Nonpositively curved developments of billiards

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Abstract. We prove existence of compact nonpositively (negatively) curved developments for a large class of Euclidean (hyperbolic) billiards in arbitrary dimension, including all convex polytopal ones.

Introduction

Given a trajectory in a polytopal billiard table, one can develop it into a straight line by attaching a mirror image of the table at each point the trajectory hits the boundary, and instead of reflecting, continuing through into the other copy. The developed trajectory is surrounded by a sequence of adjacent copies of the billiard table. Regardless of its simplicity, it is an important idea relating billiards to geodesic flows.

It is reasonable to ask for more, namely for developments of billiards which work simultaneously for all trajectories. These are spaces obtained from a number of copies of the table, so that each side of each copy is glued to a side of exactly one other copy by an identity map (this produces spaces called pseudomanifolds). Such a construction allows to develop all trajectories, perhaps to closed or self-intersecting piecewise straight lines, as long as they do not hit corners of the table. To incorporate other trajectories into the picture, it is convenient to demand that the development of the table is nonpositively curved, and extend (non-uniquely) trajectories to geodesics. It was suggested by D. Burago [Bu], that such developments, if compact, may be useful in proving results about complexity of billiards trajectories.

Existence of nonpositively curved compact developments is fairly clear for two-dimensional tables, and was established in [BuFKK] for a class of three-dimensional tables (containing all convex polytopes). In [SNPC] existence of such developments was established for the case when the table is a simplex of any dimension. The present paper, extending methods of [SNPC], proves existence of compact nonpositively curved developments for a large class of tables in arbitrary dimension, including convex polytopal tables. We state our main result at the end of this introduction, after discussing certain technical terms involved in its formulation.

The method we use both in [SNPC] and in the present paper is based on the notion of combinatorial nonpositive curvature and on new constructions of developable simplices of groups. Spaces that result from our constructions, the so called k-systolic simplicial complexes, are not necessarily nonpositively curved, and their study has uncovered several exotic phenomena in geometric group theory. But if the integer parameter k is large enough (depending on dimension), then n-dimensional k-systolic simplicial complexes are $CAT(0)$ for the standard piecewise Euclidean metric (see [SNPC]). The idea of finding sufficient combinatorial conditions on a polyhedral complex implying that it is $CAT(0)$ is taken up again in the present paper.

From a different point of view, the present paper provides a rich source of new $CAT(\kappa)$ spaces for $\kappa = -1, 0, 1$. Since these arise from simplices of groups, they come with large groups of isometries, thus are interesting for geometric group theory. It is this aspect which makes us consider as developments a class of spaces going beyond pseudomanifolds, in which a copy of a table may be adjacent to more than one other copy along a face of codimension 1.

We focus on billiard tables with constant curvature $-1, 0, 1$, even though it would be natural to allow tables with variable curvature, with an appropriate convexity of the boundary. We do not discuss this more general setting here, since it requires addressing additional technical points. We do not see it to bring essentially new challenges. The interplay between metric and combinatorial (or group theoretical) aspects of the question is subtle and rewarding already in the constant curvature case, which does cover many examples. However the ideas and techniques presented here can definitely be applied in a more general setting, for example to pseudomanifolds with boundary and piecewise constant curvature metrics. Hopefully this leads to interesting applications.

Also, we do not go in any details into the dynamical aspects of the subject due to lack of expertise.

To state our main result we need at least rough description of the terms involved. We refer the reader to Section 4 for precise definitions and for slightly more general statements (Theorems 4.4 and 4.5). For us, a *billiard table* is a riemannian manifold of constant sectional curvature with stratified boundary, with all strata geodesically convex, and such that near boundary the table looks locally like an intersection of half-spaces. Stratification of the table is *locally injective* if different germs of local strata belong to different (global) strata. One does indeed need some assumption of this type, as examples from Section 5, of tables with no nonpositively curved developments show.

Main Theorem. Let B be a Euclidean (respectively, hyperbolic) compact billiard table with convex polytopal boundary and with locally injective stratification. Then B admits a finite nonpositively curved (respectively, with curvature ≤ -1) development.

