Thin Buildings

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0. Integer thickness

Let (W, S) be a Coxeter system. Let Δ be a simplex with codimension 1 faces labelled by elements of S, and let Δ' be its first barycentric subdivision. Each $T \subseteq S$ generates a subgroup W_T of W called a special subgroup; also, T corresponds to a face Δ_T of Δ (the intersection of codimension 1 faces labelled by elements of T). The Davis chamber D is the subcomplex of Δ' spanned by barycentres of faces Δ_T for which W_T is finite ($\mathcal{F}$ will denote the set of subsets $T \subseteq S$ such that W_T is finite). To every $T \subseteq S$ we assign a face of the Davis chamber: $D_T = D \cap \Delta_T$. The Davis realization Σ of the Coxeter complex is $W \times D/\sim$, where $(w, p) \sim (u, q)$ if and only if for some T we have $p = q \in D_T$ and $w^{-1}u \in W_T$. The action of W on the first factor descends to an action on Σ; we will use $w\sigma$ to denote the image of $w \times \sigma$ in Σ. The images of $w \times D$ in Σ are called chambers. The action of W on Σ is simply transitive on the set of chambers.

A Tits building X_{Tits} with Weyl group W is a set with a W-valued distance function d, satisfying certain conditions (cf [Ron]). Its Davis' incarnation is $X = X_{Tits} \times D/\sim$, where $(x, p) \sim (y, q)$ if and only if for some T we have $p = q \in D_T$ and $d(x, y) \in W_T$. The images of $x \times D$ in X are called chambers.

We will consider only buildings of uniformly bounded thickness, ie such that for some constant $N > 0$, any $s \in S$ and any $x \in X_{Tits}$ there are no more than N elements $y \in X_{Tits}$ satisfying $d(x, y) = s$. If this number of s-neighbours of x is equal to q for all pairs (x, s), then we say that the building has uniform thickness $q + 1$. We denote such building $X(q)$ (for a right-angled Coxeter group it is unique).

Uniformly bounded thickness is equivalent to X being uniformly locally finite. Thus we can consider (reduced) L^2-(co)homology of X. This is obtained from the complex of L^2 (co)chains on X with the usual (co)boundary operators ∂, δ. These operators are in fact adjoint to each other, so that the (co)homology can be identified with $L^2\mathcal{H}^*(X)$, the space of harmonic (co)chains ("reduced" means that we divide the kernel by the closure of the image).

Assume now that X_{Tits} comes from a BN-pair in a group G. Then G acts by simplicial automorphisms on X. We can assume that G acts faithfully and is locally compact (possibly taking the closure of its image in $Aut(X)$ in the compact-open topology). We use G to measure the size of $L^2\mathcal{H}^i(X)$ via the von Neumann dimension. To do this, we first express $L^2C^i(X)$ as $\oplus_{\sigma^i \subset D} L^2(G\sigma^i)$. Then we notice that $L^2(G\sigma^i)$ is naturally isomorphic to $L^2(G)^{G_{\sigma^i}}$. We find it convenient to multiply this isomorphism by a suitable scalar factor in order to make it isometric. Then the space $L^2(G)^{G_{\sigma^i}}$ is embedded into $L^2(G)$, giving us finally an embedding of left G-modules $L^2C^i(X) \hookrightarrow \oplus_{\sigma^i \subset D} L^2(G)$. In particular, $L^2\mathcal{H}^i(X)$ is now embedded as a left G-module in $\oplus_{\sigma^i \subset D} L^2(G)$; we can consider the orthogonal projection onto this subspace, and define $L^2b^i(X)$ to be the von Neumann trace of that projection. Let B be the stabilizer of D in G. For each $\sigma^i \subset D$ we have

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a vector $\mathbf{1}_\sigma$ in $\oplus_{\sigma^i\subset D}L^2(G)$, having σ'th component $\mathbf{1}_B$ and other components 0. The projection onto $L^2\mathcal{H}^i(X)$ is given by a matrix whose σ'th row gives the projection of $\mathbf{1}_\sigma$ on $L^2\mathcal{H}^i(X)$, expressed as an element of $\oplus_{\sigma^i\subset D}L^2(G)$ (while applying this matrix we understand multiplication as convolution). Notice that both $\mathbf{1}_\sigma$ and the space $L^2\mathcal{H}^i(X)$ are B-invariant; so therefore will be the projection of $\mathbf{1}_\sigma$ on $L^2\mathcal{H}^i(X)$.

1. Real thickness

For a $w\in W$ we denote by $d(w)$ the length of the shortest word in the generators S representing w. For a chamber $c=w\times D$ of Σ we put $d(c)=d(w)$. For every simplex $\sigma\subset\Sigma$ there is a unique chamber $c\supseteq\sigma$ with smallest d; we put $d(\sigma)=d(c)$.

For a real number $t>0$ we equip the set $\Sigma^{(i)}$ of i-simplices in Σ with the measure $\mu_t(\sigma)=t^{d(\sigma)}$. We also pick (arbitrarily) orientations of simplices in D, and extend them W-equivariantly to orientations of all simplices in Σ. This allows us to identify chains and cochains with functions. We put

$$L^2_tC^i(\Sigma)=L^2_tC_i(\Sigma)=L^2(\Sigma^{(i)},\mu_t).$$

We now define $\delta^i:L^2_tC^i(\Sigma)\to L^2_tC^{i+1}(\Sigma)$ by

$$\delta^i(f)(\tau^{i+1})=\sum_{\sigma^i\subset\tau^{i+1}}[\tau:\sigma]f(\sigma)$$

and $\partial^t_i:L^2_tC_i(\Sigma)\to L^2_tC_{i-1}(\Sigma)$ by

$$\partial^t_i(f)(\eta^{i-1})=\sum_{\sigma^i\supset\eta^{i-1}}[\eta:\sigma]t^{d(\sigma)-d(\eta)}f(\sigma)$$

(here $[\alpha:\beta]=\pm1$ tells us whether orientations of α and β agree or not). We have

$$\begin{aligned}\langle\delta^i(f),g\rangle_t&=\sum_{\tau^{i+1}}\Big(\sum_{\sigma^i\subset\tau^{i+1}}[\tau:\sigma]f(\sigma)\overline{g(\tau)}t^{d(\tau)}\Big)\\&=\sum_{\sigma^i}f(\sigma)\overline{\Big(\sum_{\tau^{i+1}\supset\sigma^i}[\tau:\sigma]t^{d(\tau)-d(\sigma)}g(\tau)\Big)}t^{d(\sigma)}\\&=\langle f,\partial^t_i(g)\rangle_t.\end{aligned}$$

That is, $\delta^*=\partial^t$ as operators on $L^2_tC^*(\Sigma)$. It follows that $(\partial^t)^2=0$ (since $\delta^2=0$), and we can consider (reduced) L^2_t-(co)homology:

$$L^2_tH^i(\Sigma)=\ker\delta^i/\overline{\operatorname{im}\delta^{i-1}},\quad L^2_tH_i(\Sigma)=\ker\partial^t_i/\overline{\operatorname{im}\partial^t_{i+1}}$$

Since $\delta^*=\partial^t$, $(\partial^t)^*=\delta$ we have $L^2_tC^i(\Sigma)=\ker\partial^t_i\oplus\overline{\operatorname{im}\delta^{i-1}}=\ker\delta^i\oplus\overline{\operatorname{im}\partial^t_{i+1}}$ (orthogonal direct sums). It follows that

$$L^2_tH^i(\Sigma)\cong L^2_t\mathcal{H}^i(\Sigma)\cong L^2_tH_i(\Sigma),$$

where $L^2_t\mathcal{H}^i(\Sigma)$ is the space $\ker\delta^i \cap \ker\partial^t_i$ of harmonic i-cochains.

Remark

When $t = q$ is a positive integer the above (co)chain complexes coincide with B-invariant L^2 (co)chain complexes of a building $X(q)$ with Weyl group W—granted that such building exists.

2. Hecke algebra

We deform the usual scalar product on $\mathbf{C}[W]$ into $\langle\, ,\rangle_t$:

$$\langle \sum_{w\in W} a_w\delta_w, \sum_{w\in W} b_w\delta_w\rangle_t = \sum_{w\in W} a_w\overline{b_w}t^{d(w)}. \tag{2.1}$$

We also correspondingly deform the multiplication: for $w \in W$, $s \in S$ we put

$$\delta_w\delta_s = \begin{cases} \delta_{ws} & \text{if } d(ws) > d(w); \\ t\delta_{ws} + (t-1)\delta_w & \text{if } d(ws) < d(w). \end{cases} \tag{2.2}$$

This indeed extends to an associative multiplication on $\mathbf{C}[W]$, cf. [Bourb]. Using (2.2) and induction on $d(v)$ one easily shows

$$\delta_w\delta_v = \delta_{wv} \qquad \text{if } d(wv) = d(w) + d(v). \tag{2.3}$$

We keep the involution on $\mathbf{C}[W]$ independent of t:

$$\Big(\sum_{w\in W} a_w\delta_w\Big)^* = \sum_{w\in W} \overline{a_{w^{-1}}}\delta_w. \tag{2.4}$$

Proposition 2.1

The above scalar product, multiplication and involution define a Hilbert algebra structure on $\mathbf{C}[W]$ (in the sense of [Dix, A.54]); we use $\mathbf{C}_t[W]$ to indicate this structure.

Proof. We begin with involutivity: $(xy)^* = y^*x^*$. One checks it using (2.2) and (2.3) for $x = \delta_w$, $y = \delta_s$ considering two cases: $d(ws) < d(w)$, $d(ws) > d(w)$. Then one checks it for $x = \delta_w$, $y = \delta_u$ by induction on $d(u)$. Finally, linearity extends the result to general x, y. From involutivity and (2.2) we immediately get

$$\delta_s\delta_w = \begin{cases} \delta_{sw} & \text{if } d(sw) > d(w); \\ t\delta_{sw} + (t-1)\delta_w & \text{if } d(sw) < d(w). \end{cases} \tag{2.5}$$

We now recall and prove the conditions (i)–(iv) of [Dix] defining a Hilbert algebra.

(i) $\langle x, y\rangle_t = \langle y^*, x^*\rangle_t$.

