Buildings have finite asymptotic dimension

Jan Dymara [1] and Thomas Schick [2]

The goal of this note is to prove the following theorem, generalizing some results of Matsnev [Theorem 3.22, Mat].

Theorem 1.
The asymptotic dimension of any building is finite and equal to the asymptotic dimension of an apartment in that building.

Generally we use definitions and notation as in [D]. In particular, (W, S) is a finitely generated Coxeter system, C is a building with Weyl group W, $|C|$ is the Davis realization of C. We will, however, confuse the Coxeter group and its abstract Coxeter complex, denoting both by W; in particular, $|W|$ denotes the Davis complex. The W-valued distance in C will be denoted δ_C, while δ will be the gallery distance (i.e., $\delta = \ell \circ \delta_C$, where $\ell(w)$ is the shortest length of a word in generators S representing w). Basic properties of minimal galleries in buildings can be found in [R], [G]. We fix a chamber $B \in C$ and define the B-based *folding map* as $\pi: C \to W$, $\pi(c) = \delta_C(B, c)$. We also use π for the geometric realization $|C| \to |W|$ of this map. The word 'building' in the statement of Theorem 1 can be understood either as the discrete metric space (C, δ), or as the $CAT(0)$ metric space $(|C|, d)$ (these spaces are quasi–isometric). Neither thickness nor local finiteness of C is assumed.

Recall that a metric space X has asymptotic dimension $\leq n$ if for any $d > 0$ there exist $n + 1$ families $\mathcal{U}^0, \ldots, \mathcal{U}^n$ of subsets of X such that: (1) $\bigcup_i \mathcal{U}^i$ is a uniformly bounded cover of X, and (2) for every i any two sets $U, U' \in \mathcal{U}^i$ are *d-disjoint*: $d(U, U') := \inf\{d(x, y) \mid x \in U, y \in U'\} \geq d$. Basic properties of asymptotic dimension can be found in [BD]. We only need the definition and the fact that Coxeter groups have finite asymptotic dimension (cf. [Theorem B, DJ]).

Now we start the proof of Theorem 1. Let n be the asymptotic dimension of the apartment W of C. Because W embeds isometrically into C, the asymptotic dimension of C is at least n. We have to prove that its asymptotic dimension is $\leq n$.

Fix therefore $d > 0$. Let $\mathcal{U}^0, \ldots, \mathcal{U}^n$ be uniformly bounded families of $2d$-disjoint sets in $|W|$ such that $\bigcup_i \mathcal{U}^i$ is a cover of $|W|$. For $U \in \mathcal{U}^i$ let $N_d(U) = \{x \in |W| \mid d(x, U) < d\}$, and let $\operatorname{conv} N_d(U)$ be the convex hull of this set in the $CAT(0)$ space $|W|$. Note that $\operatorname{diam}(\operatorname{conv} N_d(U)) \leq \operatorname{diam}(U) + 2d$. Let $\mathcal{C}_U$ be the set of path–connected components of $\pi^{-1}(\operatorname{conv} N_d(U))$. Put $\mathcal{V}_U = \{\pi^{-1}(U) \cap A \mid A \in \mathcal{C}_U\}$ and $\mathcal{V}^i = \bigcup_{U \in \mathcal{U}^i} \mathcal{V}_U$. It is obvious that $\bigcup_i \mathcal{V}^i$ is a cover of $|C|$.

Claim: $\mathcal{V}^i$ are uniformly bounded families of d-disjoint sets.

Evidently, the claim implies Theorem 1. To prove the claim, we first establish d-disjointness. Let $x \in V \in \mathcal{V}_U$, $x' \in V' \in \mathcal{V}_{U'}$ with $V \neq V'$ and $U, U' \in \mathcal{U}^i$. There are two cases: $U \neq U'$ and $U = U'$. In the first case $d(x, x') \geq d(\pi(x), \pi(x'))$. But $\pi(x) \in U$, $\pi(x') \in U'$ and $d(U, U') \geq 2d$. Therefore $d(V, V') \geq 2d > d$. For the second case, assume that $U = U'$. Suppose also that $V = \pi^{-1}(U) \cap A$ and $V' = \pi^{-1}(U) \cap A'$, for some $A, A' \in \mathcal{C}_U$. Then the geodesic segment $[x, x']$ is not entirely contained in $\pi^{-1}(\operatorname{conv} N_d(U))$ (otherwise A and A', hence V and V', coincide). Let $p \in [x, x'] \setminus \pi^{-1}(\operatorname{conv} N_d(U))$; we have $d(x, x') = d(x, p) + d(p, x') \geq d(\pi(x), \pi(p)) + d(\pi(p), \pi(x')) \geq d + d \geq d$.