The paper is organized as follows. In Section 1 we discuss polytopal complexes and the (combinatorial) condition of k-largeness for them. In Section 2 we extend from simplicial to polytopal piecewise spherical complexes results of [SNPC] asserting that k-large with sufficiently large k implies $CAT(1)$ property of the piecewise spherical metric. In Section 3 we show how to induce large developments of arbitrary polytopes from large developments of simplices. Existence of the former follows then from existence of the latter (proved in

[SNPC]). In Section 4 we define our billiard tables and construct their developments. In Section 5 we gather some further remarks and speculations pertaining to the geometric side of the subject of developing billiards. Finally, two appendices are intended to make the paper self contained. The first of them is just a review of systolic complexes. The second presents a lemma concerning face complexes, a tool used to define and study the notion of k-largeness.

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1. Large polytopal complexes.

A *convex polytope* π is the convex hull of a finite set of points in a real vector space. *Faces* of π are also convex polytopes and form the *face poset* of π. In this section we pay no attention to metric aspects of convex polytopes, viewing them merely as combinatorial objects.

A *polytopal complex* X is a space obtained by glueing convex polytopes via combinatorial equivalences of their faces (i.e., homeomorphisms preserving cell structure), together with the decomposition of X into faces (so that all faces of all polytopes forming X are faces of X). We also assume that:

(1) different faces of the same polytope are not identified;

(2) intersection of two faces of X is either empty or a single face of X.

Note that simplicial complexes are examples of polytopal complexes.

1.1 Definition (Face complex). A set of faces of a polytopal complex X is *joinable* if all these faces are contained in a single face of X. Given a joinable set S of faces, *the face spanned by* S is the unique minimal face of X containing all faces of S (uniqueness follows from assumption (2) above). *The face complex* $\Phi(X)$ of a polytopal complex X is the simplicial complex whose vertices $[\tau]$ correspond to all faces τ of X and whose simplices correspond to all sets of faces that are joinable in X.

The notions of joinability and a face complex have been invented by F. Haglund and the second author, superseding our earlier attempts. The face complex is a "fattening" of a polytopal complex X to a simplicial complex, in particular it preserves the homotopy type of X (though this fact plays no role in the present paper). The face complex of X seems to be a useful substitute of the barycentric subdivision, and it allows to apply certain simplicial techniques to study X.

Now we discuss links in polytopal complexes. Given a polytope π and its proper face σ, the *link* π_σ of π at σ is defined as follows. Consider the poset of those faces of π which properly contain σ. This poset is the face poset of a convex polytope of dimension $\dim\pi - \dim\sigma - 1$, and we take this polytope as the link π_σ. Given a polytopal complex X and its face σ, the *link* X_σ of X at σ is the polytopal complex obtained from polytopes π_σ, for all faces π of X properly containing σ, by glueing them by natural identifications of faces induced by the corresponding identifications of faces in X. Accordingly, the face poset of X_σ is isomorphic to the face poset of those faces of X that properly contain σ (this can be used as alternative definition of the link X_σ).

Consider the simplicial map $e_\sigma : \Phi(X_\sigma) \to \Phi(X)$ defined at vertices by $e_\sigma([\pi_\sigma]) = [\pi]$. By definition of the link, this clearly extends to the well defined simplicial map on $\Phi(X_\sigma)$, which is injective.

1.2 Lemma. For any face σ of a polytopal complex X the image of the face complex $\Phi(X_\sigma)$ under the map e_σ is a full subcomplex of $\Phi(X)$.

Proof: Let $[\tau^i_\sigma] : i = 1, \dots, m$ be a set of vertices in $\Phi(X_\sigma)$, and suppose that their images $[\tau^i]$ span a simplex of $\Phi(X)$ which we denote δ. Then the set $\tau^i : i = 1, \dots, m$ is joinable in X, and it spans a face in X which we denote τ. Since this face contains σ, it follows that the set $\tau^i_\sigma : i = 1, \dots, m$ is joinable in X_σ (all faces from this set are contained in τ_σ). Consequently, vertices $[\tau^i_\sigma] : i = 1, \dots, m$ span a simplex of $\Phi(X_\sigma)$, denoted β. Since $\delta = e_\sigma(\beta)$, the lemma follows.