This is a straightforward calculation (using $d(w) = d(w^{-1})$).

(ii) $\langle xy, z\rangle_t = \langle y, x^*z\rangle_t$.

Due to linearity we are reduced to the case $y = \delta_w$, $z = \delta_u$, $x = \delta_v$. First one treats the case $v = s \in S$, directly using (2.5); this requires four sub-cases, depending on comparison of $d(sw)$ with $d(w)$ and $d(su)$ with $d(u)$. Then one does an easy induction on $d(v)$.

(iii) For any $x \in \mathbf{C}_t[W]$ the map $\mathbf{C}_t[W] \ni y \mapsto xy \in \mathbf{C}_t[W]$ is continuous.

One checks first that $y \mapsto \delta_s y$ is continuous, directly using (2.5); then by compositions and linearity one gets continuity of $y \mapsto xy$.

(iv) The products xy, over all $x, y \in \mathbf{C}_t[W]$ are dense in $\mathbf{C}_t[W]$.

This is immediate, since we have a unit element δ_1 in $\mathbf{C}_t[W]$. $\diamond$

Corollary 2.2

The coefficient of δ_1 in ab is equal to $\langle a, b^* \rangle_t$.

Proof. That coefficient is equal to $\langle ab, \delta_1 \rangle_t$, which by (ii) and (i) is $\langle b, a^* \rangle_t = \langle a, b^* \rangle_t$.$\diamond$

As in [Dix, A.54] we get two von Neumann algebras U_t, V_t: they are weak closures of $\mathbf{C}_t[W]$ acting on its completion L^2_t by left (respectively right) multiplication.

As in [Dix, A.57] we put $\mathbf{C}_t[W]'$ to be the algebra of all bounded elements of L^2_t; bounded means that left (or, equivalently, right) multiplication by the element is bounded on $\mathbf{C}_t[W]$ (so, extends to a bounded operator on L^2_t and defines an element of U_t or V_t).

As in [Dix, A.60] we have natural traces tr on U_t, V_t: if $B \in U_t$ (or $B \in V_t$) is self-adjoint and positive, we ask whether $B^{\frac{1}{2}} = a\cdot$ (resp., $B^{\frac{1}{2}} = \cdot a$) for an $a \in \mathbf{C}_t[W]'$. If so, we put $\operatorname{tr} B = \|a\|^2_t$; otherwise $\operatorname{tr} B = +\infty$. Of course, the $a = \sum_{w \in W} a_w \delta_w$ we are asking for has to be self-adjoint: $a_w = \overline{a_{w^{-1}}}$, so that by Corollary 2.2 $\|a\|^2_t$ is equal to the coefficient of δ_1 in a^2. Thus B is the multiplication by the bounded self-adjoint element $b = a^2$, and its trace is given by the coefficient of δ_1 in b.

Suppose now that we are given a closed subspace Z of $\oplus_{i=1}^l L^2_t$, such that the orthogonal projection P_Z onto Z is an element of $M_{l\times l} \otimes V_t$. To calculate the trace of P_Z we first need to identify it as a matrix. So, we take the standard basis $\{e_i\}$ of $\oplus_{i=1}^l L^2_t$ (e_i has δ_1 as the i'th coordinate, and other coordinates 0), and apply P_Z to it. We expand the results in the basis $\{e_i\}$: let $a_i^j \in L^2_t$ be the j'th coordinate of $P_Z(e_i)$. Then we take the coefficient of δ_1 in a_i^i and sum over i. The number we get is the trace of P_Z.

3. L^2_t-Betti numbers

It will be convenient to realize L^2_t as $L^2(W, \nu_t)$, where $\nu_t(w) = t^{d(w)}$. For any Coxeter group Γ (we have W as well as its subgroups W_T in mind) we have the generating function $\Gamma(x) = \sum_{\gamma \in \Gamma} x^{d(\gamma)}$. For a finite Γ it is a polynomial, in general it is a rational function. We denote by ρ_Γ the radius of convergence of the series defining $\Gamma(x)$.

As in the case of buildings (cf. section 0), we have $L^2_t C^i(\Sigma) = \bigoplus_{\sigma^i \subset D} L^2(W\sigma^i, \mu_t)$. Now $L^2(W\sigma^i, \mu_t)$ can be identified with $L^2(W, \nu_t)^{W_{T(\sigma)}}$ ($T(\sigma)$ is the largest subset of S such that $\sigma \subseteq D_{T(\sigma)}$) via the map ϕ given by $\phi(f)(w) = \frac{1}{\sqrt{W_{T(\sigma)}(t)}} f(w\sigma)$ (we distorted the natural map by the factor $\frac{1}{\sqrt{W_{T(\sigma)}(t)}}$ in order to make it isometric). Finally, $L^2(W, \nu_t)^{W_{T(\sigma)}}$ is a subspace of $L^2(W, \nu_t) = L^2_t$, so that we get an isometric embedding

$$\Phi : L^2_t C^i(\Sigma) \hookrightarrow \bigoplus_{\sigma^i \subset D} L^2_t = C^i(D) \otimes L^2_t$$

Let $\mathcal{L}$ denote the algebra U_t acting diagonally on the left on $\oplus_{\sigma \subset D} L^2_t = C^*(D) \otimes L^2_t$; let $\mathcal{R}$ be $\operatorname{End}(C^*(D)) \otimes V_t$ acting on the same space on the right. The von Neumann algebras $\mathcal{L}$ and $\mathcal{R}$ are commutants of each other.

Lemma 3.1
The projection of L^2_t onto $L^2(W\sigma,\mu_t) = L^2(W,\nu_t)^{W_{T(\sigma)}}$ is given by the right Hecke t-multiplication by

$$p_{T(\sigma)} = \frac{1}{W_{T(\sigma)}(t)} \sum_{w\in W_{T(\sigma)}} \delta_w \tag{3.1}$$

Proof. Put $T = T(\sigma)$. The subspace onto which we project consists of those elements of L^2_t which are right W_T-invariant; this is equivalent to being invariant under right Hecke t-multiplication by $\frac{1}{1+t}(\delta_1+\delta_s)$ for all $s\in T$ (to check this one splits W into pairs $\{w, ws\}$, and calculates for each pair separately using (2.2)). As a result, this subspace is $\mathcal{L}$-invariant, so that the projection P_T onto it is an element of $\mathcal{R}$. It follows that P_T is given by right Hecke t-multiplication by $P_T(\delta_1)$. The latter is clearly of the form $C\sum_{w\in W_T}\delta_w$, where C is a constant such that

$$\langle \delta_1 - C\sum_{w\in W_T}\delta_w, C\sum_{w\in W_T}\delta_w\rangle_t = 0.$$

This gives $C = \|\sum_{w\in W_T}\delta_w\|_t^{-2} = \left(\sum_{w\in W_T} t^{d(w)}\right)^{-1} = \frac{1}{W_T(t)}$. $\diamond$

Lemma 3.2
$\mathcal{L}$ preserves the subspace $L^2_tC^i(\Sigma)\subset C^*(D)\otimes L^2_t$ and commutes with δ and ∂^t.

Proof. The first claim follows from Lemma 3.1 (and actually was a step in the proof of that lemma). To prove the second part notice that δ is an element of $\mathcal{R}$: the matrix with V_t-coefficients describing δ has non-zero $\sigma\tau$'th entry if and only if σ is a codimension 1 face of τ; the entry is then $\sqrt{\frac{W_{T(\sigma)}(t)}{W_{T(\tau)}(t)}}\delta_1$. It follows that δ commutes with $\mathcal{L}$. So therefore does its adjoint ∂^t. $\diamond$

Corollary 3.3
$L^2_tC^i(\Sigma)$, $L^2_t\mathcal{H}^i(\Sigma)$, $\ker\delta^i$, $\ker\partial^t_i$, $\overline{\operatorname{im}\delta^i}$, $\overline{\operatorname{im}\partial^i_t}$ *are $\mathcal{L}$-invariant; therefore, orthogonal projections onto these spaces are elements of $\mathcal{R}$.*

We use tr to denote the tensor product of the usual matrix trace on $\operatorname{End}(C^*(D))$ and the von Neumann trace on V_t as described in section 3. We put

$$b^i_t = L^2_tb^i(\Sigma) = \operatorname{tr}\left(\text{projection onto } L^2_t\mathcal{H}^i(\Sigma)\right) \tag{3.2}$$

$$c^i_t = L^2_tc^i(\Sigma) = \operatorname{tr}\left(\text{projection onto } L^2_tC^i(\Sigma)\right) \tag{3.3}$$

$$\chi_t = \sum_i(-1)^ib^i_t = \sum_i(-1)^ic^i_t \tag{3.4}$$

The sums in (3.4) give the same value by the standard algebraic topology argument. It follows from Lemma 3.1 that $c^i_t = \sum_{\sigma^i\subset D}\frac{1}{W_{T(\sigma)}(t)}$. Grouping together simplices σ with the same $T(\sigma)$ and using formula (5) from [ChD] we obtain the following result (cf. [Se]).

Corollary 3.4

$$\chi_t = \frac{1}{W(t)}$$

Fact 3.5

If $t = q$ is a positive integer, b_t^i coincides with the i'th L^2 Betti number of a BN-pair-supporting building $X(q)$ with Weyl group W (if such building exists).

Proof. For $t = q$, $L_q^2 C^i(\Sigma)$ coincides with B-invariant elements of $L^2 C^i(X(q))$. By the concluding remarks of section 0, the matrix of the projection onto $L^2\mathcal{H}^i(X(q))$ has B-invariant entries—so that it coincides with the one we use to define b_t^i. Hence the conclusion. $\diamond$

Suppose now that the pair $(D, \partial D = D \cap \partial\Delta)$ is a generalized homology n-disc (ie it is a homology manifold with boundary, with relative homology groups the same as those of an n-disc modulo its boundary). Then each $D_T = D \cap \Delta_T$ is also a homology $(n - |T|)$-disc (for $T \in \mathcal{F}$). We can now use wD_T, $w \in W$, $T \in \mathcal{F}$, as a homology cellular structure on Σ (denoted Σ_{ghd}). The cell D_T has the form of an o_T-centred cone; we put $d(wD_T) = d(wo_T)$, and define μ_t, (co)chain complexes, the embedding Φ, the U_t-module structure and the numbers $b_t^i(\Sigma_{ghd})$ in essentially the same way as for the original triangulation of Σ.

