Non-absolute gage integrals for multifunctions with values in an arbitrary Banach space

Luisa Di Piazza

University of Palermo (Italy)

Integration, Vector Measures and Related Topics VI

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Joint results with Kazimierz Musial

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\( \mathcal{L} \) denotes the family of all Lebesgue measurable subsets of [0, 1] and if \( A \in \mathcal{L} \), then \( |A| \) denotes its Lebesgue measure.
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Henstock integral for real valued functions

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- A **partition of** $[0, 1]$ is a finite collection of pairs $P = \{(I_1, t_1), \ldots, (I_p, t_p)\}$, where $I_1, \ldots, I_p$ are nonoverlapping intervals of $\mathcal{I}$, $t_j \in [0, 1]$, $j = 1, \ldots, p$, and $\bigcup_{j=1}^{p} I_j = [0, 1]$. If $t_j \in I_j$, $j = 1, \ldots, p$ we say that $P$ is a **Perron partition of** $[0, 1]$. 

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Given a gauge \( \delta \), a partition \( \{(I_1, t_1), \ldots, (I_p, t_p)\} \) is said to be **\( \delta \)-fine** if \( I_j \subset (t_j - \delta(t_j), t_j + \delta(t_j)) \), \( j = 1, \ldots, p \).
Definition

A function \( h : [0, 1] \to \mathbb{R} \) is said to be **Henstock-Kurzweil-integrable**, or simply **HK-integrable**, on \([0, 1]\) if there exists \( a \in \mathbb{R} \) with the following property: for every \( \varepsilon > 0 \) there exists a gauge \( \delta \) on \([0, 1]\) such that

\[
\left| \sum_{j=1}^{p} h(t_j)|l_j| - a \right| < \varepsilon,
\]

for each \( \delta \)-fine **Perron partition** \( \{(l_j, t_j) : j = 1, \ldots, p\} \) of \([0, 1]\).

We set \((HK) \int_0^1 hdt := a\).
$X$ is a general Banach space with its dual $X^*$.
\begin{itemize}
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\item $cb(X)$ all non-empty closed convex and bounded subsets of $X$
\end{itemize}
Multifunctions

- $X$ is a general Banach space with its dual $X^*$
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- We consider on $cb(X)$ the **Minkowski addition** 
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  (A \oplus B := \{a + b : a \in A, b \in B\})
  \]
  and the standard multiplication by scalars
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- We consider on $cb(X)$ the **Minkowski addition**
  \[(A \oplus B : = \{a + b : a \in A, b \in B\})\]
  and the standard multiplication by scalars
- $d_H$ is the Hausdorff metric on $cb(X)$
For each $C \in cb(X)$ the **support function of $C$** is denoted by $s(\cdot, C)$ and defined on $X^*$ by

$$s(x^*, C) = \sup\{\langle x^*, x \rangle : x \in C\},$$

for each $x^* \in X^*$.
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A multifunction is map $\Gamma : [0, 1] \rightarrow cb(X)$
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- A **multifunction** is map $\Gamma: [0, 1] \to cb(X)$

- A function $f: [0, 1] \to X$ is called a **selection of $\Gamma$** if $f(t) \in \Gamma(t)$, for every $t \in [0, 1]$. 
A multifunction \( \Gamma : [0, 1] \to cb(X) \) is said to be **scalarly measurable** if for every \( x^* \in X^* \), the function \( s(x^*, \Gamma(\cdot)) \) is measurable.
A multifunction \( \Gamma : [0, 1] \to cb(X) \) is said to be **scalarly measurable** if for every \( x^* \in X^* \), the function \( s(x^*, \Gamma(\cdot)) \) is measurable.

A multifunction \( \Gamma : [0, 1] \to cb(X) \) is said to be **scalarly integrable** (resp. **scalarly HK-integrable**) if \( s(x^*, \Gamma(\cdot)) \) is integrable (resp. HK-integrable) for every \( x^* \in X^* \).
Definition

A scalarly HK-integrable multifunction $\Gamma : [0, 1] \to cb(X)$ is said to be **Henstock-Kurzweil-Pettis integrable** (or simply **HKP-integrable**) in $cb(X)$, $[ck(X), cwk(X)]$ if for each $I \in \mathcal{I}$ there exists a set $\Phi_{\Gamma}(I) \in cb(X)$ $[ck(X), cwk(X)$, respectively$]$ such that

$$s(x^*, \Phi_{\Gamma}(I)) = \left(\text{HK}\right) \int_{I} s(x^*, \Gamma(t)) \, dt$$

for every $x^* \in X^*$. We write $\left(\text{HKP}\right) \int_{I} \Gamma(t) \, dt := \Phi_{\Gamma}(I)$ and call $\Phi_{\Gamma}(I)$ the **Henstock-Kurzweil-Pettis integral** of $\Gamma$ over $I$. 

