$\underline{E}G$ for systolic groups

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Abstract

We prove that if a group G is systolic, i.e. if it acts properly and cocompactly on a systolic complex X, then an appropriate Rips complex constructed from X is a finite model for $\underline{E}G$.

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1 Introduction

Systolic complexes and systolic groups were introduced by T. Januszkiewicz and J. Świątkowski in [3] and independently by F. Haglund in [1]. Systolic complexes are simply–connected simplicial complexes satisfying certain link conditions (which will be recalled in Definition 2.2). Some of their properties are very similar to the properties of CAT(0) metric spaces, therefore one calls them complexes of simplicial nonpositive curvature. In particular it was shown in [3], Theorem 4.1(1), that they are contractible. Thus if a group G is systolic, which by definition means that it acts properly and cocompactly on a systolic complex X, and if G is torsion free, then X is a finite model for EG.

Similarly, if G acts properly on a CAT(0) space X and if G is torsion free, then X is a model for EG. If we do not assume that G is torsion free, then the stabilizer of any point in X is finite and the fixed point set of any finite subgroup of G is contractible (in particular nonempty). This means that X is the so called *model for* $\underline{E}G$ — the classifying space for finite subgroups [4].

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There are other families of groups G, which admit nice models for $\underline{E}G$. For example, if G is word–hyperbolic, and if S is a finite generating set for G, then for sufficiently large real number d the Rips complex $P_d(G,S)$ is a model for $\underline{E}G$. What makes this model attractive for applications is that it is a *finite* model, i.e. the action of G on it is cocompact. See [4] for details.

In this paper we give an explicit finite model for $\underline{E}G$ for a systolic group G. We prove that an appropriate Rips complex of any systolic complex X on which G acts properly is a model for $\underline{E}G$. We define the Rips complex in our context as follows.

Definition 1.1. Let X be any simplicial complex. For any $n \geq 1$, the *Rips complex* X_n is a simplicial complex with the same set of vertices as X and with a simplex spanned on any subset $S \subset X^{(0)}$ such that $\mathrm{diam}(S) \leq n$ in $X^{(1)}$ (where edges have length 1). If G acts on X properly (and cocompactly), then the natural extension of this action to X_n is also proper (and cocompact).

Our main result is the following.

Theorem 1.2. *Let X be a systolic complex on which a group G acts properly. Then for $n \geq 5$ the Rips complex X_n is a finite dimensional model for $\underline{E}G$. If additionally G acts cocompactly on X then X_n is a finite model for $\underline{E}G$.*

Theorem 1.2 extends and its proof is based on the following coarse fixed point theorem for systolic complexes (which also explains the appearance of the constant 5 in the above formulation).

Theorem 1.3 ([5], Theorem 1.2)**.** *Let H be a finite group acting by simplicial automorphisms on a systolic complex X. Then there exists a bounded subcomplex $Y \subset X$ which is invariant under the action of H and whose diameter is ≤ 5.*

To apply Theorem 1.3, let H be a group acting by automorphisms on a simplicial complex X. Then the fixed point set of the action of H on X is a subcomplex of the barycentric subdivision X' of X. Denote this subcomplex by $\mathrm{Fix}_H X'$. Similarly denote the fixed point set of the action of H on the Rips complex X_n by $\mathrm{Fix}_H X'_n$. It is a subcomplex of X'_n. By Theorem 1.3, if X is systolic, H is finite and $n \geq 5$, then $\mathrm{Fix}_H X'_n$ is nonempty.

Now the proof of Theorem 1.2 reduces to the following.

Proposition 1.4. *Let H be any group acting by automorphisms on a systolic complex X. Then for any $n \geq 1$ the complex $Fix_H X'_n$ is either empty or contractible.*

The remaining part of this paper is devoted to the proof of Proposition 1.4. This will be done without using the contractibility of systolic complexes [3]. In fact, by applying Proposition 1.4 to the case of H trivial and $n = 1$, we reprove that systolic complexes are contractible (since $X_1 = X$ by flagness of systolic complexes).