It remains to check uniform boundedness. Let $V \in \mathcal{V}_U$; then $V \subseteq A$ for some path–connected component A of $\pi^{-1}(\operatorname{conv} N_d(U))$. It is enough to find a uniform bound on the diameter of A. We would like to make A gallery–connected. Since path–connected sets are usually not gallery–connected, we perform an auxiliary thickening construction. Let X be a subset of $|C|$ or of $|W|$. We put $T(X) = \bigcup\{Res(p) \mid p \in X\}$, where $Res(p) = \bigcup\{|c| \mid p \in |c|\}$. Observe that $Res(p)$ is a geometric realization of a spherical building, and

[1] Instytut Matematyczny, Uniwersytet Wrocławski, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland, dymara@math.uni.wroc.pl. Partially supported by MNiSW grant N201 012 32/0718 and by DFG grant 436 POL 113/110/0-1.

[2] Mathem. Institut., Georg-August-Universität Göttingen, Bunsenstrasse 3, 37073 Göttingen, Germany, schick@uni-math.gwdg.de. Partially supported by DFG grant 436 POL 113/110/0-1.

spherical buildings are gallery–connected; therefore, any two points in A belong to chambers which can be connected by a gallery in $T(A)$. Since $T(A) \subseteq \pi^{-1}\big(T(\operatorname{conv} N_d(U))\big)$, the set A is contained in a gallery–connected component of $\pi^{-1}\big(T(\operatorname{conv} N_d(U))\big)$. Observe that, uniformly in U, the diameter of $T(\operatorname{conv} N_d(U))$ is bounded by $R + 2d + 2\kappa$, where R is the uniform bound on diameters of elements of the families $\mathcal{U}^i$, and κ is the diameter of the realization of a chamber. Since the distances d and δ are quasi–isometric, it remains to prove the following lemma.

Lemma 1.

For any $N > 0$ there exists $M > 0$ such that if U is a subset of W of δ-diameter $\leq N$, then any gallery–connected component V of $\pi^{-1}(U)$ has δ-diameter $\leq M$.

Lemma 1 follows form the next two lemmas.

Lemma 2.

Let W be a Coxeter group. For any $R > 0$ there exists $D = D(R)$ such that for any subset $U \subset |W|$ of diameter R satisfying $d(|1|, U) > D$ there exists a codimension-one face of $|1|$ such that the wall containing that face separates $|1|$ from U.

Proof. For $s \in S$, let M_s be the wall containing the s-face of $|1|$, and let M_s^- be the open half-space with boundary M_s that does not intersect $|1|$. Let r be greater than twice the diameter of a chamber, and let b be the barycentre of $|1|$.

We denote with $B_r(b)$ the open distance ball around b. We claim that $\{M_s^- \setminus B_r(b) \mid s \in S\}$ is an open cover of $|W| \setminus B_r(b)$. If a point $q \in |W| \setminus B_r(b)$ is an interior point of a chamber, then we can consider a minimal gallery from 1 to that chamber; this gallery starts by crossing some wall M_s, and then $q \in M_s^-$. In general, we apply the above argument to the chamber c in $Res(q)$ that is δ-closest to $|1|$. Any other chamber c' in $Res(q)$ can be connected to 1 by a minimal gallery passing through c, so that if a wall M_s separates 1 from c then it also separates 1 from c'. Therefore M_s separates b from all points in the interior of $Res(q)$, in particular from q.

Intersecting open sets from the family $\{M_s^- \setminus B_r(b) \mid s \in S\}$ with the distance sphere $S_r(b)$, we obtain an open cover of this compact metric space. Let ϵ be the Lebesgue number of this cover. Put $D = Rr/\epsilon$. We claim that the Lebesgue number of the cover $\{M_s^- \setminus B_D(b) \mid s \in S\}$ of $|W| \setminus B_D(b)$ is at least R. Indeed, let U' be a subset of $|W| \setminus B_D(b)$ of diameter $\leq R$. Because W is a $CAT(0)$-space, the set U of intersection points of $S_r(b)$ and geodesic intervals from b to points in U' has diameter $\leq \epsilon$. Thus $U \subset M_s^-$ for some $s \in S$. Now if a point $q \in U'$ did not belong to M_s^-, then $S_r(b) \cap [b, q]$ would not be in M_s^- either (because b, q belong to the convex set $|W| \setminus M_s^-$), contradicting $U \subset M_s^-$. QED(Lemma 2)

For $X \subseteq C$ or $X \subseteq W$ we denote by $T(X)$ the union of all spherical residues that intersect X. Let U be a subset of W of δ-diameter $\leq N$. There exists $R > 0$ depending only on N (and W) such that the d-diameter of $|T(U)|$ is $\leq R$. Iterated application of Lemma 2 provides a minimal gallery $\gamma = (1, w_1, \ldots, w_k)$ such that the wall between w_i and w_{i+1} separates w_i from $T(U)$ and $d(|w_k|, |T(U)|) \leq D$. Note that this separation property implies that every chamber which meets U can be joint to $|1|$ by a minimal gallery which is a concatenation $\gamma\delta$, i.e. extending γ.