4. Dual cells

We used so far the triangulation of Σ which originated from the barycentric subdivision of a simplex. We will use notation Σ_{st} to remind that we have this standard triangulation in mind. In this section we will introduce other cell structures on Σ. This will make our discussion of Poincaré duality in the next section look pretty standard.

To each $T \in \mathcal{F}$ there corresponds a face Δ_T of Δ, whose barycentre o_T is a vertex of the Davis chamber D. We define $\langle T \rangle$ as the union of all simplices $\sigma \subset \Sigma$ such that $\sigma \cap D_T = o_T$ (recall that $D_T = D \cap \Delta_T$). As a simplicial complex, $\langle T \rangle$ is an o_T-centred cone over Σ_T; since W_T was taken to be finite, Σ_T is a sphere and $\langle T \rangle$ is a disc of dimension $|T|$. The boundary of $\langle T \rangle$ is cellulated by $w\langle U \rangle$, for all possible $T \subset U \subseteq S$, $w \in W_T$. The complex Σ expressed as the union of $w\langle T \rangle$, over all $w \in W$, $T \subseteq S$ with finite W_T, is a cellular complex that we denote Σ_d.

We now put $d(w\langle T \rangle) = d(wo_T)$, and define the measures μ_t on the set $\Sigma_d^{(i)}$ of i-dimensional cells of Σ_d by $\mu_t(\langle a \rangle) = t^{d(\langle a \rangle)}$. Then

$$L_t^2 C^i(\Sigma_d) = L_t^2 C_i(\Sigma_d) \simeq L^2(\Sigma_d^{(i)}, \mu_t),$$

We now define $\delta^i : L_t^2 C^i(\Sigma_d) \to L_t^2 C^{i+1}(\Sigma_d)$ by

$$\delta^i(f)(\langle \tau \rangle^{i+1}) = \sum_{\langle \sigma \rangle^i \subset \langle \tau \rangle^{i+1}} [\langle \tau \rangle : \langle \sigma \rangle] f(\langle \sigma \rangle)$$

and $\partial_i^t : L_t^2 C_i(\Sigma_d) \to L_t^2 C_{i-1}(\Sigma_d)$ by

$$\partial_i^t(f)(\langle \eta \rangle^{i-1}) = \sum_{\langle \sigma \rangle^i \supset \langle \eta \rangle^{i-1}} [\langle \eta \rangle : \langle \sigma \rangle] t^{d(\langle \sigma \rangle) - d(\langle \eta \rangle)} f(\langle \sigma \rangle).$$

The discussion from section 1 can be continued, and supplies us with $L^2_t\mathcal{H}^i(\Sigma_d)$. Now we wish to bring in the Hecke algebra. We pick (arbitrarily) orientations of the cells $\langle T\rangle$ ($T \in \mathcal{F}$), and extend these to orientations of all cells in Σ_d as follows: $w\langle T\rangle$ is the oriented cell which is the image of the oriented cell $\langle T\rangle$ by w, with orientation changed by a factor of $(-1)^{d(w)}$. Using these orientations, we identify $L^2_tC^*(\Sigma_d)$ with $\oplus_{T\in\mathcal{F}}L^2(W\langle T\rangle,\mu_t)$. For every $T \in \mathcal{F}$ we define a map $\psi_T : L^2(W\langle T\rangle,\mu_t) \to L^2_t$ by the formula

$$\psi_T(f) = \sum_{w\in W^T} f(w\langle T\rangle)(-1)^{d(w)}\sqrt{W_T(t^{-1})}\delta_w a_T, \tag{4.1}$$

where $W^T = \{w \in W \mid \forall u \in W_T, d(wu) \geq d(w)\}$ (the set of T-*reduced* elements), and

$$a_T = \frac{1}{W_T(t^{-1})}\sum_{u\in W_T}(-t)^{-d(u)}\delta_u. \tag{4.2}$$

Putting together these maps we get a map $\Psi : L^2_tC^*(\Sigma_d) \to \oplus_{T\in\mathcal{F}}L^2_t$.

Lemma 4.1
1) For all $s \in T$ we have $\delta_s a_T = -a_T$.
2) For all $u \in W_T$ we have $\delta_u a_T = (-1)^{d(u)}a_T$.
3) For all $U \subseteq T$ we have $a_U a_T = a_T$.

Proof. 1) Let $w \in W$ be such that $d(sw) > d(w)$. Then $\delta_s\delta_w = \delta_{sw}$ (by (2.3)). We then have

$$\begin{aligned}\delta_s(\delta_w - \frac{1}{t}\delta_{sw}) &= \delta_{sw} - \frac{1}{t}(\delta_s\delta_s)\delta_w = \delta_{sw} - \frac{1}{t}(t\delta_1 + (t-1)\delta_s)\delta_w \\ &= (1 - \frac{t-1}{t})\delta_{sw} - \delta_w = -(\delta_w - \frac{1}{t}\delta_{sw})\end{aligned}$$

Since a_T is a linear combination of expressions of the form $\delta_w - \frac{1}{t}\delta_{sw}$, 1) follows.

2) Follows from 1) by induction on $d(u)$.

3) $$\begin{aligned}a_U a_T &= \frac{1}{W_U(t^{-1})}\sum_{u\in W_U}(-t)^{-d(u)}\delta_u a_T = \frac{1}{W_U(t^{-1})}\sum_{u\in W_U}(-t)^{-d(u)}(-1)^{d(u)}a_T \\ &= \frac{1}{W_U(t^{-1})}\left(\sum_{u\in W_U}t^{-d(u)}\right)a_T = a_T\end{aligned}$$ $\diamond$

Lemma 4.2
1) For every $T \in \mathcal{F}$ the map ψ_T is an isometric embedding.
2) The projection of L^2_t onto the image of ψ_T is given by right Hecke t-multiplication by a_T.

Proof. 1) The squared norm of a summand from the right hand side of (4.1) is

$$\|f(w\langle T\rangle)(-1)^{d(w)}\sqrt{W_T(t^{-1})}\delta_w a_T\|^2_t = |f(w\langle T\rangle)|^2 W_T(t^{-1})\|\delta_w a_T\|^2_t.$$

Since w is T-reduced, we have $\delta_w \delta_u = \delta_{wu}$ for all $u \in W_T$. Therefore

$$\|\delta_w a_T\|_t^2 = \left\| \frac{1}{W_T(t^{-1})} \sum_{u \in W_T} (-t)^{-d(u)} \delta_{wu} \right\|_t^2 = \left| \frac{1}{W_T(t^{-1})} \right|^2 \sum_{u \in W_T} |-t|^{-2d(u)} t^{d(wu)}$$

$$= t^{d(w)} \frac{1}{W_T(t^{-1})^2} \sum_{u \in W_T} t^{-d(u)} = t^{d(w)} \frac{1}{W_T(t^{-1})}.$$

2) Due to $a_T a_T = a_T$, it is clear that right Hecke t-multiplication by a_T is a projection. Let $w \in W$; write $w = vu$ where $u \in W_T$ and v is T-reduced. Then $\delta_w a_T = \delta_v \delta_u a_T = (-1)^{d(u)} \delta_v a_T$. This shows that images of ψ_T and right Hecke t-multiplication by a_T on finitely supported functions coincide. Since ψ_T is isometric, the same is true for the L^2_t-completions. $\diamond$

Denote by $\mathcal{L}$ the algebra U_t acting diagonally on the left on $\oplus_{T \in \mathcal{F}} L^2_t$, and by $\mathcal{R}$ its commutant $M_{|\mathcal{F}|}(\mathbf{C}) \otimes V_t$ (acting on the right). It follows from Lemma 4.2 that the image of Ψ is $\mathcal{L}$-invariant. In other words, we have a U_t-module structure on $L^2_t C^*(\Sigma_d)$, defined by the condition that the isometric embedding $\Psi : L^2_t C^*(\Sigma_d) \to \oplus L^2_t$ is a morphism of U_t-modules. Thus, we can think of $L^2_t C^*(\Sigma_d)$ as if it was a submodule of $\oplus_{T \in \mathcal{F}} L^2_t$.

Lemma 4.3
$\delta : L^2_t C^*(\Sigma_d) \to L^2_t C^*(\Sigma_d)$ is (a restriction of) an element of $\mathcal{R}$. For $U \subset T \in \mathcal{F}$ satisfying $|T| = |U| + 1$, the UT-entry of this element is

$$[\langle T \rangle, \langle U \rangle] \sqrt{\frac{W_T(t^{-1})}{W_U(t^{-1})}}\, a_T$$

Proof. Consider the pair of cells $w\langle U \rangle$, $w\langle T \rangle$. We have $[w\langle T \rangle, w\langle U \rangle] = [\langle T \rangle, \langle U \rangle]$. We can assume that w is U-reduced, and write it as vu, where v is T-reduced and $u \in W_T$. Let $f \in L^2_t C^{\dim\langle U \rangle}(\Sigma_d)$. The summand in $\psi_U(f)$ corresponding to the cell $w\langle U \rangle$ is

$$f(w\langle U \rangle)(-1)^{d(w)} \sqrt{W_U(t^{-1})}\, \delta_w a_U.$$

The summand in $\psi_T(\delta f)$ corresponding to the contribution of $f(w\langle U \rangle)$ to $(\delta f)(w\langle T \rangle)$ is

$$[\langle T \rangle, \langle U \rangle] f(w\langle U \rangle)(-1)^{d(v)} \sqrt{W_T(t^{-1})}\, \delta_v a_T.$$

Now $\delta_w a_U a_T = \delta_w a_T = \delta_v \delta_u a_T = (-1)^{d(u)} \delta_v a_T$, and the lemma follows. $\diamond$

Corollary 4.4
The subspaces $L^2_t C^i(\Sigma_d)$, $L^2_t \mathcal{H}^i(\Sigma_d)$, $\ker \delta^i$, $\ker \partial^t_i$, $\overline{\operatorname{im} \delta^i}$, $\overline{\operatorname{im} \partial^i_t}$ of $\oplus_{T \in \mathcal{F}} L^2_t$ are $\mathcal{L}$-invariant; therefore, orthogonal projections onto these spaces are elements of $\mathcal{R}$.