We now use the face complex $\Phi(X)$ to define the notion(s) of largeness for polytopal complexes X. This is both based on and generalizes the notion of largeness for simplicial complexes, introduced under the name of k-largeness (with integer parameter $k \geq 4$) in [SNPC]. We recall this in Appendix A to this paper. In [SNPC] largeness serves as a local curvature-like condition satisfied by links of a simplicial complex, leading to rich theory of simplicial nonpositive curvature. In this paper we use largeness in the context of spherical polytopal complexes, to get criteria of $CAT(1)$ condition for them (see Section 2).

1.3 Definition. Given integer $k \geq 4$, a polytopal complex X is *k-large* if its face complex $\Phi(X)$ is a k-large simplicial complex.

1.4 Remark. Observe that, due to Proposition B.1 from Appendix B, the above definition of k-largeness applied to simplicial complexes coincides with the original simplicial definition of k-largeness from [SNPC].

1.5 Lemma. If X is a k-large polytopal complex then any of its links X_σ is also k-large.

Proof: $\Phi(X_\sigma)$ is isomorphic with its image under the map e_σ in $\Phi(X)$, and the latter is full in $\Phi(X)$ due to Lemma 1.2. Since $\Phi(X)$ is k-large, and since any full subcomplex of a k-large simplicial complex is k-large (see Lemma A.2(1) from Appendix A), it follows that $\Phi(X_\sigma)$ is k-large, hence the lemma.

2. Large implies $CAT(1)$.

In this section we deal with metric aspects of polytopal complexes. We focus on piecewise spherical complexes and study the relationship between their combinatorial largeness and metric largeness (the $CAT(1)$ property). Results of this section both use and generalize analogous results from Section 14 in [SNPC] concerning piecewise spherical simplicial complexes. We refer the reader to [BH], Sections I.7 and II.5, for a detailed background concerning spherical polyhedral complexes.

A *spherical convex polytope* is the convex hull of a finite set C of points in the unit sphere (we assume that C is contained in some open hemisphere). Faces of a convex spherical polytope π are also convex spherical polytopes, and we view π as equipped with the family (poset) of its faces.

A *spherical polytopal complex* is a polytopal complex formed by glueing spherical convex polytopes via certain isometric identifications of their faces. We will assume that the set Shapes(X) of isometry classes of the faces of a spherical polytopal complex X is finite. Due to this assumption, each spherical polytopal complex is equipped with the piecewise spherical metric for which it is a complete geodesic metric space ([BH], Theorem 7.50).

Links of spherical polytopal complexes carry the structure of spherical polytopal complexes as follows. Given a face σ in a spherical convex polytope π contained in a unit sphere S, consider an interior point p of σ and the set of all unit tangent vectors to π at p that are orthogonal to σ and point towards π (i.e. an initial segment of a geodesic in S started at p in the direction of such vector is contained in π). This set of vectors forms a spherical convex polytope which is combinatorially the link π_σ and which we take as *spherical link* of π at σ. The above procedure applied to all polytopes in the link X_σ of a spherical polytopal complex clearly turns this link into spherical polytopal complex.

We now turn to the $CAT(1)$ condition. We are interested in it since it characterizes links in nonpositively (respectively, negatively) curved piecewise Euclidean (respectively, hyperbolic) complexes (see [BH], Theorem 5.2 on p. 206, and Section 4 of this paper). A convenient characterization of the $CAT(1)$ condition for piecewise spherical complexes is given by the following (see [BH], Theorem 5.4(7) on p. 206).

2.1 Criterion. If X is spherical polytopal complex then X is $CAT(1)$ iff neither X nor any of its spherical links contains a closed geodesic of length less than 2π.

Our main result in this section is the following.

2.2 Theorem. Let Π be a finite set of isometry classes of spherical convex polytopes. Then there is a natural number $k \geq 4$, depending only on Π, such that if X is a k-large spherical polytopal complex with Shapes(X) $\subset \Pi$ then X is $CAT(1)$.

Remarks.

(1) For the case of spherical *simplicial* complexes Theorem 2.2 has been proven in [SNPC], Theorem 14.1(1). Proof of Theorem 2.2 we present follows the same lines as the proof of Theorem 14.1 in [SNPC], but requires additional arguments.