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Henstock-Kurzweil-Pettis integrability

**Definition**

A scalarly HK-integrable multifunction $\Gamma : [0, 1] \to cb(X)$ is said to be **Henstock-Kurzweil-Pettis integrable** (or simply **HKP-integrable**) in $cb(X)$, $[ck(X), cwk(X)]$ if for each $I \in \mathcal{I}$ there exists a set $\Phi_{\Gamma}(I) \in cb(X)$ $[ck(X), cwk(X)$, respectively] such that

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A scalarly integrable multifunction \( \Gamma : [0, 1] \to cb(X) \) is said to be **Pettis integrable** (or simply **P-integrable** in \( cb(X), [ck(X), cwk(X)] \) if for each \( E \in \mathcal{L} \) there exists a set \( \Phi_{\Gamma}(E) \in cb(X) [ck(X), cwk(X), \text{ respectively}] \) such that

\[
s(x^*, \Phi_{\Gamma}(E)) = \int_E s(x^*, \Gamma(t)) \, dt \quad \text{for every } x^* \in X^*.
\]

We write \((P) \int_E \Gamma(t) \, dt := \Phi_{\Gamma}(E)\) and call \( \Phi_{\Gamma}(E) \) the **Pettis integral of \( \Gamma \) over \( E \).**
Proposition

(i) Let $\Gamma : [0, 1] \to cwk(X)$ be a scalarly HK-integrable multifunction. Then $\Gamma$ is HKP-integrable in $cwk(X)$ if and only if for each $I \in \mathcal{I}$ the mapping

$$x^* \mapsto (HK) \int_I s(x^*, \Gamma(t)) \, dt$$

is $\tau(X^*, X)$-continuous (where $\tau(X^*, X)$ is the Mackey topology on $X^*$).
Proposition

(i) Let $\Gamma : [0, 1] \rightarrow \text{cwk}(X)$ be a scalarly HK-integrable multifunction. Then $\Gamma$ is HKP-integrable in $\text{cwk}(X)$ if and only if for each $I \in \mathcal{I}$ the mapping

$$x^* \mapsto (\text{HK}) \int_I s(x^*, \Gamma(t)) \, dt$$

is $\tau(X^*, X)$-continuous (where $\tau(X^*, X)$ is the Mackey topology on $X^*$).

(ii) Let $\Gamma : [0, 1] \rightarrow \text{ck}(X)$ be a scalarly HK-integrable multifunction. Then $\Gamma$ is HKP-integrable in $\text{ck}(X)$ if and only if for each $I \in \mathcal{I}$ the mapping

$$x^* \mapsto (\text{HK}) \int_I s(x^*, \Gamma(t)) \, dt$$

is $\tau_c(X^*, X)$-continuous (where $\tau_c(X^*, X)$ is the topology on $X^*$ of uniform convergence on elements of $\text{ck}(X)$).
Selections of HKP-integrable multifunctions

**Proposition**

Let \( \Gamma : [0, 1] \to \text{cwk}(X) \) be a multifunction HKP-integrable in \( \text{cwk}(X) \). Then there exists an HKP-integrable selection \( f \) of \( \Gamma \). Moreover each scalarly measurable selection \( f \) of \( \Gamma \) is HKP-integrable.

**Sketch of the proof.**

Since \( \Gamma \) is scalarly HK-integrable, it is scalarly measurable. So by a remarkable result of Cascales-Kadets-Rodriguez (2010) we have the existence of a scalarly measurable selection \( f \) of \( \Gamma \).

Then, for each \( x^* \in X^* \) we have

\[
- s(-x^*, \Gamma(t)) \leq x^* f(t) \leq s(x^*, \Gamma(t))
\]

and the HK-integrability of the function \( x^* f \) follows.
**Proposition**

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Sketch of the proof. Since $\Gamma$ is scalarly HK-integrable, it is scalarly measurable. So by a remarkable result of Cascales-Kadets-Rodriguez (2010) we have the existence of a scalarly measurable selection $f$ of $\Gamma$. Then, for each $x^* \in X^*$ we have

$$-s(-x^*, \Gamma(t)) \leq x^* f(t) \leq s(x^*, \Gamma(t)).$$

$$0 \leq x^* f(t) + s(-x^*, \Gamma(t)) \leq s(x^*, \Gamma(t)) + s(-x^*, \Gamma(t)).$$

and the HK-integrability of the function $x^* f$ follows.
Moreover for each $I \in \mathcal{I}$

$$-(HK) \int_I s(-x^*, \Gamma(t)) \, dt \leq (HK) \int_I x^* f(t) \, dt \leq (HK) \int_I s(x^*, \Gamma(t)) \, dt.$$ 