Our proof may seem more sophisticated than the original proof [3], but the reason for this is that we deal at the same time with contractibility of the systolic complex X and with contractibility of its Rips complexes.

In fact, our proof is simpler than the original proof. By using the methods of Section 4 (not present in [3]), we are able to avoid writing down explicit homotopies.

Note that if Theorem 1.3 could be strengthened to guarantee a true fixed point instead of an invariant subcomplex, then under the hypothesis of Theorem 1.2 we would get a stronger assertion: Proposition 1.4 would imply that the original complex X is a model for $\underline{E}G$.

The paper is organized as follows. In Section 2 we recall the classical simplicial nonpositive curvature notions and results from [3]. In Section 3 we introduce the key notion of the paper, the expansion by projection, and establish its basic properties. In Section 4 we present two abstract ways of producing homotopies in geometric realizations of posets, which will be needed later. The proof of Proposition 1.4 occupies Section 5.

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2 Systolic complexes

Let us recall (from [3]) the definition of a systolic complex and a systolic group.

Definition 2.1. A subcomplex K of a simplicial complex X is called *full* in X if any simplex of X spanned by vertices of K is a simplex of K. The *span* of a subcomplex $K \subset X$ is the smallest full subcomplex of X containing K. We will denote it by $\mathrm{span}(K)$. A simplicial complex X is called *flag* if any set of vertices, which are pairwise connected by edges of X, spans a simplex in X. A simplicial complex X is called *k–large*, $k \geq 4$, if X is flag and there are no embedded cycles of length $< k$, which are full subcomplexes of X (i.e. X is flag and every simplicial loop of length $< k$ and ≥ 4 "has a diagonal").

Definition 2.2. A simplicial complex X is called *systolic* if it is connected, simply connected and links of all simplices in X are 6–large. A group Γ is called *systolic* if it acts cocompactly and properly by simplicial automorphisms on a systolic complex X. (*Properly* means X is locally finite and for each compact subcomplex $K \subset X$ the set of $\gamma \in \Gamma$ such that $\gamma(K) \cap K \neq \emptyset$ is finite.)

Recall [3], Proposition 1.4, that systolic complexes are themselves 6–large. In particular they are flag.

Now we briefly treat the definitions and facts concerning convexity.

Definition 2.3. For every pair of subcomplexes (usually vertices) A, B in a simplicial complex X denote by $|A, B|$ ($|ab|$ for vertices $a, b \in X$) the combinatorial distance between $A^{(0)}, B^{(0)}$ in $X^{(1)}$, the 1–skeleton of X. The *diameter* $\mathrm{diam}(A)$ is the maximum of $|a_1a_2|$ over vertices a_1, a_2 in A.

A subcomplex K of a simplicial complex X is called *3–convex* if it is a full subcomplex of X and for every pair of edges ab, bc such that $a, c \in K$, $|ac| = 2$, we have $b \in K$. A nonempty subcomplex K of a systolic complex X is called *convex* if it is connected and links of all simplices in K are 3–convex subcomplexes of links of those simplices in X.

In Lemma 7.2 of [3] authors conclude that convex subcomplexes of a systolic complex X are contractible, full and 3–convex in X. For a subcomplex $Y \subset X$, $n \geq 0$, *the combinatorial ball $B_n(Y)$ of radius n around Y* is the span of $\{p \in X^{(0)} \colon |p, Y| \leq n\}$. (Similarly $S_n(Y) = \mathrm{span}\{p \in X^{(0)} \colon |p, Y| = n\}$.) If Y is convex (in particular, if Y is a simplex) then $B_n(Y)$ is also convex, as proved in [3], Corollary 7.5. The intersection of a family of convex subcomplexes is convex and we can define the *convex hull* of any subcomplex $Y \subset X$ as the intersection of all convex subcomplexes of X containing Y. We denote the convex hull of Y by $\mathrm{conv}(Y)$.