Lemma 3.

Let U and γ be given as just described. Recall that B is the "base" chamber in C, with $\pi(B) = |1|$. For any chamber c meeting $\pi^{-1}(U)$ there is a minimal gallery from B to c whose π-projection extends γ.

For any gallery–connected component $V \subset C$ of $\pi^{-1}(U)$ there exists a chamber $e \in \pi^{-1}(w_k)$ such that any minimal gallery from B to a chamber in V whose π-projection prolongs γ passes through e.

Proof. For each chamber $c \in T(V)$ let $P(c)$ be the set of all minimal galleries from b to c that are of the form $\Gamma\Delta$, where $\pi(\Gamma) = \gamma$. We first show that this set is not empty. Choose an arbitrary minimal gallery from B to c with π-projection α, and a minimal gallery of the form $\gamma\delta$ from $|1|$ to $\pi(c)$. α and $\gamma\delta$ are minimal galleries with the same extremities, therefore are related by a sequence of Tits moves (cf. [Chap 4, Proposition 5, Bourb], or [Theorem 2.11, Ron]). This sequence lifts to a sequence of moves relating the original minimal gallery to a minimal $\Gamma\Delta$ with π-projection $\gamma\delta$, as required.

Next we claim that if $\Gamma_1\Delta_1, \Gamma_2\Delta_2 \in P(c)$, then $\Gamma_1 = \Gamma_2$. In fact, $\pi(\Delta_1)$ and $\pi(\Delta_2)$ are minimal galleries in W with the same extremities, therefore again are related by a sequence of Tits moves. As above, this sequence lifts to a sequence of moves relating Δ_1 and Δ_2' (and keeping extremities fixed), where Δ_2' has the same π-projection as Δ_2. Now $\Gamma_1\Delta_2'$ and $\Gamma_2\Delta_2$ have the same extremities and the same π-projections, so that they coincide.

Therefore we can define $e(c)$ as $\Gamma \cap \pi^{-1}(w_k)$ for some $\Gamma\Delta \in P(c)$, and then $e(c)$ does not depend on the choice of $\Gamma\Delta$. We will now show that, for $c \in V$, $e(c)$ does not depend on c. Since V is gallery connected, it is enough to check this independence for adjacent $c, c' \in V$. There are three cases:

1) $\delta(w_k, \pi(c)) = \delta(w_k, \pi(c'))$. Then there is a chamber $c'' \in T(V)$ adjacent to both c and c' whose π-projection is closer to w_k than that of c. And for any $\Gamma\Delta \in P(c'')$ we have $\Gamma\Delta c \in P(c)$, $\Gamma\Delta c' \in P(c')$.
2) $\delta(w_k, \pi(c)) = \delta(w_k, \pi(c')) + 1$. Then for any $\Gamma\Delta \in P(c)$ we have $\Gamma\Delta c' \in P(c')$.
3) $\delta(w_k, \pi(c)) = \delta(w_k, \pi(c')) - 1$. This is symmetric to 2), we just switch c and c'.

Finally, $e = e(c)$, where $c \in V$, does not depend on c and is as claimed. QED(Lemma 3)

Let L is the maximal gallery distance between chambers of d-distance $\leq D + \kappa$, where κ is the diameter of a realization of a chamber. As a result of Lemma 3, every chamber in V is at δ-distance $< L + N$ from e, hence V has δ-diameter $< 2L + 2N$. This proves Lemma 1 and therefore Theorem 1.

The result leaves as an open task to determine the precise asymptotic dimension of a Coxeter complex W as above. These are particularly nice $CAT(0)$-complexes. In this context, it is plausible to expect that the asymptotic dimension coincides with the classical (microscopic) dimension for a suitable choice or modification of the complex W.

This coincidence has been established in model examples (coming from slightly different contexts), in particular for many homogeneous manifolds in [Corollary 3.6, CG] and in [Theorem 12, BD2]. For simply connected Riemannian manifolds with curvature bounded by $c < 0$, the result has been proved by Grave [Theorem 6.20, Gra]. For spaces with cocompact action of the isometry group which are Gromov hyperbolic, a related result has been obtained by Buyalo and Lebedeva. It is not known to us whether these results extend to non-positive curvature, or whether one can find counterexamples in this wider class. However, observe that it is not even known that the asymptotic dimension of every finite dimensional $CAT(0)$-space is finite.

Nonetheless, we conjecture that the answer to the following question is yes, at least in many good cases. Given this belief, it would be even more interesting to find counterexamples.

Question: Is it true that for the Coxeter complex W as considered in this note, the asymptotic dimension and the virtual cohomological dimension, i.e. the dimension of its Bestvina complex [Best] coincide? Note that the latter is also given by the dimension of the $CAT(0)$-boundary (compare [BM]).

It follows from [Corollary 4.11, Dr] that the asymptotic dimension of W is not smaller than its virtual cohomological dimension.

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