5. Invariance

In this section we prove that $L^2_t H^*(\Sigma_d) \simeq L^2_t H^*(\Sigma_{st})$ ($\simeq L^2_t H^*(\Sigma_{ghd})$, if the latter exists) as U_t-modules. It will be convenient for us to work with homology rather than cohomology; since they are both isomorphic to the U_t-module of harmonic cochains, it makes no difference.

We start by fixing orientation conventions. Let us pick arbitrary orientations of the dual cells $\langle T\rangle$ for all $T \in \mathcal{F}$. We extend these orientations to all dual cells as in section 3 ($w\langle T\rangle$ is oriented by $(-1)^{d(w)}$ times the orientation of $\langle T\rangle$ pushed forward by w). For $T \in \mathcal{F}$ of cardinality k, let $\langle T\rangle \cap D^{(k)}$ be the set of all k-simplices of Σ_{st} contained in $\langle T\rangle \cap D$. We orient every element of $\langle T\rangle \cap D^{(k)}$ by the restriction of the chosen orientation of $\langle T\rangle$. We then extend these orientations W-equivariantly to a portion of Σ_{st} (and put arbitrary equivariant orientations on the rest of Σ_{st}). Notice that if a k-simplex σ is contained in $w\langle T\rangle$ (where T has cardinality k), then the orientation of σ agrees with $(-1)^{d(\sigma)}$ times that of $w\langle T\rangle$. Orientations being chosen, we treat (co)chains as functions on the set of cells/simplices.

Now we define a topological embedding of Hilbert spaces $\theta : L^2_t C^*(\Sigma_d) \to L^2_t C^*(\Sigma_{st})$.

Definition

Let $f \in L^2_t C^k(\Sigma_d)$, $\sigma \in \Sigma^{(k)}_{st}$.

1) If there exists $\langle\alpha\rangle \in \Sigma^{(k)}_d$ such that $\sigma \subseteq \langle\alpha\rangle$ (there is at most one such $\langle\alpha\rangle$), then

$$\theta f(\sigma) = (-1)^{d(\sigma)} t^{d(\langle\alpha\rangle)-d(\sigma)} f(\langle\alpha\rangle).$$

2) If there is no $\langle\alpha\rangle$ as in 1), we put $\theta f(\sigma) = 0$.

Lemma 5.1

$$\partial^t \theta = \theta \partial^t.$$

Proof. We will show that for all $f \in L^2_t C^k(\Sigma_d)$, $\sigma \in \Sigma^{(k)}_{st}$ we have $\partial^t \theta f(\sigma) = \theta \partial^t f(\sigma)$. There are two cases to consider.

1. Suppose that there exists $\langle\alpha\rangle \in \Sigma^{(k)}_d$ such that $\sigma \subseteq \langle\alpha\rangle$. Then

$$(5.1)\quad \begin{aligned} \theta\partial^t f(\sigma) &= (-1)^{d(\sigma)} t^{d(\langle\alpha\rangle)-d(\sigma)} \partial^t f(\langle\alpha\rangle) \\ &= (-1)^{d(\sigma)} t^{d(\langle\alpha\rangle)-d(\sigma)} \sum_{\langle\beta\rangle^{k+1}\supset\langle\alpha\rangle} [\langle\beta\rangle : \langle\alpha\rangle] t^{d(\langle\beta\rangle)-d(\langle\alpha\rangle)} f(\langle\beta\rangle) \\ &= (-1)^{d(\sigma)} \sum_{\langle\beta\rangle^{k+1}\supset\langle\alpha\rangle} [\langle\beta\rangle : \langle\alpha\rangle] t^{d(\langle\beta\rangle)-d(\sigma)} f(\langle\beta\rangle). \end{aligned}$$

On the other hand,

$$(5.2)\quad \partial^t \theta f(\sigma) = \sum_{\tau^{k+1}\supset\sigma} [\tau : \sigma] t^{d(\tau)-d(\sigma)} \theta f(\tau).$$

Notice that if $\theta f(\tau) \neq 0$ then there exists a (necessarily unique) dual cell $\langle\beta\rangle^{k+1} \supset \tau$, so that (5.2) equals

$$(5.3)\quad \begin{aligned} &\sum_{\langle\beta\rangle^{k+1}\supset\langle\alpha\rangle} [\tau : \sigma] t^{d(\tau)-d(\sigma)} (-1)^{d(\tau)} t^{d(\langle\beta\rangle)-d(\tau)} f(\langle\beta\rangle) \\ &= \sum_{\langle\beta\rangle^{k+1}\supset\langle\alpha\rangle} [\tau : \sigma] (-1)^{d(\tau)} t^{d(\langle\beta\rangle)-d(\sigma)} f(\langle\beta\rangle). \end{aligned}$$

Now (5.3) and (5.1) are equal because $[\tau:\sigma]=(-1)^{d(\tau)}(-1)^{d(\sigma)}[\langle\beta\rangle:\langle\alpha\rangle]$.

2. The smallest dual cell $\langle\alpha\rangle$ containing σ is of dimension $m>k$. Then $\theta\partial^t f(\sigma)=0$. On the other hand,

$$\partial^t\theta f(\sigma)=\sum_{\tau^{k+1}\supset\sigma}[\tau:\sigma]t^{d(\tau)-d(\sigma)}\theta f(\tau).$$

Let $\tau^{k+1}\supset\sigma$, and let $\langle\beta\rangle\supset\langle\alpha\rangle$ be the smallest dual cell containing τ. If $\theta f(\tau)\neq 0$, then $\dim\langle\beta\rangle=k+1$, which forces $\langle\beta\rangle=\langle\alpha\rangle$ and $m=\dim\langle\alpha\rangle=k+1$. Thus, we are reduced to the case $m=k+1$. In this case, there are exactly two simplices $\sigma_\pm\in\Sigma_{st}^{(k+1)}$, $\sigma_\pm\subset\langle\alpha\rangle$, $\sigma_\pm\supset\sigma$. Since $\sigma_\pm$ is oriented by $(-1)^{d(\sigma_\pm)}$ times the orientation of $\langle\alpha\rangle$, we have

$$(-1)^{d(\sigma_+)}[\sigma_+:\sigma]=-(-1)^{d(\sigma_-)}[\sigma_-:\sigma]. \tag{5.4}$$

Therefore

$$\begin{aligned}\partial^t\theta f(\sigma)&=[\sigma_+:\sigma]t^{d(\sigma_+)-d(\sigma)}\theta f(\sigma_+)+[\sigma_-:\sigma]t^{d(\sigma_-)-d(\sigma)}\theta f(\sigma_-)\\&=[\sigma_+:\sigma]t^{d(\sigma_+)-d(\sigma)}(-1)^{d(\sigma_+)}t^{d(\langle\alpha\rangle)-d(\sigma_+)}f(\langle\alpha\rangle)\\&\quad+[\sigma_-:\sigma]t^{d(\sigma_-)-d(\sigma)}(-1)^{d(\sigma_-)}t^{d(\langle\alpha\rangle)-d(\sigma_-)}f(\langle\alpha\rangle)\\&=((-1)^{d(\sigma_+)}[\sigma_+:\sigma]+(-1)^{d(\sigma_-)}[\sigma_-:\sigma])t^{d(\langle\alpha\rangle)}f(\langle\alpha\rangle)\\&=0.\end{aligned}$$

◇

Lemma 5.2
θ is a morphism of U_t-modules.

Proof. The U_t-module structures on $L_t^2C^k(\Sigma_d)$ and on $L_t^2C^k(\Sigma_{st})$ are defined via embeddings Ψ and Φ. We will compare Ψ and $\Phi\circ\theta$. Let $f\in L_t^2C^k(\Sigma_d)$; $\Psi(f)$ is a collection of $\psi_T(f)$, where

$$\psi_T(f)=\sum_{w\in W^T}f(w\langle T\rangle)(-1)^{d(w)}\sqrt{W_T(t^{-1})}\delta_w a_T. \tag{5.5}$$

The part of θf corresponding to $\psi_T(f)$ is supported by the set of W-translates of simplices $\sigma\in\langle T\rangle\cap D^{(k)}$, and is mapped by Φ into $\oplus_{\sigma\in\langle T\rangle\cap D^{(k)}}L_t^2$. The component indexed by σ is $\sum_{w\in W}\theta f(w\sigma)\delta_w$ (notice that the stabiliser of σ is trivial), ie

$$\sum_{w\in W}(-1)^{d(w\langle T\rangle)}t^{d(w\langle T\rangle)-d(w\sigma)}f(w\langle T\rangle)\delta_w. \tag{5.6}$$

Comparing (5.5) and (5.6) with the help of (4.2), we get that $\psi_T(f)$ agrees with (every component of) the corresponding part of $\Phi(\theta f)$, up to a multiplicative factor of $\sqrt{W_T(t^{-1})}$. This implies the lemma. ◇

Theorem 5.3
The map θ induces an isomorphism of U_t-modules $L_t^2H_*(\Sigma_d)\simeq L_t^2H_*(\Sigma_{st})$.

Proof. Lemmas 5.1 and 5.2 imply that θ induces a morphism of U_t-modules on homology. We have to check that it is an isomorphism of vector spaces.

Let K_* be the image of θ. It is a subcomplex of $(L_t^2 C_*(\Sigma_{st}), \partial^t)$. A k-chain $c \in L_t^2 C_*(\Sigma_{st})$ is in K_* if and only if the following two conditions hold:

1) c is supported by the union of k-dimensional dual cells $\bigcup \Sigma_{st}^{(k)}$;

2) if $\sigma^k, \tau^k \subseteq \langle\alpha\rangle^k$, then $c(\sigma) = (-t)^{d(\tau)-d(\sigma)} c(\tau)$. We need to show that the inclusion $K_* \hookrightarrow L_t^2 C_*(\Sigma_{st})$ induces an isomorphism on (reduced) homology.