We include the proof of the following well known lemma, since it does not appear elsewhere.

Lemma 2.4. $\mathrm{diam}(\mathrm{conv}(Y)) = \mathrm{diam}(Y)$.

Proof. If Y is unbounded then there is nothing to prove. Otherwise, denote $d = \mathrm{diam}(Y)$. The inequality $\mathrm{diam}(\mathrm{conv}(Y)) \geq d$ is obvious. For the other direction, let y_1, y_2 be any two vertices in $\mathrm{conv}(Y)$. We want to prove that $|y_1y_2| \leq d$. Observe that for any vertex $y \in Y$ the ball $B_d(y)$ is convex and contains Y, hence by the definition of the convex hull we have $\mathrm{conv}(Y) \subset B_d(y)$. This means that $|yy_1| \leq d$. Thus Y is contained in $B_d(y_1)$ and by convexity of balls we have $\mathrm{conv}(Y) \subset B_d(y_1)$. We get $|y_1 y_2| \leq d$, as desired. □

The paper [2] of F. Haglund and J. Świątkowski contains a proof of the following proposition (Proposition 4.9), which will be used throughout the present paper.

Proposition 2.5. *A nonempty full subcomplex Y of a systolic complex X is convex if and only if $Y^{(1)}$ is geodesically convex in $X^{(1)}$ (i.e. if all geodesics in $X^{(1)}$ joining vertices of Y lie in $Y^{(1)}$).*

We will need a crucial projection lemma ([3], Lemma 7.7). The *residue* of a simplex σ in X is the union of all simplices in X, which contain σ.

Lemma 2.6. *Let Y be a convex subcomplex of a systolic complex X and let σ be a simplex in $B_1(Y)$ disjoint with Y. Then the intersection of the residue of σ and of the complex Y is a simplex (in particular it is nonempty).*

Definition 2.7. The simplex as in Lemma 2.6 is called the *projection* of σ onto Y.

3 Expansion by projection

The proof of contractibility of systolic complexes given by T. Januszkiewicz and J. Świątkowski in [3] uses Lemma 2.6 and the notion of projection (Definition 2.7). To be able to deal with the Rips complex we need to extend this notion: we need to be able to project not only simplices, but all convex subcomplexes. In this section we introduce the necessary definitions for this and establish the basic properties of the corresponding notions.

Definition 3.1. Let Y be a convex subcomplex of a systolic complex X and let σ be a simplex in $B_1(Y)$. The *expansion by projection* of σ (denoted by $e_Y(\sigma)$) is a simplex in $B_1(Y)$ defined in the following way. If $\sigma \subset Y$ then $e_Y(\sigma) = \sigma$. Otherwise $e_Y(\sigma)$ is the join of $\sigma \cap S_1(Y)$ (which is nonempty) and its projection (c.f. Definition 2.7) onto Y.

Remark 3.2. Observe that $\sigma \subset e_Y(\sigma)$. Moreover, by Lemma 2.6, $e_Y(\sigma) \cap Y$ is nonempty.

Definition 3.3. Let Y be a convex subcomplex of a systolic complex X and let Z be a convex subcomplex in $B_1(Y)$. The *expansion by projection* of Z (denoted by $e_Y(Z)$) is a subcomplex of $B_1(Y)$ defined in the following way. Let $e_Y(Z)$ be the span of the union of $e_Y(\sigma)$ over all maximal (with respect to inclusion) simplices $\sigma \subset Z$. Clearly this definition extends Definition 3.1.

Remark 3.4. Observe that $Z \subset e_Y(Z)$. Moreover $e_Y(Z) \cap Y$ is nonempty. Note that $e_Y(Z)$ does not have to be convex.

Remark 3.5. Let g be an automorphism of X which leaves Y and Z invariant. Then g leaves also $e_Y(Z)$ invariant.

The following property of the expansion by projection is not at all obvious.

Lemma 3.6. $\mathrm{diam}(e_Y(Z)) \leq \max\{\mathrm{diam}(Z), 1\}$.