So by previous characterization $f$ is HKP-integrable. \qed

By the symbol $\mathcal{S}_{\text{HKP}}(\Gamma)$ we denote the family of all selections of $\Gamma$ that are HKP-integrable.
Theorem

Let $\Gamma : [0, 1] \rightarrow cwk(X)$ be a scalarly measurable multifunction. Then $\Gamma$ is HKP-integrable in $cwk(X)$ if and only if each scalarly measurable selection $f$ of $\Gamma$ is HKP-integrable.
Theorem

Let \( \Gamma : [0, 1] \to cwk(X) \) be a scalarly measurable multifunction. Then \( \Gamma \) is HKP-integrable in \( cwk(X) \) if and only if each scalarly measurable selection \( f \) of \( \Gamma \) is HKP-integrable.

Theorem

If \( \Gamma : [0, 1] \to ck(X) \) is scalarly HK-integrable, then TFAE:

1. \( \Gamma \) is HKP-integrable in \( ck(X) \) and \( \Phi \Gamma(I) := \bigcup I \in I \Phi \Gamma(I) \) is relatively compact
2. Each scalarly measurable selection of \( \Gamma \) is HKP-integrable and has norm relatively compact range of its integral
3. Each scalarly measurable selection of \( \Gamma \) is HKP-integrable and has continuous primitive.
Theorem

Let \( \Gamma : [0, 1] \to cwk(X) \) be a scalarly measurable multifunction. Then \( \Gamma \) is HKP-integrable in \( cwk(X) \) if and only if each scalarly measurable selection \( f \) of \( \Gamma \) is HKP-integrable.

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**Theorem**

Let $\Gamma : [0, 1] \to \text{cw}k(X)$ be a scalarly measurable multifunction. Then $\Gamma$ is HKP-integrable in $\text{cw}k(X)$ if and only if each scalarly measurable selection $f$ of $\Gamma$ is HKP-integrable.

**Theorem**

If $\Gamma : [0, 1] \to \text{ck}(X)$ is scalarly HK-integrable, then TFAE:

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Let $\Gamma : [0, 1] \to \text{cwk}(X)$ be a scalarly measurable multifunction. Then $\Gamma$ is HKP-integrable in $\text{cwk}(X)$ if and only if each scalarly measurable selection $f$ of $\Gamma$ is HKP-integrable.

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3. Each scalarly measurable selection of $\Gamma$ is HKP-integrable and has continuous primitive.
A Decomposition Theorem

**Theorem**

A scalarly HK-integrable multifunction $\Gamma : [0, 1] \to \text{ck}(X)[\text{cwk}(X)]$ is **HKP-integrable** in $\text{ck}(X)[\text{cwk}(X)]$ if and only if $S_{HKP}(\Gamma) \neq \emptyset$ and for every $f \in S_{HKP}(\Gamma)$ the multifunction $G : [0, 1] \to \text{ck}(X)[\text{cwk}(X)]$ defined by

$$\Gamma(t) = G(t) + f(t)$$

is **Pettis integrable** in $\text{ck}(X)[\text{cwk}(X)]$. 
We know that for Pettis integrable functions the space $c_0$ is that space which makes problems: if $c_0 \subset X$ isomorphically, then there are $X$-valued scalarly integrable functions that are not Pettis integrable.
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In case of HKP-integral a similar role to spaces not containing $c_0$ is played by weakly sequentially complete separable Banach spaces.
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In case of HKP-integral a similar role to spaces not containing $c_0$ is played by weakly sequentially complete separable Banach spaces. Let us recall that $X$ is called *weakly sequentially complete* if each weakly Cauchy sequence in $X$ is weakly convergent. It is known that no weakly sequentially complete Banach space can contain an isomorphic copy of $c_0$. 

(Gordon 1989): A separable Banach space $X$ is weakly sequentially complete if and only if each $X$-valued scalarly HK-integrable function $f: [0, 1] \to X$ is HKP integrable.
We know that for Pettis integrable functions the space $c_0$ is that space which makes problems: if $c_0 \subset X$ isomorphically, then there are $X$-valued scalarly integrable functions that are not Pettis integrable.