(2) In combination with the existence results from Section 3, Theorem 2.2 gives many examples of $CAT(1)$ spaces.

(3) Estimates for k in our proof of Theorem 2.2 (below in this section) are not explicit.

To prove Theorem 2.2 we need two preliminary results, which both require some preparation. Given a closed geodesic γ in a spherical polytopal complex X, the *size* of γ is the number of maximal nontrivial subsegments in γ contained in a single face of X. Note that this number is always finite since any local geodesic of finite length in X is the concatenation of a finite number of segments, each contained in a single face ([BH, Corollary 7.29, p. 110]). The following theorem is a reformulation of [BH, Theorem 7.28, p. 109] or [B, Lemma 1].

2.3 Proposition. Given a finite set $\mathcal{S}$ of isometry classes of convex spherical polytopes, there is a natural number N (depending on $\mathcal{S}$) such that if a closed geodesic γ in a spherical polytopal complex X with $\text{Shapes}(X) \subset \mathcal{S}$ has length less than 2π then its size is less than N.

A polytopal complex X is ∞*-large* if its face complex $\Phi(X)$ is ∞-large (see Appendix A). The next result is an extension of Proposition 14.3 of [SNPC] from simplicial to polytopal setting. Proof presented below is based on different arguments and is much shorter.

2.4 Proposition. Let X be a spherical polytopal complex and suppose it is ∞-large. Then X contains no closed local geodesic.

Proof: Suppose to the contrary that X contains a closed local geodesic γ. Split γ into maximal segments contained in a single face of X, and denote these segments by $a_1, \ldots, a_m$. Denote initial point of any segment a_i by p_{i-1} and terminal point by p_i, with $p_0 = p_m$. For $i = 1, \ldots, m$, let τ_i be the face of X containing p_i in its interior, and let σ_i be the face of X containing the interior of a_i in its interior. Note that τ_{i-1}, τ_i are both the proper faces of σ_i and that they span σ_i. This means in particular that for each i vertices $[\tau_i], [\tau_{i+1}]$ in the face complex $\Phi(X)$ are distinct and connected with an edge.

Consider the *immersed cycle* in the 1-skeleton of $\Phi(X)$ defined by the sequence $[\tau_0], [\tau_1], \ldots, [\tau_m]$ of vertices, and denote it by $\hat{\gamma}$. We claim that $\hat{\gamma}$ is a 2-*geodesic* cycle. (Both concepts of an immersed cycle and a 2-geodesic cycle are defined in Appendix A, just before Lemma A.3).

To see this, note first that no two faces σ_i, σ_{i+1} in X are joinable. This is because faces of X are convex and the local geodesic γ enters interior of σ_{i+1} immediately after leaving interior of σ_i. Consequently, no two consecutive edges of $\hat{\gamma}$ are contained in a simplex of $\Phi(X)$.

But since the face complex $\Phi(X)$ is ∞-large it does not admit a 2-geodesic immersed cycle (by Lemma A.3 of Appendix A). This concludes the proof.

Proof of Theorem 2.2:

Denote by $\mathcal{S}$ the union of Π and the set of isometry classes of all spherical links of all polytopes from Π. Consider all spherical polytopal complexes K, with $\text{Shapes}(K) \subset \mathcal{S}$, containing a closed geodesic γ of length less than 2π. For any geodesic γ as above let L_γ be the union of those faces of K whose interiors are intersected by γ. Put Φ_γ to be the full span of the face complex $\Phi(L_\gamma)$ in $\Phi(K)$. Note that simplicial complexes Φ_γ, for all geodesics γ as above, have universally bounded number of vertices. Indeed, this number is bounded by $N \cdot M$, where N is the number asserted by Theorem 2.3 and M is the maximal number of faces in polytopes from $\mathcal{S}$. Consequently, up to simplicial isomorphism there are only finitely many simplicial complexes Φ_γ.

For a flag complex Φ_γ denote by $l(\Phi_\gamma)$ the largest integer $k \geq 4$ for which Φ_γ is k-large. Note that every flag complex is 4-large and that, due to Proposition 2.4, Φ_γ is never ∞-large. Thus the number $l(\Phi_\gamma)$ is well defined. (We do not care for those complexes Φ_γ which are not flag.)