In fact, since by Remark 3.4 we have $Z \subset e_Y(Z)$, this is an equality unless Z is a single vertex of Y.

Before giving the proof we need to establish some facts about the distance between maximal simplices in convex subcomplexes.

Lemma 3.7. *Let Z be a convex subcomplex in a systolic complex X. Let d be the diameter of Z. Assume $d \geq 2$. Let σ, τ be any maximal simplices of Z and let v be any vertex of Z. Then*

(1) $|\sigma, v| \leq d - 1$,

(2) $|\sigma, \tau| \leq d - 2$.

Proof. First we prove assertion (1). We do this by contradiction. Assume $|\sigma, v| = d$. This means that $\sigma \subset S_d(v)$, so $e_{B_{d-1}(v)}(\sigma)$ (the expansion by projection onto $B_{d-1}(v)$, c.f. Definition 3.1 and Remark 3.2) is a simplex strictly greater than σ. All vertices in $e_{B_{d-1}(v)}(\sigma)$ lie on some 1–skeleton geodesics from v to vertices in σ. Hence by Proposition 2.5 and by convexity of Z we have $e_{B_{d-1}(v)}(\sigma) \subset Z$. Thus σ is not maximal in Z, contradiction.

Now we prove assertion (2). We do this again by contradiction. Assume $|\sigma, \tau| > d - 2$. By (1) this implies that $|\sigma, v| = d - 1$ for all $v \in \tau$. Thus $\tau \subset S_{d-1}(\sigma)$. As before, by Proposition 2.5 and by convexity of Z we get $e_{B_{d-2}(\sigma)}(\tau) \subset Z$. Since $e_{B_{d-2}(\sigma)}(\tau)$ is strictly greater than τ, we obtain contradiction with maximality of τ. □

Proof of Lemma 3.6 Denote by d the diameter of Z. Suppose $d \geq 2$ (otherwise the lemma is obvious). Take any $v, w \in e_Y(Z)$. We must prove that $|vw| \leq d$. If $v, w \in Z$ then there is nothing to prove. Now assume that $v \in Z, w \notin Z$. Thus there exists a maximal simplex $\sigma \subset Z$ such that $w \in e_Y(\sigma)$. By Lemma 3.7(1) we have $|\sigma, v| \leq d - 1$, hence there exists a vertex $s \in \sigma$ such that $|vs| \leq d - 1$. Since $|sw| \leq 1$, we are done.

Now assume that both $v, w \notin Z$. Thus there exist maximal simplices $\sigma, \tau \subset Z$ such that $v \in e_Y(\sigma), w \in e_Y(\tau)$. By Lemma 3.7(2) there exist vertices $s \in \sigma, t \in \tau$ such that $|st| \leq d-2$. Since $|vs| \leq 1$ and $|wt| \leq 1$, we are done. $\square$

We end this section with a lemma which though seems technical, nevertheless lies at the heart of the proof of Proposition 1.4, which will be presented in Section 5. This lemma states, roughly speaking, that expanding by projection has not too bad monotonicity properties (although usually it is not true that $Z \subset Z'$ implies $e_Y(Z) \subset e_Y(Z')$ or $e_Y(Z) \supset e_Y(Z')$).

Lemma 3.8. *Let $Z_1 \subset Z_2 \subset \ldots \subset Z_n \subset B_1(Y)$ be an increasing sequence of convex subcomplexes of $B_1(Y)$. Then the intersection*

$$\Big(\bigcap_{i=1}^{n} e_Y(Z_i)\Big) \cap Y$$

is nonempty.

