$ (such numbers clearly exist by definition of maximal points and because $B \neq B^2$ and we choose 7 to assure that we integrate over an interval of length greater than the period 2π).
Let s be the homeomorphism: $\mathbf{R}^+ \cup \{\infty\} \to (0, 1]$ given by $s(t) = \frac{2}{\pi}\arctan(t)$, for $t \in \mathbf{R}^+$ and $s(\infty) = 1$.
For i as above define:

$$z_i^+ = \inf\left\{ s\left(\frac{R(A, n_i, y)}{r(A, n_i, y)}\right) : n_i < y < y_i \right\}$$

$$z_i^- = \inf\left\{ s\left(\frac{L(A, x, m_i)}{l(A, x, m_i)}\right) : x_i < x < m_i \right\}$$

(with the convention: $c/0 = \infty$ for $c > 0$) and:

$$z_i = \min\{z_i^+, z_i^-\}.$$

Then we define a 2π-periodic function $\tilde{H}(A) : \mathbf{R} \to \mathbf{R}^+$:

$$\tilde{H}(A)(t) = \begin{cases} p^*(A)(t) + z_i \cdot \frac{\epsilon}{4} \; if \;\; t = x_i + 2n\pi \;\; or \;\; t = y_i + 2n\pi \\ \qquad\qquad\qquad\qquad for \;\; some \;\; i \in \{1, ..., k\} \;\; and \;\; n \in \mathbf{Z} \\ \qquad p^*(A)(t) \qquad in \;\; other \;\; cases \end{cases}$$

This function induces a function $\hat{H}(A)$ on S^1 by the formula $\hat{H}(A)(x) = \tilde{H}(A)(t)$, where t is an arbitrary number such that $p(t) = x$. Unfortunately $\hat{H}(A)(x)$ is not always ≤ 1, hence $B(\hat{H}(A))$ (where the map B was defined at the begining of Section 3) does not need to be contained in B^2 but we can "stretch" it using the method of Section 4:

Definition 27 *Define the map* $H : C_0(2) \setminus \{B^2\} \to C_0(2) \setminus \{B^2\}$ *by:*

$$H(A) = \tilde{a}(A)(conv(B(\hat{H}(A)))),$$

where the map

$$\tilde{a}(A) = k^{-1} \begin{bmatrix} a_1^{-1} & 0 \\ 0 & a_2^{-1} \end{bmatrix} k \in GL(2, \mathbf{R}),$$

$k \in O(2)$, *is a right "stretching" as defined in Section 4.*

Lemma 28 *The map* H *is an* $O(2)$*-equivariant, continuous,* ϵ*-close to the identity map from* $(C_0(2) \setminus \{B^2\})$ *to* $C_0(2) \setminus B(C^1(S^1, \mathbf{R}^+))$*, i.e. its image is disjoint with that of* F*.*

A sketch of a proof of the last lemma can be found in [O].

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