Since there are only finitely many simplicial complexes Φ_γ (up to simplicial isomorphism), define

$$k = \max\{l(\Phi_\gamma) : \ \Phi_\gamma \text{ is flag}\} + 1.$$

We will show that this k is as required.

Let X be a k-large spherical polytopal complex with $\mathrm{Shapes}(X) \subset \Pi$, and suppose it is not $CAT(1)$. Then, due to Criterion 2.1, certain complex K equal either to X or to some link of X contains a closed geodesic γ of length less than 2π. These K and γ lead to one of the complexes Φ_γ as above.

However, since X is k-large, K is also k-large (by Lemma 1.5), and thus the face complex $\Phi(K)$ is also k-large. Since Φ_γ is by definition a full subcomplex in $\Phi(K)$, it is also k-large by Lemma A.2(1), and in particular it is flag. This contradicts the definition of k above, proving that X is $CAT(1)$, as required.

3. Induced polytopes of groups

In this section we return to combinatorial (and not metric) aspects of polytopal complexes. We deal with polytopes of groups and their developments, in particular showing existence of many finite and k-large developments of this form, for arbitrary convex polytopes and arbitrary k. Our approach builds upon a similar result for simplices (Theorem A.5 in Appendix A), via a natural operation of inducing polytopes of groups from simplices of groups (described below).

The notions of a simplex of groups and its development with respect to a morphism, as described in Appendix A, extend without any change to those of a *polytope of groups equipped with a morphism* and its *development*. Analogues of Properties A.4 also hold in this setting.

We now describe the notion of a polytope of groups *induced* from a simplex of groups. Let π be a convex polytope, and let Δ_π be the simplex whose codimension 1 faces are in a (fixed) bijective correspondence with the codimension 1 faces of π. For a face τ of π, let $s_1, \dots, s_m$ be the codimension 1 faces of π that contain τ. Put $\hat{\tau}$ to be the intersection of the codimension 1 faces of Δ_π corresponding to $s_1, \dots, s_m$ (we also declare $\hat{\pi} = \Delta_\pi$).

Let $\mathcal{G} = \{G_\sigma\}$ be a simplex of groups over Δ_π equipped with a morphism $m : \mathcal{G} \to G$. Define a polytope of groups $\mathcal{H}$ over π by putting $H_\tau := G_{\hat{\tau}}$ for any face τ of π. Put also $H := G$ and denote by μ the corresponding morphism from $\mathcal{H}$ to H. We call the polytope of groups $\mathcal{H}$ and the morphism μ obtained from $\mathcal{G}$ and m as above the *induced polytope of groups* and the *induced morphism*. To emphasize the fact that $\mathcal{H}$ and μ are determined by π, $\mathcal{G}$ and m, we use the following notation: $\mathcal{G}^\pi := \mathcal{H}$ and $m^\pi := \mu$.

Our main result in this section is the following

3.1 Proposition. Let π be a convex polytope and Δ_π the associated simplex. Let $\mathcal{G}$ be a simplex of groups over Δ_π equipped with a morphism $m : \mathcal{G} \to G$, and suppose that the development $D(\mathcal{G}, m)$ is a k-large simplicial complex, for some $k \geq 4$. Let $\mathcal{G}^\pi$ be the induced polytope of groups over π, and $m^\pi : \mathcal{G}^\pi \to G$ the induced morphism. Then the development $D(\mathcal{G}^\pi, m^\pi)$ is a k-large polytopal complex.

Proof: To avoid too many indices in the notation, we denote the induced polytope of groups $\mathcal{G}^\pi$ by $\mathcal{H}$, and the induced morphism m^π by μ. We need to show that the development $D(\mathcal{H}, \mu)$ is a polytopal (and not multi-polytopal) complex, and that it is k-large.

First assertion above follows from the following.

Claim 1. Intersection of any two faces in the development $D(\mathcal{H},\mu)$ is either empty or a single face.