Let $m_t : L_t^2 C_*(\Sigma_{st}) \to L_{t^{-1}}^2 C_*(\Sigma_{st})$ be the isomorphism (of Hilbert spaces) $m_t f(\sigma) = t^{d(\sigma)} f(\sigma)$. Instead of working directly with K_*, $L_t^2 C_*(\Sigma_{st})$ and ∂^t, we will work with $L_* = m_t(K_*)$, $E_* = L_{t^{-1}}^2 C_*(\Sigma_{st}) = m_t(L_t^2 C_*(\Sigma_{st}))$ and $\partial = m_t \partial^t m_t^{-1}$. The advantage is that

$$\begin{aligned}
\partial g(\sigma) &= m_t \partial^t m_t^{-1} g(\sigma) = t^{d(\sigma)} \partial^t m_t^{-1} g(\sigma) = t^{d(\sigma)} \sum_{\tau^{k+1} \supset \sigma} [\tau:\sigma] t^{d(\tau)-d(\sigma)} m_t^{-1} g(\tau) \\
&= \sum_{\tau^{k+1} \supset \sigma} [\tau:\sigma] t^{d(\tau)} t^{-d(\tau)} g(\tau) = \sum_{\tau^{k+1} \supset \sigma} [\tau:\sigma] g(\tau).
\end{aligned} \tag{5.7}$$

To check whether $c \in E_*$ is in L_* we use 1) and the following version of 2):

2′) if $\sigma^k, \tau^k \subseteq \langle\alpha\rangle^k$, then $c(\sigma) = (-1)^{d(\tau)-d(\sigma)} c(\tau)$.

Lemma 5.4

Let $c \in E_k$. If $\partial c \in L_*$, then there exists a $d \in E_{k+1}$ such that $c - \partial d \in L_*$. Moreover, there is a constant C depending only on W and t such that d can be chosen so that $\|d\| \le C\|c\|$.

Proof. We construct, by descending induction on m ($m \ge k$), cochains $d_m \in E_{k+1}$ such that $c - \partial d_m$ is supported on dual cells of dimensions at most m. For $m \ge \dim \Sigma$ we put $d_m = 0$. Suppose that d_m is already constructed, where $m > k$. For every dual m-cell $\langle\alpha\rangle$, let c_α be the restriction of $c - \partial d_m$ to $\langle\alpha\rangle$ (ie, if $c - \partial d_m = \sum a_\sigma \sigma$, then $c_\alpha = \sum_{\sigma \subseteq \langle\alpha\rangle} a_\sigma \sigma$). Let $\sigma^k \subset \mathrm{int}\langle\alpha\rangle$. Then σ appears in ∂c_α and in $\partial c = \partial(c - \partial d)$ with the same coefficient, due to the inductive assumption. But, since $\partial c \in L_*$, this coefficient is 0. As a result, $c_\alpha \in Z_k(\langle\alpha\rangle, \partial\langle\alpha\rangle)$ (where ∂ denotes topological boundary). Since $H_k(\langle\alpha\rangle, \partial\langle\alpha\rangle) = 0$ (recall that $m = \dim\langle\alpha\rangle > k$), we can find $d_\alpha \in C_{k+1}(\langle\alpha\rangle)$ such that $c_\alpha - \partial d_\alpha \in C_k(\partial\langle\alpha\rangle)$. Moreover, we can choose d_α so that $\|d_\alpha\| \le C_1 \|c_\alpha\|$, for some constant C_1 depending only on W and t. Due to uniform local finiteness of Σ, we deduce $\|\sum_{\langle\alpha\rangle} d_\alpha\| \le C_2\|c\|$ for some constant C_2. We put $d_{m-1} = d_m + \sum_{\langle\alpha\rangle \in \Sigma_d^m} d_\alpha$, and $d = d_k$.

The estimate $\|d\| \le C\|c\|$ clearly follows from the construction. The chain $c - \partial d = \sum b_\sigma \sigma$ is supported by the union of dual cells of dimensions at most k. Let us check that it satisfies the condition 2′). Suppose that $\sigma^{k-1} \subset \mathrm{int}\langle\alpha\rangle^k$. There are exactly two k-simplices $\sigma_\pm \subset \langle\alpha\rangle$ such that $\sigma \subset \sigma_\pm$. The coefficient of σ in $\partial(c - \partial d) = \partial c$ is 0 (because $\partial c \in L_*$), and, on the other hand, is equal to $[\sigma_+ : \sigma] b_{\sigma_+} + [\sigma_- : \sigma] b_{\sigma_-}$. Using (5.4) we get $b_{\sigma_+} = (-1)^{d(\sigma_+)-d(\sigma_-)} b_{\sigma_-}$. This holds for all $\sigma^{k-1} \subset \mathrm{int}\langle\alpha\rangle^k$, which implies that $c - \partial d$ satisfies 2′). Hence $c - \partial d \in L_*$. $\diamond$(lemma)

We are ready to check that the inclusion $\iota : L_* \hookrightarrow E_*$ induces an isomorphism ι_* on (reduced) homology. To show that ι_* is surjective, suppose that $c \in E_*$ is closed: $\partial c = 0$.

Then $\partial c \in L_*$, and, by Lemma 5.4, there exists $d \in E_*$ such that $c - \partial d \in L_*$. We get $[c] = \iota_*[c - \partial d]$.

To show that ι_* is 1-1, suppose that $l \in L_*$, $\partial l = 0$ and $\iota_*[l] = 0$, ie $l = \lim \partial e_n$ for some sequence of $e_n \in E_*$. Applying Lemma 5.4 to $c = l - \partial e_n$, we get that there exist $f_n \in E_*$, $f_n \to 0$ such that $l - \partial e_n - \partial f_n \in L_*$. But, since $l \in L_*$, we deduce that $\partial(e_n + f_n) \in L_*$. Now we apply Lemma 5.4 to $c = e_n + f_n$ to get $g_n \in E_*$ such that $h_n = e_n + f_n - \partial g_n \in L_*$. We have

$$\partial h_n = \partial e_n + \partial f_n - \partial\partial g_n.$$

The last term is 0, the middle term converges to 0 since ∂ is bounded and $f_n \to 0$, so that, finally,

$$\lim \partial h_n = \lim \partial e_n = l.$$

This means that $[l] = 0$ in $H_*(L_*)$.

We have shown that $(L_*, \partial) \hookrightarrow (E_*, \partial)$ induces an isomorphism on homology. Therefore so does the inclusion $(K_*, \partial^t) \hookrightarrow (L^2_t C_*(\Sigma_{st}), \partial^t)$. The theorem follows. $\diamond$

Let us now assume that D is a generalized homology disc. Then, along the same lines as above, one shows $L^2_t H^*(\Sigma_{st}) \simeq L^2_t H^*(\Sigma_{ghd})$ (as U_t-modules). More precisely, one defines $\theta : L^2_t H^*(\Sigma_{ghd}) \to L^2_t H^*(\Sigma_{st})$ by $\theta f(\sigma) = f(\alpha)$ if $\sigma^k \subseteq \alpha^k \in \Sigma^{(k)}_{ghd}$, and $\theta f(\sigma) = 0$ if no such α^k exists. The proof of $\partial^t \theta = \theta \partial^t$ is similar to that of Lemma 5.1, and it is clear that θ is a U_t-morphism. A chain $c \in L^2_t C_k(\Sigma_{st})$ is in the image K_* of θ if and only if

1) c is supported by $\bigcup \Sigma^{(k)}_{ghd}$;

2) if $\sigma^k, \tau^k \subseteq \alpha^k \in \Sigma^{(k)}_{ghd}$, then $c(\sigma) = c(\tau)$.

These conditions do not change under m_t, and the rest of the proof of Theorem 5.3 can be repeated with dual cells replaced by cells (the only other change will be $[\sigma_+ : \sigma] = -[\sigma_- : \sigma]$ instead of the more complicated (5.4)). We get

Theorem 5.5

Suppose that D is a generalized homology disc. Then $L^2_t H^*(\Sigma_{ghd}) \simeq L^2_t H^*(\Sigma_{st}) \simeq L^2_t H^*(\Sigma_d)$ as (graded) U_t-modules.

6. Poincaré Duality

Let us define a map $D : L^2_t \to L^2_{t^{-1}}$ by

(6.1) $$D\left(\sum a_w \delta_w\right) = \sum (-t)^{d(w)} a_w \delta_w.$$

Direct calculation shows that D is an isometric isomorphism of Hilbert spaces. Notice that D maps $\mathbf{C}_t[W]$ onto $\mathbf{C}_{t^{-1}}[W]$. It is easy to check that D preserves the relations defining Hecke multiplication: if $d(ws) > d(w)$, then

$$D(\delta_w \delta_s) = D(\delta_{ws}) = (-t)^{d(ws)} \delta_{ws} = (-t)^{d(w)} \delta_w (-t\delta_s) = D(\delta_w) D(\delta_s);$$

if $d(ws) < d(w)$, then

$$\begin{aligned} D(\delta_w \delta_s) &= D(t\delta_{ws} + (t-1)\delta_w) = t(-t)^{d(ws)} \delta_{ws} + (t-1)(-t)^{d(w)} \delta_w \\ &= (-t)^{d(w)+1} t^{-1} \delta_{ws} + (-t)^{d(w)+1}(t^{-1} - 1)\delta_w = (-t)^{d(w)} \delta_w (-t)\delta_s \\ &= D(\delta_w) D(\delta_s). \end{aligned}$$

Hence, D restricts to an isometric isomorphism of Hilbert algebras $\mathbf{C}_t[W]$ and $\mathbf{C}_{t^{-1}}[W]$. In particular, D preserves products: for all $x, y \in \mathbf{C}_t[W]$, we have $D(xy) = D(x)D(y)$. Passing to limits with y in the norm $\|\cdot\|_t$, we deduce that the map $D : L^2_t \to L^2_{t^{-1}}$ is a morphism of left modules over the algebra morphism $D : \mathbf{C}_t[W] \to \mathbf{C}_{t^{-1}}[W]$. Then passing to limits with x in the weak operator topology, we deduce that $D : L^2_t \to L^2_{t^{-1}}$ is a morphism of left modules over the von Neumann algebra isomorphism $D : U_t \to U_{t^{-1}}$. Analogous statements hold for the right module structures. Finally, since D preserves the coefficient in front of δ_1, it preserves dimensions of (left) submodules of L^2_t.