Proof. If $Z_1 \cap Y$ is nonempty then any vertex $v \in Z_1 \cap Y$ belongs to the required intersection. Otherwise take any maximal (in Z_1) simplex $\sigma_1 \subset Z_1$. We define inductively an increasing sequence of simplices $\sigma_i \subset Z_i$ for $i = 2, \ldots, n$. Namely choose σ_i to be any maximal simplex in Z_i containing σ_{i-1}. Take any vertex $v \in e_Y(\sigma_n) \cap Y$. Since σ_i do not lie entirely in Y, we have by definition of $e_Y(\sigma_i)$ that $v \in e_Y(\sigma_i)$ for all i. Since each σ_i is maximal in the corresponding Z_i, this implies that $v \in e_Y(Z_i)$ for all i, hence v belongs to the required intersection. $\square$

4 Homotopies

We will use the following well known results. The proof of the first proposition can be found, for example, in the paper of G. Segal [6]. However, for completeness, we give an indication of an argument.

Proposition 4.1 ([6], Proposition 1.2)**.** *If $\mathcal{C}, \mathcal{D}$ are posets and $F_0, F_1 \colon \mathcal{C} \to \mathcal{D}$ are functors (i.e. they respect the order) such that for each $c \in \mathcal{C}$ we have $F_0(c) \leq F_1(c)$, then the maps induced by F_0, F_1 on geometric realizations of $\mathcal{C}, \mathcal{D}$ are homotopic. Moreover this homotopy is constant on the geometric realization of the subposet of $\mathcal{C}$ of objects on which F_0 and F_1 agree.*

Proof. We need to extend the natural homotopy on vertices of geometric realizations to higher skeleta. This is done by performing the so called prism subdivision of the cells of the homotopy. Then the homotopy can be realized simplicially, it can be explicitly written down. $\square$

In the next proposition we will consider a functor $F: \mathcal{C}' \to \mathcal{C}$ from the flag poset $\mathcal{C}'$ of a poset $\mathcal{C}$ into the poset $\mathcal{C}$, assigning to each object in $\mathcal{C}'$, which is a chain of objects of $\mathcal{C}$, its minimal element. F is covariant if we take on $\mathcal{C}'$ the partial order inverse to the inclusion. Geometric realizations of $\mathcal{C}, \mathcal{C}'$ are homeomorphic in a canonical way (one is the barycentric subdivision of the other), which allows us to identify them.

Proposition 4.2. *The map induced by F on geometric realizations of $\mathcal{C}', \mathcal{C}$ is homotopic to identity.*

Proof. We give only a sketch. Take any simplex in the geometric realization of $\mathcal{C}'$, suppose it corresponds to a chain $c_1' \subset \ldots \subset c_n'$ (c_i' are chains of objects in $\mathcal{C}$). This simplex and its image under the map induced by F both lie in the simplex of the image, which corresponds to the chain c_n'. Thus the homotopy can be realized affinely on each simplex. $\square$

5 Nonempty fixed point sets are contractible

As observed in the Introduction, Theorem 1.2 is implied by Theorem 1.3 and Proposition 1.4. Thus to prove Theorem 1.2 it is enough to prove Proposition 1.4, which we do in this section.

Let us give an outline of the proof. Suppose the fixed point set we are considering is nonempty. We define an increasing sequence of subcomplexes exhausting the Rips complex, with an invariant simplex as the first subcomplex. We then prove that the intersection of the fixed point set with a subcomplex from our family is homotopy equivalent to the intersection of the fixed point set with the subsequent subcomplex. Since we know that the first of those intersections is contractible, it follows by induction that any of the intersections is contractible. Since we choose an exhausting family, this means that the whole fixed point set is contractible.

We define now this exhausting family.

Definition 5.1. Let X be any simplicial complex. Let $\sigma \subset X_n$ be any simplex in the Rips complex of X for some $n \geq 1$. Let $A \subset X_n^{(0)} = X^{(0)}$ be the set of vertices of σ. Recall that $B_i(A)$ is the combinatorial ball of radius i around A in X. Now define an increasing sequence of full subcomplexes $D_i(\sigma) \subset X_n'$, where $i \geq 0$, in the following way. Let $D_{2i}(\sigma)$ be the span of all vertices in X_n' corresponding to simplices in X_n, which have all their vertices in $B_i(A)$ (i.e. $D_{2i}(\sigma)$ is equal to the barycentric subdivision of the span in X_n of vertices in $B_i(A) \subset X$). Let $D_{2i+1}(\sigma)$ be the span of those vertices in

X'_n, which correspond to those simplices in X_n that have all their vertices in $B_{i+1}(A)$ and at least one vertex in $B_i(A)$ (where the balls are taken in X).