In case of HKP-integral a similar role to spaces not containing $c_0$ is played by weakly sequentially complete separable Banach spaces. Let us recall that $X$ is called **weakly sequentially complete** if each weakly Cauchy sequence in $X$ is weakly convergent. It is known that no weakly sequentially complete Banach space can contain an isomorphic copy of $c_0$.

(Gordon 1989): **A separable Banach space $X$ is weakly sequentially complete if and only if each $X$-valued scalarly HK-integrable function $f : [0, 1] \to X$ is HKP integrable.**
Integration in weakly sequentially complete Banach spaces

We recall that a space \( Y \) determines a function \( f : [0, 1] \to X \) (resp. a multifunction \( \Gamma : [0, 1] \to cb(X) \)) if \( x^*f = 0 \) (resp. \( s(x^*, \Gamma) = 0 \)) a.e. for each \( x^* \in Y^\perp \) (the exceptional sets depend on \( x^* \)).
We recall that a space $Y$ determines a function $f : [0, 1] \to X$ (resp. a multifunction $\Gamma : [0, 1] \to cb(X)$) if $x^* f = 0$ (resp. $s(x^*, \Gamma) = 0$) a.e. for each $x^* \in Y^\perp$ (the exceptional sets depend on $x^*$).

**Theorem**

For an arbitrary Banach space $X$ TFAE:

1. $X$ is weakly sequentially complete Banach space
We recall that a space $Y$ determines a function $f : [0, 1] \rightarrow X$ (resp. a multifunction $\Gamma : [0, 1] \rightarrow \text{cb}(X)$) if $x^* f = 0$ (resp. $s(x^*, \Gamma) = 0$) a.e. for each $x^* \in Y^\perp$ (the exceptional sets depend on $x^*$).

**Theorem**

For an arbitrary Banach space $X$ TFAE:

1. $X$ is weakly sequentially complete Banach space

2. Each scalarly HK-integrable function $f : [0, 1] \rightarrow X$ that is determined by a weakly compactly generated (WCG) space is HKP-integrable
We recall that a space $Y$ determines a function $f : [0, 1] \to X$ (resp. a multifunction $\Gamma : [0, 1] \to \text{cb}(X)$) if $x^* f = 0$ (resp. $s(x^*, \Gamma) = 0$) a.e. for each $x^* \in Y^\perp$ (the exceptional sets depend on $x^*$).

Theorem

For an arbitrary Banach space $X$ TFAE:

1. $X$ is weakly sequentially complete Banach space

2. Each scalarly HK-integrable function $f : [0, 1] \to X$ that is determined by a weakly compactly generated (WCG) space is HKP-integrable

3. Each scalarly HK-integrable multifunction $\Gamma : [0, 1] \to \text{cwk}(X)[\text{ck}(X)]$ that is determined by a WCG space, is HKP-integrable in $\text{cwk}(X)$.
We recall that a Banach space $X$ has the **Schur property** if each sequence weakly convergent to 0 is also norm convergent. It is well known that each space with the Schur property is weakly sequentially complete.
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**Theorem**

For an arbitrary Banach space $X$ TFAE:

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**Theorem**

For an arbitrary Banach space $X$ TFAE:

1. $X$ has the Schur property
2. Each scalarly HK-integrable multifunction $\Gamma : [0, 1] \to ck(X)$ that is determined by a WCG space, is HKP-integrable in $ck(X)$.

Proof. 

$(1) \Rightarrow (2)$ According to previous theorem, if $\Gamma : [0, 1] \to ck(X)$ is scalarly HK-integrable and determined by a WCG space, then it is HKP-integrable in $ck(X)$. The Schur property of $X$ forces the integrability in $ck(X)$. □
Integration in Banach spaces possessing the Schur property

We recall that a Banach space $X$ has the **Schur property** if each sequence weakly convergent to 0 is also norm convergent. It is well known that each space with the Schur property is weakly sequentially complete.

**Theorem**

For an arbitrary Banach space $X$ TFAE:

1. $X$ has the Schur property
2. Each scalarly HK-integrable multifunction $\Gamma : [0, 1] \to ck(X)$ that is determined by a WCG space, is HKP-integrable in $ck(X)$.