Theorem 6.1
Suppose that the pair $(D, \partial D)$ is a generalized homology n-disc. Then $b^i_t = b^{n-i}_{t^{-1}}$.

Proof. There is a bijection $D_T \leftrightarrow \langle T\rangle$, where $T \in \mathcal{F}$; it can be unambiguously extended to $wD_T \leftrightarrow w\langle T\rangle$, a natural bijection between i-cells of Σ_{ghd} and $(n-i)$-cells of Σ_d. When w and T are not specified we write simply $\sigma \leftrightarrow \langle\sigma\rangle$. A property of this bijection which is crucial for us is: the codimension 1 faces of $\langle\tau^{i-1}\rangle$ are $\langle\sigma^i\rangle$, for $\sigma \supseteq \tau$. Let us pick orientations of all faces D_T of D, and extend them equivariantly to orientations of all cells η in Σ_{ghd}. Then we orient each dual cell $\langle\eta\rangle$ dually to the chosen orientation of η (dually with respect to a chosen orientation of Σ). These orientations are of the kind considered in section 3. With these choices we have $[\langle\sigma\rangle : \langle\tau\rangle] = \pm[\sigma : \tau]$, with the sign depending only on the dimensions of σ, τ (and n, which is fixed in our discussion).

We define the duality map $\mathcal{D} : L^2_t C^*(\Sigma_{ghd}) \to L^2_{t^{-1}} C^{n-*}(\Sigma_d)$ by

$$\mathcal{D}f(\langle\sigma\rangle) = t^{d(\sigma)} f(\sigma). \tag{6.2}$$

$\mathcal{D}$ is clearly an isometry of Hilbert spaces. We will now check that $\delta^{n-i}\mathcal{D} = \pm\mathcal{D}\partial^t_i$ (the sign depending only on i, n):

$$\delta(\mathcal{D}f)(\langle\tau^{i-1}\rangle) = \sum_{\sigma^i \supset \tau^{i-1}} [\langle\sigma\rangle : \langle\tau\rangle](\mathcal{D}f)(\langle\sigma\rangle) = \pm \sum_{\sigma^i \supset \tau^{i-1}} [\sigma : \tau] t^{d(\sigma)} f(\sigma)$$

while

$$\mathcal{D}(\partial^t f)(\langle\tau^{i-1}\rangle) = t^{d(\tau)}(\partial^t f)(\tau^{i-1}) = t^{d(\tau)} \sum_{\sigma^i \supset \tau^{i-1}} [\sigma : \tau] t^{d(\sigma)-d(\tau)} f(\sigma)$$

which proves what we wanted. It follows that $\mathcal{D}$ intertwines also the adjoint operators; consequently, it restricts to an isomorphism $\mathcal{D} : L^2_t \mathcal{H}^*(\Sigma_{ghd}) \to L^2_{t^{-1}} \mathcal{H}^{n-*}(\Sigma_d)$.

We still have to check that the Hecke dimensions of these spaces are the same.

To this end, let us now consider $L^2_t C^*(\Sigma_{ghd})$ as a subspace of $\oplus_{T\in\mathcal{F}} L^2_t$ via the embedding Φ_t (cf section 3), and $L^2_{t^{-1}} C^{n-*}(\Sigma_d)$ as a subspace of $\oplus_{T\in\mathcal{F}} L^2_{t^{-1}}$ via the embedding $\Psi_{t^{-1}}$ (cf section 4). We will check that $\mathcal{D}$ can be regarded as the restriction of the map D (applied componentwise in $\oplus_{T\in\mathcal{F}} L^2_t$). Let $f \in L^2(WD_T, \mu_t)$ be a part of a cochain on Σ_{ghd}. Then

$$\phi_T(f) = \sqrt{W_T(t)} \sum_{w\in W^T} f(wD_T)\delta_w p_T(t),$$

where $p_T(t) = \frac{1}{W_T(t)} \sum_{u\in W_T} \delta_u$. Since

$$D(p_T(t)) = \frac{1}{W_T(t)} \sum_{u\in W_T} (-t)^{d(u)}\delta_u = \frac{1}{W_T((t^{-1})^{-1})} \sum_{u\in W_T} (-t^{-1})^{-d(u)}\delta_u = a_T(t^{-1}),$$

we have

$$D(\phi_T(f)) = \sum_{w\in W^T} f(wD_T)\sqrt{W_T(t)}(-t)^{d(w)}\delta_w a_T(t^{-1}). \tag{6.3}$$

On the other hand, $(\mathcal{D}f)(w\langle T\rangle) = t^{d(w\langle T\rangle)} f(wD_T)$, and

$$\psi_T(\mathcal{D}f) = \sum_{w\in W^T} t^{d(w\langle T\rangle)} f(wD_T)(-1)^{d(w)}\sqrt{W_T(t)}\delta_w a_T(t^{-1}). \tag{6.4}$$

Since for $w \in W^T$ we have $d(w\langle T\rangle) = d(w)$, (6.3) and (6.4) are equal. This implies that $\mathcal{D}$ is in fact an isomorphism of U_t-modules. Thus, $\mathcal{D}$ preserves dimensions. Hence $b^{n-i}_{t^{-1}} = b^i_t$.
$\diamond$

7. b^0_t

Theorem 7.1 *For $t < \rho_W$ we have $b^0_t = \frac{1}{W(t)}$; for $t \geq \rho_W$ we have $b^0_t = 0$.*

Proof. We will use the cell structure Σ_d. Vertices of Σ_d are located at centres of chambers wD, thus are in bijection with W. We embed $L^2_t C^0(\Sigma_d)$ into L^2_t by $(\Psi c)(w) = (-1)^{d(w)} c(w\langle\emptyset\rangle)$. Under this embedding, all harmonic 0-cochains are constant functions, multiples of $\mathbf{1}(w) = 1$. The square of the norm of $\mathbf{1}$ is $\sum_{w\in W} t^{d(w)}$. It is finite and equal to $W(t)$ for $t < \rho_W$, and infinite if $t \geq \rho_W$. The latter means that for $t \geq \rho_W$ we have $L^2_t\mathcal{H}^0(\Sigma_d) = 0$.

To find b^0_t for $t < \rho_W$ we need to identify the projection of δ_1 on $L^2_t\mathcal{H}^0(\Sigma_d)$; it is $C\mathbf{1}$, where

$$\langle \delta_1 - C\mathbf{1}, \mathbf{1}\rangle_t = 0.$$

This gives $C = \|\mathbf{1}\|_t^{-2} = \frac{1}{W(t)}$. In accordance with the procedure described at the end of section 2, we find $b^0_t = C = \frac{1}{W(t)}$. $\diamond$

In view of Corollary 3.4, the above result makes it plausible to suspect that for $t < \rho_W$ we have $b^{>0}_t = 0$. In the next section we prove this for right angled Coxeter groups.

8. Mayer-Vietoris sequence

In this section we limit our attention to right angled Coxeter groups. This means that whenever two generators $s, s' \in S$ are related in the standard presentation, they in fact commute. If we join each pair of commuting generators by an edge, we get a graph with the set of vertices S. It is convenient to fill it, gluing in a simplex whenever we can see its 1-skeleton in the graph. The resulting simplicial complex is denoted L, and the Coxeter group W_L. The Davis chamber D can be identified with the cone CL' over the first barycentric subdivision of L. We say that a subcomplex $K \subseteq L$ is full, if whenever it contains all vertices of a simplex of L, it contains the simplex as well. Full subcomplexes K correspond to subsets of S and thus to special subgroups W_K of W_L. The Davis complex of W_K is naturally embedded in Σ_{W_L}: we first embed $D_K = CK'$ in $D_L = CL'$, and then extend W_K-equivariantly. We abbreviate Σ_{W_L} as Σ_L, and use W_L and Σ_L interchangeably while referring to Betti numbers.

Let $L = L_1 \cup L_2$, where L_1, L_2 and (consequently) $L_0 = L_1 \cap L_2$ are full subcomplexes of L. We can embed W_{L_i} into W_L, and Σ_{L_i} into Σ_L; then $\Sigma_L = W_L\Sigma_{L_1} \cup W_L\Sigma_{L_2}$, $W_L\Sigma_{L_1} \cap W_L\Sigma_{L_2} = W_L\Sigma_{L_0}$. We have a short exact sequence of cochain complexes

$$0 \to L^2_tC^*(\Sigma_L) \to L^2_tC^*(W_L\Sigma_{L_1}) \oplus L^2_tC^*(W_L\Sigma_{L_2}) \to L^2_tC^*(W_L\Sigma_{L_0}) \to 0,$$

from which we get the long exact Mayer-Vietoris sequence:

$$\begin{aligned} &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \ldots \to L^2_tH^{i-1}(W_L\Sigma_{L_0}) \to \\ &\to L^2_tH^i(\Sigma_L) \to L^2_tH^i(W_L\Sigma_{L_1}) \oplus L^2_tH^i(W_L\Sigma_{L_2}) \to L^2_tH^i(W_L\Sigma_{L_0}) \to \ldots \end{aligned} \tag{8.1}$$

Since we work with reduced cohomology, the exactness of the above sequence requires some extra care, see eg [L,1.22]. Notice that $W_L\Sigma_{L_i}$ is the disjoint union of $w\Sigma_{L_i}$, $w \in W_L/W_{L_i}$. The L^2_t norm on $w\Sigma_{L_i}$ is $t^{d/2}$ times the L^2_t norm on Σ_{L_i}, where d is the length of the shortest element of wW_{L_i}. In particular, if $L^2_tH^p(\Sigma_{L_i}) = 0$, then $L^2_tH^p(W_L\Sigma_{L_i}) = 0$.

Corollary 8.1
Suppose that $b^{>0}_t(\Sigma_{L_i}) = 0$ for $i = 0, 1, 2$. Then $b^{>1}_t(\Sigma_L) = 0$.