In case of a flag complex X for $n = 1$ we have $X_1 = X$ and the subcomplexes $D_i(\sigma)$ are combinatorial balls in X' around the barycentric subdivision of σ.

Remark 5.2. Notice that $\bigcup_{i=0}^{\infty} D_i(\sigma) = X'_n$. Moreover, any compact subcomplex of X'_n is contained in some $D_i(\sigma)$.

Proof of Proposition 1.4 Assume that $\mathrm{Fix}_H X'_n$ is nonempty. Let $\sigma \subset X_n$ be a maximal H–invariant simplex in X_n. Denote the set of vertices of σ in $X_n^{(0)} = X^{(0)}$ by A. We claim that the span of A in X is convex. Otherwise, by Lemma 2.4, the vertices of $\mathrm{conv}(A)$ in X span a simplex in X_n, which is also H–invariant and strictly greater than σ, contradiction. Let $D_i(\sigma) \subset X'_n$ be as in Definition 5.1. In the further discussion we will use an abbreviated notation $D_i = D_i(\sigma)$.

We will prove the following three assertions.

(i) $D_0 \cap \mathrm{Fix}_H X'_n$ is contractible,

(ii) the inclusion $D_{2i} \cap \mathrm{Fix}_H X'_n \subset D_{2i+1} \cap \mathrm{Fix}_H X'_n$ is a homotopy equivalence,

(iii) the identity on $D_{2i+2} \cap \mathrm{Fix}_H X'_n$ is homotopic to a mapping with image in $D_{2i+1} \cap \mathrm{Fix}_H X'_n \subset D_{2i+2} \cap \mathrm{Fix}_H X'_n$.

Suppose for a moment that (i)–(iii) hold. We will show how this implies the theorem. We will prove by induction on k the following.

Claim. $D_k \cap \mathrm{Fix}_H X'_n$ is contractible.

For $k = 0$ this is stated in assertion (i). Suppose we have proved the claim for some $k \geq 0$. If k is even, $k = 2i \geq 0$, then assertion (ii) implies the claim for $k = 2i + 1$. If k is odd, $k = 2i + 1$, then the identity mapping from assertion (iii) is homotopic to the mapping with image in a contractible subspace, hence the identity mapping is homotopically trivial. This proves the claim for $k = 2i + 2$. We have thus completed the induction step.

By Remark 5.2, the image of any sphere mapped into $\mathrm{Fix}_H X'_n$ is contained in some $D_i \cap \mathrm{Fix}_H X'_n$, which is contractible. Thus all homotopy groups of $\mathrm{Fix}_H X'_n$ are trivial and since $\mathrm{Fix}_H X'_n$ is a simplicial complex, it is contractible, by Whitehead's Theorem, as desired. To complete the proof we must now prove assertions (i)–(iii).

Assertion (i). Since D_0 is the barycentric subdivision of the simplex $\sigma \subset X_n$ and the barycenter of σ belongs to $\mathrm{Fix}_H X_n'$, we have that $D_0 \cap \mathrm{Fix}_H X_n'$ is a cone over the barycenter of σ, hence it is contractible.