**Proof.** (1) $\Rightarrow$ (2) According to previous theorem, if $\Gamma : [0, 1] \to ck(X)$ is scalarly HK-integrable and determined by a WCG space, then it is HKP-integrable in $cwk(X)$. The Schur property of $X$ forces the integrability in $ck(X)$. □
Definition

A multifunction $\Gamma : [0, 1] \rightarrow cb(X)$ is said to be **Henstock** (resp. **McShane** integrale), if there exists a set $\Phi_{\Gamma}[0, 1] \in cb(X)$ with the following property: for every $\varepsilon > 0$ there exists a gauge $\delta$ on $[0, 1]$ such that for each $\delta$–fine Perron partition (resp. partition) $\{(l_1, t_1), \ldots, (l_p, t_p)\}$ of $[0, 1]$, we have

$$dH(\Phi_{\Gamma}[0, 1], \bigoplus_{i=1}^{p} \Gamma(t_i) | I_i) < \varepsilon.$$
Definition

A multifunction $\Gamma : [0, 1] \to cb(X)$ is said to be **Henstock** (resp. **McShane**) integrable, if there exists a set $\Phi_{\Gamma}[0, 1] \in cb(X)$ with the following property: for every $\varepsilon > 0$ there exists a gauge $\delta$ on $[0, 1]$ such that for each $\delta$–fine Perron partition (resp. partition) $\{(l_1, t_1), \ldots, (l_p, t_p)\}$ of $[0, 1]$, we have

$$d_H\left(\Phi_{\Gamma}[0, 1], \bigoplus_{i=1}^{p} \Gamma(t_i)|l_i|\right) < \varepsilon.$$
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$$d_H\left(\Phi_\Gamma [0, 1], \bigoplus_{i=1}^{p} \Gamma(t_i)|l_i|\right) < \varepsilon .$$

We write then $(H) \int_{0}^{1} \Gamma(t) \, dt := \Phi_\Gamma [0, 1]$ (resp. $(MS) \int_{0}^{1} \Gamma(t) \, dt := \Phi_\Gamma [0, 1]$).
Remarks:

From the definition and the completeness of the Hausdorff metric in \( cwk(X)[ck(X)] \), it is easy to see that if a \( cwk(X)[ck(X)] \)-valued multifunction is Henstock integrable, then also \( \Phi_{\Gamma}[0,1] \in cwk(X)[ck(X)] \).
Remarks:

- From the definition and the completeness of the Hausdorff metric in $cwk(X)[ck(X)]$, it is easy to see that if a $cwk(X)[ck(X)]$-valued multifunction is Henstock integrable, than also $\Phi_\Gamma[0,1] \in cwk(X)[ck(X)]$.

- Each McShane integrable multifunction, is also Henstock integrable (with the same values of the integrals).
According to Hörmander's equality

$$d_H\left( K, \bigoplus_{i=1}^{p} \Gamma(t_i)|l_i| \right) = \sup_{\|x^*\| \leq 1} \left| s(x^*, K) - \sum_{i=1}^{p} s(x^*, \Gamma(t_i))|l_i| \right|.$$
According to Hörmander’s equality

\[
\text{d}_H\left( K, \bigoplus_{i=1}^p \Gamma(t_i) \middle| I_i \right) = \sup_{\|x^*\| \leq 1} \left| s(x^*, K) - \sum_{i=1}^p s(x^*, \Gamma(t_i)) \middle| I_i \right|.
\]

Let us consider the embedding \( j : cb(X) \to l_\infty(B(X^*)) \) defined by

\[
j(K)(x^*) = s(x^*, K).
\]
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The images \( j(cb(X)), j(ck(X)) \) and \( j(cwk(X)) \) are closed cones of \( l_{\infty}(B(X^*)) \). So, if \( z \in l_{\infty}(B(X^*)) \) is the value of the Henstock integral of \( j \circ \Gamma \), then there exists a set \( K \in cb(X) [ck(X), cwk(X)] \) with \( j(K) = z \).
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The images \( j(cb(X)) \), \( j(ck(X)) \) and \( j(cwk(X)) \) are closed cones of \( l_\infty(B(X^*)) \). So, if \( z \in l_\infty(B(X^*)) \) is the value of the Henstock integral of \( j \circ \Gamma \), then there exists a set \( K \in cb(X)[ck(X), cwk(X)] \) with \( j(K) = z \).

Therefore: a multifunction \( \Gamma : [0, 1] \rightarrow cb(X) \) is Henstock (or McShane) integrable if and only if the single valued function \( j \circ \Gamma : [0, 1] \rightarrow l_\infty(B(X^*)) \) is Henstock (or McShane) integrable in the usual sense.
If \( \Gamma : [0, 1] \to cb(X)[cwk(X), ck(X)] \) is Henstock integrable, then it is also Henstock-Kurzweil-Pettis integrable in \( cb(X)[cwk(X), ck(X)] \).
Henstock and McShane integrals for multifunctions

- If $\Gamma : [0, 1] \rightarrow cb(X)[cwk(X), ck(X)]$ is Henstock integrable, then it is also Henstock-Kurzweil-Pettis integrable in $cb(X)[cwk(X), ck(X)]$.