Theorem 8.2
Let W be a right angled Coxeter group. For $t < \rho_W$ we have $b^0_t(W) = \chi_t(W) = \frac{1}{W(t)}$ and $b^{>0}_t(W) = 0$.

Proof. We prove this by induction on the number of vertices of L.

1) If L is a simplex, then $\Sigma_{L,d}$ is a cube; its L^2_t cohomology coincides with the usual cohomology and is concentrated in dimension 0.

2) If L is not a simplex, we can find two vertices $a, b \in L$ not connected by an edge; we put $L_1 = \bigcup\{\sigma \mid a \notin \sigma\}$, $L_2 = \bigcup\{\sigma \mid b \notin \sigma\}$ and $L_0 = L_1 \cap L_2$. These have fewer vertices, and so $L^2_tH^{>0}(\Sigma_{L_i}) = 0$ for $t < \rho(W_{L_i})$ $(i = 0, 1, 2)$. Since $L_i \subset L$, we have $\rho(W_{L_i}) > \rho(W_L)$. Therefore we have $L^2_tH^{>0}(\Sigma_{L_i}) = 0$ for $t < \rho(W_L)$. It follows from Corollary 8.1 that $L^2_tH^{>1}(\Sigma_L) = 0$ (still for $t < \rho(W_L)$), while from Corollary 3.4 and Theorem 7.1 we conclude that

$$b^0_t(\Sigma_L) = \chi_t(W_L) = b^0_t(\Sigma_L) - b^1_t(\Sigma_L).$$

Thus $b^1_t(\Sigma_L) = 0$. ◇

Corollary 8.3
Assume that L is a generalized homology $(n-1)$-sphere (ie D is a generalized homology n-disc); then for $t < \frac{1}{\rho(W_L)}$ we have $b^n_t = 0$, while for $t > \frac{1}{\rho(W_L)}$ the L^2_t cohomology is concentrated in dimension n and $b^n_t = \chi_t = \frac{1}{W_L(t)}$.

Proof. It follows from Theorems 8.2 and 7.1 via Poincaré duality (Theorem 6.1). $\diamond$

Proposition 8.4

Let $K \subset L$ be a full subcomplex. Then the dimension of the $U_t(L)$-module $L^2_t H^q(W_L \Sigma_K)$ is the same as the dimension of the $U_t(K)$-module $L^2_t H^q(\Sigma_K)$ (ie is equal to $b^q_t(K)$).

Proof. A harmonic q-cochain on $W_L\Sigma_K = \bigcup\{w\Sigma_K \mid w \in W_L/W_K\}$ is the same as a collection of harmonic q-cochains on $w\Sigma_K$. In order to calculate dimensions, we embed everything in $V = \oplus_{\sigma\subset D_L} L^2_t(W_L)$. Let $\mathbf{1}_\sigma \in V$ have δ_1 as its coordinate with index σ, and 0 on all other coordinates. As we project $\mathbf{1}_{\sigma^q}$ on $L^2_t\mathcal{H}^q(W_L\Sigma_K)$, we get in fact a harmonic cochain supported on Σ_K—harmonic cochains supported on other components of $W_L\Sigma_K$ are orthogonal to $\mathbf{1}_{\sigma^q}$, so also to its projection. We can as well project $\mathbf{1}_{\sigma^q}$ on $L^2_t\mathcal{H}^q(\Sigma_K)$ inside $\oplus L^2_t(W_K)$, so that the projection matrices are the same (apart for the case $\sigma \not\subset K$, which gives 0 in the first case and does not appear in the second), and traces coincide. $\diamond$

9. Chain homotopy contraction

In this section we will describe a simplicial version of the geodesic contraction of Σ with respect to the Moussong metric. We will consider the chain complex $C_*(\Sigma_{st})$ equipped with the boundary operator ∂ given by (5.7). Henceforth we write Σ for Σ_{st}, and we denote by b the barycentre of the basic chamber D. Recall that Σ can be equipped with a W-invariant, $CAT(0)$ metric d_M, the Moussong metric (cf. [M]). From now on, all balls, geodesics etc. will be considered with respect to d_M (unless explicitly stated otherwise). Besides $CAT(0)$, the following property of the Moussong metric will be useful for us: for every $R > 0$ there exists a constant $N(R)$ such that any ball in Σ of radius R intersects at most $N(R)$ chambers. Let A be the length of the longest edge in Σ.

Theorem 9.1

There exists a linear map $H : C_*(\Sigma) \to C_{*+1}(\Sigma)$, and constants C, R, with the following properties:

a) if $v \in \Sigma^{(0)}$, then $\partial H(v) = v - b$;

b) if σ is a simplex of positive dimension, then $\partial H(\sigma) = \sigma - H(\partial\sigma)$;

c) for every simplex σ, $\|H(\sigma)\|_{L^\infty} < C$;

d) if γ is a geodesic from a vertex of a simplex σ to b, then $\operatorname{supp}(H(\sigma)) \subseteq B_R(\operatorname{image}(\gamma))$.

Proof. We will construct, for all integers $i \geq 0$, linear maps $h_i : C_*(\Sigma) \to C_*(\Sigma)$, $H_i : C_*(\Sigma) \to C_{*+1}(\Sigma)$ such that:

1) $h_0 = \mathrm{id}$;
2) $\partial h_i = h_i\partial$;
3) $\partial H_i = h_i - H_i\partial - h_{i+1}$;
4) $\exists C_k, \forall\sigma \in \Sigma^{(k)}, \forall i \geq 0, \|H_i(\sigma)\|_{L^\infty} < C_k$;
5) $\exists R_k, \forall\sigma \in \Sigma^{(k)}, \forall i \geq 0$, if γ is a geodesic from a vertex of σ to b, then $\operatorname{supp}(h_i(\sigma))$, $\operatorname{supp}(H_{i-1}(\sigma))$ (if $i > 0$) and $\operatorname{supp}(H_i(\sigma))$ are contained in $B_{R_k}(\gamma(i))$ (or in $B_{R_k}(b)$, if $i > \operatorname{length}(\gamma)$);
6) if $i \geq \operatorname{diam}(\sigma \cup \{b\})$, then $h_i(\sigma) = 0$ (unless $\dim\sigma = 0$, in which case $h_i(\sigma) = b$) and $H_i(\sigma) = 0$.

The construction will be by induction on the chain degree k. Throughout this prove, we will say that a family of chains is uniformly bounded if they have uniformly bounded support diametres and L^∞ norms.

1. $k = 0$.

Let $v \in \Sigma^{(0)}$, let $\gamma_v : [0, l] \to \Sigma$ be a geodesic such that $\gamma_v(0) = v$, $\gamma_v(l) = b$. We put $h_0(v) = v$, $h_i(v) = b$ if $i \geq l$, and we choose a vertex within distance A from $\gamma_v(i)$ and declare it to be $h_i(v)$ in the remaining cases. We have $d(h_i(v), h_{i+1}(v)) \leq 1 + 2A$. Now, up to the isometric action of W, there are only finitely many pairs of vertices (y, z) satisfying $d(y, z) < 1 + 2A$. In every isometry class of such pairs we choose a pair (y, z) and we fix a 1-chain $H(y, z)$, $\partial H(y, z) = y - z$; we then extend H to the W-orbit of (y, z) using the W-action (and making choices if stabilisers are non-trivial). In the case $y = z$ we choose $H(y, y) = 0$. Notice that the chosen 1-chains H are uniformly bounded. Finally, we put $H_i(v) = H(h_i(v), h_{i+1}(v))$.

2. $k \to (k + 1)$.

Let $\sigma \in \Sigma^{(k+1)}$. Then, due to 2), $\partial h_i(\partial\sigma) = h_i(\partial\partial\sigma) = 0$. Thus, $h_i(\partial\sigma)$ is a cycle. Moreover, we claim that as we vary σ, the cycles thus obtained are uniformly bounded. In fact, as a consequence of 5), every simplex from the support of $h_i(\partial\sigma)$ is within R_k of one of the points $\gamma_v(i)$, where v runs through vertices of σ and, by $CAT(0)$ comparison, the $k+2$ points $\gamma_v(i)$ are within $2A$ of each other. Whence uniform boundedness of supports of $h_i(\partial\sigma)$. Uniform boundedness of L^∞ norms follows from 4). Up to the isometric W-action on Σ, there are finitely many possible values of $h_i(\sigma)$. As in step 1, we fix $(k + 1)$-chains $h_i(\sigma)$, $\partial h_i(\sigma) = h_i(\partial\sigma)$, so that they are uniformly bounded (and are 0 whenever $h_i(\partial\sigma) = 0$).

To define $H_i(\sigma)$, we consider the chain $h_i(\sigma) - H_i(\partial\sigma) - h_{i+1}(\sigma)$. It is a cycle:

$$\begin{aligned}
&\partial(h_i(\sigma) - H_i(\partial\sigma) - h_{i+1}(\sigma)) = \partial h_i(\sigma) - \partial H_i(\partial\sigma) - \partial h_{i+1}(\sigma) \\
&= h_i(\partial\sigma) - \big(h_i(\partial\sigma) - H_i(\partial\partial\sigma) - h_{i+1}(\partial\sigma)\big) - h_{i+1}(\sigma) = 0.
\end{aligned}$$

Again, all such chains (as we vary σ) are uniformly bounded, and we can choose $H_i(\sigma)$, satisfying $\partial H_i(\sigma) = h_i(\sigma) - H_i(\partial\sigma) - h_{i+1}(\sigma)$, in a uniformly bounded way. As before, we put $H_i(\sigma) = 0$ whenever we have to chose it so that it has boundary 0 (so as to satisfy 6)).

Now that we have a family of maps satisfying 1)–6), we put $H(\sigma) = \sum_{i \geq 0} H_i(\sigma)$. The sum is always finite because of 6), and it is easy to check the conditions a)–d):

a),b) follow from 1), 3) and 6);

c) follows from 4) and 5): since the supports of $H_i(\sigma)$ are uniformly bounded and "move along" a geodesic γ with constant speed as i grows, only a finite number of $H_i(\sigma)$ contribute to a coefficient of a fixed simplex τ in the chain $H(\sigma)$; moreover, because of 4), each contibution is smaller than $C_{\dim\sigma}$;

d) is a consequence of 5). $\diamond$

10. Vanishing below ρ

Let H be a map as in Theorem 9.1.