Proof of Claim 1: Let $[\tau_1,g_1],[\tau_2,g_2]$ be faces in $D(\mathcal{H},\mu)$, and suppose they both contain a face $[\rho,g]$. Consider the simplices $[\hat{\tau}_1,g_1]$ and $[\hat{\tau}_2,g_2]$ in $D(\mathcal{G},m)$. Their intersection contains the simplex $[\hat{\rho},g]$, and thus is nonempty. Since $D(\mathcal{G},m)$ is a true simplicial complex, the intersection $[\hat{\tau}_1,g_1]\cap[\hat{\tau}_2,g_2]$ is a simplex, and we denote it $[\delta,h]$.

Now, since $[\hat{\rho},g]$ is a face of $[\delta,h]$, it follows that $g^{-1}h\in G_{\hat{\rho}}$, and thus $[\hat{\rho},g]=[\hat{\rho},h]$. Let $s_1,\dots,s_m$ be the codimension 1 faces of π such that $\delta=\hat{s}_1\cap\ldots\cap\hat{s}_m$. Since $[\hat{\rho},h]$ is a face of $[\delta,h]$, $\hat{\rho}$ is contained in all $\hat{s}_i$ above. Put $\sigma:=s_1\cap\ldots\cap s_m$ and note that, by what was said above, it is a face of π that contains ρ. But then the face $[\sigma,h]$ in $D(\mathcal{H},\mu)$ contains $[\rho,h]=[\rho,g]$.

Since the inclusion $[\rho,g]\subset[\sigma,h]$ holds, by the same argument, for every face $[\rho,g]$ in the intersection $[\tau_1,g_1]\cap[\tau_2,g_2]$, the claim will follow if we prove that $[\sigma,h]\subset[\tau_1,g_1]\cap[\tau_2,g_2]$. To do this, note that by definition of σ we have $\hat{\sigma}\subset\delta$. Consequently, for $i=1,2$ we have $\hat{\sigma}\subset\delta\subset\hat{\tau}_i$, which implies that $\sigma\subset\tau_i$. Furthermore, since $[\delta,h]\subset[\hat{\tau}_i,g_i]$, we have $h^{-1}g_i\in G_\delta<G_{\hat{\sigma}}=H_\sigma$, and hence $[\sigma,h]=[\sigma,g_i]\subset[\tau_i,g_i]$. Thus Claim 1 follows.

To prove k-largeness of $D(\mathcal{H},\mu)$, consider a map i from the vertex set of $\Phi(D(\mathcal{H},\mu))$ to the vertex set of $D(\mathcal{G},m)$, defined by

$$i([[\tau,g]])=[[\hat{\tau},g]].$$

This map extends uniquely to a well defined injective simplicial map $i:\Phi(D(\mathcal{H},\mu))\to\Phi(D(\mathcal{G},m))$ (we omit straightforward arguments).

Claim 2. The image of $\Phi(D(\mathcal{H},\mu))$ under i is a full subcomplex in $\Phi(D(\mathcal{G},m))$.

Proof of Claim 2: Let $[\tau_1,g_1],\dots,[\tau_m,g_m]$ be a set of faces in $D(\mathcal{H},\mu)$ such that the vertices

$$i([[\tau_1,g_1]]),\dots,i([[\tau_m,g_m]])$$

span a simplex of $\Phi(D(\mathcal{G},m))$. We need to show that the set $[\tau_1,g_1],\dots,[\tau_m,g_m]$ is joinable.

By definition of a face complex, the set of simplices $[\hat{\tau}_1,g_1],\dots,[\hat{\tau}_m,g_m]$ in $D(\mathcal{G},m)$ corresponding to vertices $i([[\tau_1,g_1]]),\dots,i([[\tau_m,g_m]])$ in $\Phi(D(\mathcal{G},m))$ is joinable. It follows that these simplices are all contained in a single simplex, say $[\rho,g]$, in $D(\mathcal{G},m)$. By the definition of $D(\mathcal{G},m)$, this means that for $i=1,\dots,m$ we have $g^{-1}g_i\in G_{\hat{\tau}_i}$, and hence $[\hat{\tau}_i,g_i]=[\hat{\tau}_i,g]$. This however implies that $[\tau_i,g_i]=[\tau_i,g]$ for $i=1,\dots,m$, and thus this set of faces is joinable in $D(\mathcal{H},\mu)$, since they are all contained in the face $[\pi,g]$. Hence Claim 2.