- If $\Gamma : [0, 1] \rightarrow cb(X)[cwk(X), ck(X)]$ is McShane integrable, then it is also Pettis integrable in $cb(X)[cwk(X), ck(X)]$. 
Equi-integrability.

In the theory of Lebesgue integration \textit{uniform integrability} plays an essential role. It’s counterpart in the theory of gauge integrals is the notion of \textit{equi-integrability}.
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**Definition**

We recall that a family of real valued HK-integrable (or McShane integrable) functions \( \{g_\alpha : \alpha \in \mathbb{A}\} \) is **Henstock** (resp. **McShane**) equi-integrable on \([0, 1]\) whenever for every \( \varepsilon > 0 \) there is a gauge \( \delta \) such that

\[
\sup \left\{ \left| \sum_{j=1}^{p} g_\alpha(t_j)|l_j| - (HK) \int_{0}^{1} g_\alpha \, dt \right| : \alpha \in \mathbb{A} \right\} < \varepsilon,
\]

for each \( \delta \)-fine Perron partition (resp. partition) \( \{(l_j, t_j) : j = 1, \ldots, p\} \) of \([0, 1]\).
Equi-integrability

Given a multifunction \( \Gamma : [0, 1] \to cb(X) \) we set

\[
Z_\Gamma := \{ s(x^*, \Gamma(\cdot)) : \|x^*\| \leq 1 \},
\]

Proposition A scalarly HK–integrable (resp. integrable) multifunction \( \Gamma : [0, 1] \to cb(X) \), is Henstock (resp. McShane) integrable iff the family \( Z_\Gamma \) is Henstock (resp. McShane) equi-integrable.
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**Proposition**

A scalarly HK–integrable (resp. integrable) multifunction $\Gamma : [0, 1] \to cb(X)$, is Henstock (resp. McShane) integrable iff the family $Z_\Gamma$ is Henstock (resp. McShane) equi-integrable.
Selections of Henstok or McShane integrable multifunctions.

By \( S_H(\Gamma) \) \( [S_{MS}(\Gamma), S_P(\Gamma)] \) we denote the family of all scalarly measurable selections of \( \Gamma \) that are Henstock \([\text{McShane, Pettis}]\) integrable.
Selections of Henstok or McShane integrable multifunctions.

By $S_H(\Gamma)$ [or $S_{MS}(\Gamma)$, $S_P(\Gamma)$] we denote the family of all scalarly measurable selections of $\Gamma$ that are Henstock [McShane, Pettis] integrable.

Theorem

If $\Gamma : [0, 1] \to \text{cwk}(X)$ is Henstock integrable, then $S_H(\Gamma) \neq \emptyset$. 
Selections of Henstok or McShane integrable multifunctions.

By $S_H(\Gamma) \cup S_{MS}(\Gamma) \cup S_P(\Gamma)$ we denote the family of all scalarly measurable selections of $\Gamma$ that are Henstock [McShane, Pettis] integrable.

**Theorem**

If $\Gamma : [0, 1] \rightarrow cwk(X)$ is Henstock integrable, then $S_H(\Gamma) \neq \emptyset$.

**Sketch of the proof.** In the first part we proceed in a way similar to that of Cascales-Kadets-Rodriguez (2009) for Pettis integrable multifunctions.
Let \( \Gamma : [0, 1] \rightarrow cwk(X) \) be Henstock integrable. Since
\[
H := \left( H \right) \int_0^1 \Gamma(t) \, dt \in cwk(X),
\]
there exists a **strongly exposed point** \( x_0 \in H \). Assume that \( x_0^* \in B(X^*) \) is such that
\[
x_0^*(x) > x_0^*(x_0)
\]
for every \( x \in H \setminus \{x_0\} \) and the sets
\[
\{ x \in H : x_0^*(x) > x_0^*(x_0) - \alpha \}, \; \alpha \in \mathbb{R},
\]
form a neighborhood basis of \( x_0 \) in the norm topology on \( H \).
Let \( \Gamma : [0, 1] \to cwk(X) \) be Henstock integrable. Since \( H := \int_0^1 \Gamma(t) \, dt \in cwk(X) \), there exists a strongly exposed point \( x_0 \in H \). Assume that \( x_0^* \in B(X^*) \) is such that \( x_0^*(x) > x_0^*(x_0) \) for every \( x \in H \setminus \{x_0\} \) and the sets \( \{x \in H : x_0^*(x) > x_0^*(x_0) - \alpha\} \), \( \alpha \in \mathbb{R} \), form a neighborhood basis of \( x_0 \) in the norm topology on \( H \).