Theorem 10.1

Suppose that $t > \frac{1}{\rho_W}$. Then the map H extends to a bounded operator $H : L^2_t C_*(\Sigma) \to L^2_t C_{*+1}(\Sigma)$.

Proof. Unspecified summmations will be over $\Sigma^{(k)}$. N_k will denote the number of k-simplices in a chamber.

Let $a = \sum a_\sigma \sigma \in L^2_t C_k(\Sigma)$. We know that for every simplex σ, $\|H(\sigma)\|_{L^\infty} < C$. Also

$$\begin{aligned} \sum |a_\sigma| = \sum |a_\sigma| t^{d(\sigma)/2} t^{-d(\sigma)/2} &\leq \left(\sum |a_\sigma|^2 t^{d(\sigma)}\right)^{1/2} \left(\sum t^{-d(\sigma)}\right)^{1/2} \\ &\leq \|a\|_t \left(N_k W(t^{-1})\right)^{1/2} < +\infty, \end{aligned}$$

so that $\sum a_\sigma H(\sigma)$ is pointwise convergent to a chain $H(a) \in L^\infty C_{k+1}(\Sigma)$. We want to estimate $\|H(a)\|_t$. Let us write $\tau \prec \sigma$ if τ appears with non-zero coefficient in $H(\sigma)$. We have $|H(a)_\tau| \leq \sum_{\sigma|\tau\prec\sigma} C|a_\sigma|$, so that

$$\begin{aligned} \sum |H(a)_\tau|^2 t^{d(\tau)} &\leq C^2 \sum_\tau \Big(\sum_{\sigma|\tau\prec\sigma} |a_\sigma| \Big)^2 t^{d(\tau)} \\ &\leq C^2 \sum_\tau \Big(\sum_{\sigma|\tau\prec\sigma} |a_\sigma| t^{d(\sigma)/2} t^{-\alpha\left(\frac{d(\sigma)-d(\tau)}{2}\right)} t^{-\beta\left(\frac{d(\sigma)-d(\tau)}{2}\right)} \Big)^2 \qquad (10.1) \\ &\leq C^2 \sum_\tau \Big(\sum_{\sigma|\tau\prec\sigma} |a_\sigma|^2 t^{d(\sigma)} (t^{-\alpha})^{d(\sigma)-d(\tau)} \Big) \Big(\sum_{\sigma|\tau\prec\sigma} (t^{-\beta})^{d(\sigma)-d(\tau)} \Big). \end{aligned}$$

Here α, β are positive numbers chosen so that $\alpha + \beta = 1$, $t^{-\beta} < \rho_W$.

Claim

There exists a constant C', independent of τ, such that

$$\sum_{\sigma|\tau\prec\sigma} (t^{-\beta})^{d(\sigma)-d(\tau)} \leq C' W(t^{-\beta}).$$

Proof. Recall that A is the length of the longest edge in Σ, and $N(r)$ is the maximal number of chambers intersecting a ball of radius r. The claim follows from two observations.

1. For $w_0 \in W$ let $E(w_0) = \{w \in W \mid d(w) = d(w_0) + d(w_0^{-1}w)\}$. In more geometric terms, $E(w_0)$ is the set of all w such that some gallery connecting D and wD passes through $w_0 D$. We have

$$\sum_{w\in E(w_0)} (t^{-\beta})^{d(w)-d(w_0)} = \sum_{w\in E(w_0)} (t^{-\beta})^{d(w_0^{-1}w)} \leq \sum_{w\in W} (t^{-\beta})^{d(w)} = W(t^{-\beta}).$$

2. If $\tau \prec \sigma$, then τ is at distance at most R from a geodesic joining (a vertex of) σ and b. Let us consider the union U of all galleries joining D and a fixed chamber D' containing σ. Then U is the intersection of all half-spaces containing D and D' (cf. [Ron]). Since half-spaces are totally geodesic in d_M, we have $\gamma \subseteq U$. Consequently, every point of γ lies

in a gallery joining D' and D. Therefore, if we put $B(\tau) = \{w_0 \mid w_0 D \cap B_R(\tau) \neq \emptyset\}$, then we have $\{\sigma \mid \tau \prec \sigma\} \subseteq \bigcup_{w_0 \in B(\tau)} E(w_0)D$.

Putting these together,

$$\begin{aligned}\sum_{\sigma|\tau\prec\sigma} (t^{-\beta})^{d(\sigma)-d(\tau)} &\leq \sum_{w_0\in B(\tau)} t^{-\beta(d(w_0)-d(\tau))} \sum_{w\in E(w_0)} N_k (t^{-\beta})^{d(w)-d(w_0)} \\ &\leq \sum_{w_0\in B(\tau)} t^{-\beta(d(w_0)-d(\tau))} N_k W(t^{-\beta}).\end{aligned}$$

Notice that $|d(w_0) - d(\tau)|$ does not exceed the gallery distance from $w_0 D$ to some chamber containing τ, and is therefore uniformly bounded. Also, the cardinality of $B(\tau)$ is bounded by $N(R + A)$. $\diamond$(claim)

Using the claim, we can continue the estimate (10.1):

$$\|H(a)\|_t^2 \leq C^2 C' W(t^{-\beta}) \sum_\tau \Big(\sum_{\sigma|\tau\prec\sigma} |a_\sigma|^2 t^{d(\sigma)} (t^{-\alpha})^{d(\sigma)-d(\tau)} \Big).$$

Now

$$\sum_\tau \Big(\sum_{\sigma|\tau\prec\sigma} |a_\sigma|^2 t^{d(\sigma)} (t^{-\alpha})^{d(\sigma)-d(\tau)} \Big) = \sum_\sigma \Big(|a_\sigma|^2 t^{d(\sigma)} \sum_{\tau|\tau\prec\sigma} (t^{-\alpha})^{d(\sigma)-d(\tau)} \Big),$$

so that the following lemma is all we need:

Lemma 10.2

There exists a constant K independent of σ such that $\sum_{\tau|\tau\prec\sigma} (t^{-\alpha})^{d(\sigma)-d(\tau)} < K$.

Proof. Since W acts on (Σ, d_M) isometrically, cocompactly and properly discontinuously, the word metric d on W is quasi-isometric to the metric d_M restricted to $W \simeq Wb \hookrightarrow \Sigma$. This implies that there exist constants M, m, L such that for any two points $y, z \in \Sigma$ and any chambers $D_y \ni y$, $D_z \ni z$, we have

$$Md_M(y,z) + L \geq d(D_y, D_z) \geq md_M(y,z) - L, \tag{10.2}$$

where we put $d(wD, uD) = d(w,u) = d(w^{-1}u)$.

Let v be a vertex of σ, and let $\gamma : [0,l] \to \Sigma$ be a geodesic, $\gamma(0) = v$, $\gamma(l) = b$. To each $\tau \prec \sigma$ we can assign one of the points $\gamma(i)$ $(0 \leq i \leq \lfloor l \rfloor)$ in such a way that $d_M(\tau, \gamma(i)) < R + 1$. The number of simplices to which we assign a given $\gamma(i)$ does not exceed $N(R+1)N_k$. Suppose that $\gamma(i)$ is assigned to τ. Let D_τ (resp., D_σ) be the chamber containing τ (resp., σ) such that $d(\tau) = d(D, D_\tau)$ (resp., $d(\sigma) = d(D, D_\sigma)$). Let D_i be a chamber containing $\gamma(i)$. We choose D_i so that some gallery from D to D_σ passes through D_i (see part 2 of the proof of the claim above). Using (10.2), we get

$$\begin{aligned}d(\sigma) - d(\tau) &= d(D, D_\sigma) - d(D, D_\tau) \\ &\geq d(D, D_i) + d(D_i, D_\sigma) - (d(D, D_i) + d(D_i, D_\tau)) \\ &\geq md_M(\gamma(i), v) - L - (Md_M(\tau, \gamma(i)) + L) \\ &\geq mi - (M(R+1) + 2L) = mi - P,\end{aligned}$$

where $P = M(R+1)+2L$. Remember that t^{-1} and, whence, $t^{-\alpha}$ are less than 1. Therefore

$$\begin{aligned}\sum_{\tau|\tau\prec\sigma}(t^{-\alpha})^{d(\sigma)-d(\tau)} &\leq \sum_{i=0}^{\lfloor l\rfloor} N(R+1)N_k(t^{-\alpha})^{mi-P}\\ &= N(R+1)N_k t^{\alpha P}\sum_{i=0}^{\lfloor l\rfloor}(t^{-\alpha m})^i\\ &\leq N(R+1)N_k t^{\alpha P}\frac{1}{1-t^{-\alpha m}}.\end{aligned}$$

$\diamond$(Lemma 10.2, Theorem 10.1)

Theorem 10.3

Let W be a Coxeter group. For $t < \rho_W$ we have $b_t^0(W) = \chi_t(W) = \frac{1}{W(t)}$ and $b_t^{>0}(W) = 0$.

Proof. Theorems 9.1 and 10.1 imply that for $t > \frac{1}{\rho_W}$ we have $H_{>0}(L_t^2C_*(\Sigma), \partial) = 0$. Indeed, if $c \in L_t^2C_k(\Sigma)$, $\partial s = 0$, then $c = \partial H(c)+H(\partial c) = \partial H(c)$, so that $[c] = 0$. It follows that the isomorphic complex $(L_{t^{-1}}^2C_*(\Sigma), \partial^{t^{-1}})$ also has vanishing homology in degrees > 0 (if $t^{-1} < \rho_W$). Thus, its homology is concentrated in dimension 0, and the zero'th Betti number is equal to the Euler characteristic. $\diamond$

Corollary 10.4

Assume that $(D, \partial D)$ is a generalized homology n-disc; then for $t < \frac{1}{\rho_W}$ we have $b_t^n = 0$, while for $t > \frac{1}{\rho_W}$ the L_t^2 cohomology is concentrated in dimension n and $b_t^n = \chi_t = \frac{1}{W(t)}$.

Proof. It follows from Theorems 10.3 and 7.1 via Poincaré duality (Theorem 6.1). $\diamond$

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