Assertion (ii). Let $\mathcal{C}$ be the poset of H–invariant simplices in X_n with vertices in $B_{i+1}(A)$ (ball in X) and at least one vertex in $B_i(A)$. Its geometric realization is $D_{2i+1} \cap \mathrm{Fix}_H X_n'$. Consider a functor $F\colon \mathcal{C} \to \mathcal{C}$ assigning to each object of $\mathcal{C}$ i.e. a simplex in X_n its subsimplex spanned by vertices in $B_i(A)$. Notice that this subsimplex is H–invariant (i.e. it is an object of $\mathcal{C}$) since A and hence $B_i(A)$ are H–invariant. By Proposition 4.1 the geometric realization of F is homotopic to identity (which is the geometric realization of the identity functor). Moreover this homotopy is constant on $D_{2i} \cap \mathrm{Fix}_H X_n'$. The image of the geometric realization of F is contained in $D_{2i} \cap \mathrm{Fix}_H X_n'$. Hence $D_{2i} \cap \mathrm{Fix}_H X_n'$ is a deformation retract of $D_{2i+1} \cap \mathrm{Fix}_H X_n'$, as desired.

Assertion (iii). Let $\mathcal{C}$ be the poset of H–invariant simplices in X_n' with vertices in $B_{i+1}(A)$ and let $\mathcal{C}'$ be its flag poset, with the partial order inverse to the inclusion. Let $F_0\colon \mathcal{C}' \to \mathcal{C}$ be the functor (from Proposition 4.2) assigning to each object in $\mathcal{C}'$, which is a chain of objects of $\mathcal{C}$, its minimal element. The geometric realization of both $\mathcal{C}$ and $\mathcal{C}'$ is equal to $D_{2i+2} \cap \mathrm{Fix}_H X_n'$ and by Proposition 4.2 the geometric realization of F_0 is homotopic to identity.

Now we define another functor $F_1\colon \mathcal{C}' \to \mathcal{C}$. This is the heart of the proof. First notice that since $\mathrm{span}(A)$ is convex in X, we have that the ball $B_i(A)$ is also convex. Hence for any convex subcomplex $Z \subset B_{i+1}(A)$ there exists its expansion by projection (c.f. Definition 3.3) $e_{B_i(A)}(Z)$. Now we define F_1. For any object $c' \in \mathcal{C}'$, which is a chain of objects $c_1 < c_2 < \ldots < c_k$ of $\mathcal{C}$, recall that c_j (where $1 \leq j \leq k$) are some H–invariant simplices in X_n with vertices in $B_{i+1}(A)$. Denote the set of vertices of c_j by S_j and treat it as a subset of $X^{(0)}$. Notice that the subcomplexes $\mathrm{conv}(S_j) \subset X$ are of diameter $\leq n$ (by Lemma 2.4), they form an increasing sequence and they are all contained in $B_{i+1}(A)$ by monotonicity of taking the convex hull and by convexity of balls. Thus if we define S_j' to be the set of vertices in $e_{B_i(A)}(\mathrm{conv}(S_j))$, then by Lemma 3.8 the intersection $\bigcap_{j=1}^k S_j'$ contains at least one vertex in $B_i(A)$. Also note that this intersection is contained in $B_{i+1}(A)$. Moreover, by Lemma 3.6, all the sets S_j', and hence their intersection, have diameter $\leq n$. Thus we can treat the set $\bigcap_{j=1}^k S_j'$ as a simplex in X_n with vertices in $B_{i+1}(A)$. By Remark 3.5 this simplex is H–invariant, hence it is an object in $\mathcal{C}$. We define $F_1(c')$ to be this object. In geometric realization of $\mathcal{C}$, which is $D_{2i+2} \cap \mathrm{Fix}_H X_n'$ the object $F_1(c')$ corresponds to a vertex in $D_{2i+1} \cap \mathrm{Fix}_H X_n'$, by our previous remarks. It is obvious that F_1 preserves the partial order (inverse to the inclusion on $\mathcal{C}'$), since the greater the chain, the more sets S_j'

we have to intersect.

Now notice that by Remark 3.4 for any $c' \in \mathcal{C}'$ we have $F_0(c') \subset F_1(c')$, hence by Proposition 4.1 the geometric realizations of F_0 and F_1 are homotopic. But as observed at the beginning, F_0 is homotopic to the identity. On the other hand, F_1 has image in $D_{2i+1} \cap \mathrm{Fix}_H X'_n$. Thus we are done. $\square$

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