We define \( G : [0, 1] \to cwk(X) \) by

\[
G(t) := \{x \in \Gamma(t) : x_0^*(x) = s(x_0^*, \Gamma(t))\}.
\]

Since \( \Gamma \) is Henstock integrable, then \( \Gamma \) is also HKP-integrable in \( cwk(X) \) and also \( G \) is HKP-integrable in \( cwk(X) \). Let \( g : [0, 1] \to X \) be any selection of \( G \). Then \( g \) is scalarly measurable (and of course HKP-integrable). Moreover \( x_0^*(x_0) = (HK) \int_0^1 x_0^*g(t) \, dt \).
Let $\varepsilon > 0$ and $0 < \varepsilon' < \varepsilon/2$ be arbitrary. Then, let $0 < \eta < \varepsilon'$ be such that

$$\forall x \in H \quad [|x_0^*(x) - x_0^*(x_0)| < \eta \Rightarrow \|x - x_0\| < \varepsilon'].$$  \hspace{1cm} (3)

Since $\Gamma$ is Henstock integrable and $x_0^*g$ is HK-integrable we can find a gauge $\delta$ on $[0, 1]$ such that for each $\delta$-fine Perron partition $\mathcal{P} := \{(l_1, t_1), \ldots, (l_p, t_p)\}$ of $[0, 1]$

$$d_H \left( H, \bigoplus_{i=1}^{p} \Gamma(t_i)|l_i| \right) < \eta/2$$

and

$$\left| \int_0^1 x_0^*g(t) \, dt - \sum_{i=1}^{p} x_0^*g(t_i)|l_i| \right| < \eta/2.$$  

So there exists a point $x_{\mathcal{P}} \in H$ with
Continuation of the proof

\[
\left\| \sum_{i=1}^{p} g(t_i) |l_i| - x_P \right\| < \eta/2.
\]

and so

\[
|x_0^*(x_P) - x_0^*(x_0)| \leq \left| x_0^*(x_P) - x_0^* \left( \sum_{i=1}^{p} g(t_i) |l_i| \right) \right| +
\]

\[
+ \left| \sum_{i=1}^{p} x_0^* g(t_i) |l_i| - x_0^*(x_0) \right| < \eta.
\]

Now, previous inequality yields \( \|x_P - x_0\| < \varepsilon' \)
Finally

\[ \left\| \sum_{i=1}^{p} g(t_i) l_i - x_0 \right\| \leq \left\| \sum_{i=1}^{p} g(t_i) l_i - x_P \right\| + \| x_P - x_0 \| < \varepsilon. \]
Fremlin (1994) proved that a Banach space valued function is McShane integrable if and only if it is Henstock and Pettis integrable.

Question: Is the result valid also in case of multifunctions? We will see that the answer is positive for multifunctions with compact convex values being subsets of an arbitrary Banach space.
Fremlin (1994) proved that a Banach space valued function is McShane integrable if and only if it is Henstock and Pettis integrable.

Question:
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Relations among the integrals

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We will see that answer is positive for multifunctions with compact convex values being subsets of an arbitrary Banach space.
Relations among the integrals

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1. the existence of Henstock integrable selections;
2. a Decomposition Theorem;
3. a technical (but useful) Lemma.
A Decomposition Theorem

Theorem

Let \( \Gamma : [0, 1] \to ck(X) \) be a scalarly Henstock–Kurzweil integrable multifunction. Then TFAE:

1. \( \Gamma \) is Henstock integrable;
2. \( S_H(\Gamma) \neq \emptyset \) and for every \( f \in S_H(\Gamma) \) the multifunction \( G : [0, 1] \to ck(X) \) defined by \( \Gamma(t) = G(t) + f(t) \) is McShane integrable.
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A Decomposition Theorem

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A Decomposition Theorem

**Theorem**

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$$\mathcal{Z}_\Gamma := \{s(x^*, \Gamma(\cdot)) = s(x^*, G(\cdot)) + x^*f(\cdot) : \|x^*\| \leq 1\}$$

Luisa Di Piazza
Non-absolute gage integrals for multifunctions with values in an arbitrary Banach space
Lemma

Let $\mathcal{A} = \{g_\alpha : [0, 1] \to [0, \infty) : \alpha \in S\}$ be a family of functions satisfying the following conditions:

1. $\mathcal{A}$ is Henstock equi-integrable;
2. $\mathcal{A}$ is totally bounded for the seminorm $\|||$;
3. $\mathcal{A}$ is pointwise bounded.