Returning to the proof of Proposition 3.1, note that since the simplicial complex $D(\mathcal{G},m)$ is k-large, by Proposition B.1 of Appendix B its face complex $\Phi(D(\mathcal{G},m))$ is also k-large. Moreover, by Claim 2 above, the face complex $\Phi(D(\mathcal{H},\mu))$ is isomorphic with a full subcomplex of $\Phi(D(\mathcal{G},m))$, and the latter is k-large by Lemma A.2(1) of Appendix A. Thus $\Phi(D(\mathcal{H},\mu))$ is also k-large, which finishes the proof.

Proposition 3.1 is all we need for the arguments in the next section. However, for clarity of the picture, we state below an existence result concerning polytopes of groups having large developments. It follows directly from Proposition 3.1 and from the related result for simplices of groups, namely Theorem A.5 of Appendix A.

3.2 Corollary. Let π be a convex polytope and suppose that for any codimension 1 face s of π we are given a finite group A_s. Then for any $k \geq 4$ there exist a finite groups H, and a polytope of groups $\mathcal{H} = \{H_\sigma : \ \sigma \text{ is a face of } \pi\}$ equipped with a morphism $\mu : \mathcal{H} \to H$, such that $H_\pi = \{1\}$, $H_s = A_s$ for any codimension 1 face s of π, and the development $D(\mathcal{H}, \mu)$ is a k-large polytopal complex.

4. Developments of billiards.

In this section we state and prove the main result of the paper, Theorem 4.5, about existence of finite nonpositively curved developments for a large class of billiards. This is a more detailed version of Main Theorem from Introduction. We start with recalling the terminology related to (or fixing our setting for) developments of billiards, and with describing the class of billiards we deal with.

The class of billiard tables we are interested in are the *constant curvature billiards with convex polytopal boundary*. By this we mean manifolds B of arbitrary dimension, equipped with constant curvature riemannian metric (spherical, Euclidean or hyperbolic), with boundary ∂B decomposed into totally geodesic (closed) strata, and such that locally near a boundary point it is isometric to a neighborhood of a boundary point in a convex polytope in the corresponding constant curvature model space (i.e. sphere, Euclidean space or hyperbolic space of the same dimension). Examples of such billiard tables are convex polytopes in model spaces, but there are many more. We make, at least temporarily, no restriction neither on the topology of B and its boundary strata, nor on their compactness or the number of strata.

A *development* of a billiard table B as above is a space D obtained by glueing together a family of copies of B via identity maps between some of their corresponding codimension 1 strata, so that each codimension one stratum s in each copy of B is glued to at least one codimension one stratum in some other copy of B.

More formally, given a billiard table B, let S be the set of its codimension one (boundary) strata. Let Λ be a set, and suppose that for each $s \in S$ we are given an equivalence relation $\sim_s$ on Λ. We assume that every equivalence class in $\sim_s$ has at least two elements. For each $b \in B$ denote by $\sim_b$ the equivalence relation on Λ which is the transitive closure of the union of all relations $\sim_s$, such that $b \in s$. (In particular, if b is an interior point of B then all equivalence classes of $\sim_b$ are singletons.) These data define the development D of B as

$$D = B \times \Lambda / \sim$$

where the equivalence relation $\sim$ is defined by $(b, \lambda) \sim (b', \lambda')$ iff $b = b'$ and $\lambda \sim_b \lambda'$.

Note that D is a stratified space, with strata equal to images (under the quotient map) of strata in the copies of B (where each copy of B is also viewed as a stratum). Note that, due to the way of glueing copies of B, strata of D are injective images of strata of copies of B, and thus they are homeomorphic to the latter.

By definition of a development D, every stratum of codimension one is contained in at least two top-dimensional strata (images of copies of B). To make statements of some properties of the development (like the one in the previous sentence) easier, we introduce the following terminology. A *chamber* of a development D is any of its top-dimensional strata. The *thickness* of a development D at a stratum s of codimension 1 is the number of chambers containing s. Note that, by definition, thickness of D at any stratum is ≥ 2. Special case of developments are pseudo-manifolds. A development D is a *pseudo-manifold* if its thickness at every codimension 1 stratum equals 2.