Then the family $\mathcal{A}$ is also McShane equi-integrable.
Lemma

Let $\mathcal{A} = \{g_\alpha : [0, 1] \to [0, \infty) : \alpha \in S\}$ be a family of functions satisfying the following conditions:

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Then the family $\mathcal{A}$ is also McShane equi-integrable.
Relations among the integrals

Theorem
Let $\Gamma : [0, 1] \to ck(X)$ be a multifunction. Then TFAE:

1. $\Gamma$ is McShane integrable;

Proof.
$(2) \Rightarrow (1)$ Since $\Gamma : [0, 1] \to ck(X)$ is Henstock and Pettis integrable in $ck(X)$, then $S_{MS}(\Gamma) \neq \emptyset$. Let $f$ be a McShane integrable selection $\Gamma$. It follows from the Decomposition Theorem that there exists a multifunction $G : [0, 1] \to ck(X)$ that is McShane integrable such that $\Gamma = G + f$. It follows that $\Gamma$ is also McShane integrable.

□

Luisa Di Piazza
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Let \( \Gamma : [0, 1] \rightarrow ck(X) \) be a multifunction. Then TFAE:

1. \( \Gamma \) is McShane integrable;

2. \( \Gamma \) is Henstock and Pettis integrable in \( ck(X) \).

Proof.

(2) \( \Rightarrow \) (1) Since \( \Gamma : [0, 1] \rightarrow ck(X) \) is Henstock and Pettis integrable in \( ck(X) \), then \( S_{MS}(\Gamma) \neq \emptyset \). Let \( f \) be a McShane integrable selection \( \Gamma \). It follows from the Decomposition Theorem that there exists a multifunction \( G : [0, 1] \rightarrow ck(X) \) that is McShane integrable such that \( \Gamma = G + f \). It follows that \( \Gamma \) is also McShane integrable. \( \square \)
Relations among the integrals

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Let $\Gamma : [0, 1] \to ck(X)$ be a multifunction. Then TFAE:

1. $\Gamma$ is McShane integrable;
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Proof.

$(2) \implies (1)$ Since $\Gamma : [0, 1] \to ck(X)$ is Henstock and Pettis integrable in $ck(X)$, then $S_{MS}(\Gamma) \neq \emptyset$. Let $f$ be a McShane integrable selection $\Gamma$. It follows from the Decomposition Theorem that there exists a multifunction $G : [0, 1] \to ck(X)$ that is McShane integrable such that $\Gamma = G + f$. It follows that $\Gamma$ is also McShane integrable. \[\square\]
Relations among the integrals

Theorem
Let $\Gamma : [0, 1] \to ck(X)$ be a multifunction. Then TFAE:
1. $\Gamma$ is McShane integrable;
2. $\Gamma$ is Henstock integrable and $\text{SH}(\Gamma) \subset \text{SM}(\Gamma)$;
3. $\Gamma$ is Henstock integrable and $\text{SH}(\Gamma) \subset \text{SP}(\Gamma)$;
4. $\Gamma$ is Henstock integrable and $\text{SP}(\Gamma) \neq \emptyset$. 

Luisa Di Piazza
Non-absolute gage integrals for multifunctions with values in an arbitrary Banach space
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Let \( \Gamma : [0, 1] \to ck(X) \) be a multifunction. Then TFAE:

1. \( \Gamma \) is McShane integrable;

2. \( \Gamma \) is Henstock integrable and \( S_H(\Gamma) \subset S_{MS}(\Gamma) \);

3. \( \Gamma \) is Henstock integrable and \( S_H(\Gamma) \subset S_P(\Gamma) \);

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Relations among the integrals

Theorem

Let $\Gamma : [0, 1] \rightarrow ck(X)$ be a multifunction. Then TFAE:

1. $\Gamma$ is McShane integrable;

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Relations among the integrals

Theorem

Let \( \Gamma : [0, 1] \to \text{ck}(X) \) be a multifunction. Then TFAE:

1. \( \Gamma \) is McShane integrable;
2. \( \Gamma \) is Henstock integrable and \( S_H(\Gamma) \subset S_{MS}(\Gamma) \);
3. \( \Gamma \) is Henstock integrable and \( S_H(\Gamma) \subset S_P(\Gamma) \);
4. \( \Gamma \) is Henstock integrable and \( S_P(\Gamma) \neq \emptyset \).
THANK YOU!


D. H. Fremlin, Pointwise compact sets of measurable functions, Manuscripta Math. 15 (1975), 219–242