Of special interest to us, mainly because of our methods of construction, are *developments with respect to a group.* These are those developments D which are equipped with an action of a group G such that some (and hence any) chamber of D is a strict fundamental domain for this action.

Having chosen a chamber of a development D with respect to a group G, we identify it with the table B. The pair (D, G) defines then a *table of groups* over B equipped with a morphism to G, in the same way as in Appendix A a pair (Z, G) defines a simplex of groups with a morphism. Moreover, a general notion of a *table of groups over B equipped with a morphism* is introduced in the same way as for a simplex. It allows to define, again in the same way, *development of a table of groups with respect to a morphism*, which satisfies properties analogous to Properties A.4. In particular, each development with respect to a group for B is a development of a table of groups over B with respect to a morphism. We will use the same notation $\mathcal{G}$, m and $D(\mathcal{G}, m)$ for tables of groups, morphisms, and for their developments.

Next result, the easy proof of which we omit, expresses thickness of a development of a table of groups $\mathcal{G} = \{G_\sigma : \ \sigma \text{ is a stratum of } B\}$ over B with respect to a morphism $m : \mathcal{G} \to G$, as well as the number of its chambers, in terms of groups G_σ and G.

4.1 Lemma. Let $\mathcal{G} = \{G_\sigma\}$ be a table of groups over a constant curvature billiard table B with convex polytopal boundary, equipped with a morphism $m : \mathcal{G} \to G$. Then
(1) for any stratum s of codimension 1 in B, and for any $g \in G$, thickness of the development $D(\mathcal{G}, m)$ at the stratum $[s, g]$ is equal to the index $(G_s : G_B)$;
(2) the number of chambers in the development $D(\mathcal{G}, m)$ is equal to the index $(G : G_B)$.

Since in our applications we will be mainly interested in tables of groups over B for which the group G_B is trivial, we formulate some consequences of Lemma 4.1 for this case.

4.2 Corollary. Let $\mathcal{G} = \{G_\sigma\}$ be a table of groups over a constant curvature billiard table B with convex polytopal boundary, equipped with a morphism $m : \mathcal{G} \to G$. Suppose also that $G_B = \{1\}$. Then
(1) for any stratum s of codimension 1 in B, and for any $g \in G$, thickness of the development $D(\mathcal{G}, m)$ at the stratum $[s, g]$ is equal to the order $|G_s|$ of the group G_s;
(2) $D(\mathcal{G}, m)$ is a development for B iff for each stratum s of codimension 1 the group G_s has order ≥ 2;
(3) $D(\mathcal{G}, m)$ is a pseudo-manifold iff for each stratum s of codimension 1 the group G_s has order 2;
(4) the number of chambers in the development $D(\mathcal{G}, m)$ is equal to the order $|G|$ of the group G.

We now turn to discussing piecewise constant curvature metric on a development of a billiard B. In order to ensure existence of such metric, we assume that B is compact. Due to [BH, Theorem 7.19, p. 105], under this assumption, the above metric is well defined and turns any development of B to a geodesic metric space. Our main concern in this paper is to construct nonpositively (respectively, negatively) curved finite developments (or even developments with respect to a group) for compact Euclidean (respectively, hyperbolic) billiards. A convenient way of recalling definitions of curvature bounds uses spherical links.

For constant curvature billiard tables B as above, and for their developments D, spherical links B_σ and D_σ at all their strata σ are defined in the same way as for constant curvature polytopal complexes (see Section 2 for the case of spherical polytopal complexes, or [BH, I.7] for the general case). Beware that we view these links as spherical polytopal complexes, keeping the information about both the piecewise spherical metric and the decomposition into cells, as their structure.

We recall the following local criterion for curvature bounds.

4.3 Criterion (see [BH, Theorem 5.5, p. 207]).

(1) If D is a development of a compact Euclidean billiard table B with convex polytopal boundary, then D is nonpositively curved iff links of D at all its strata are $CAT(1)$.

(2) If D is a development of a compact hyperbolic (i.e. with constant curvature -1) billiard table B with convex polytopal boundary, then D has curvature ≤ -1 iff links of D at all its strata are $CAT(1)$.

One more condition for billiard tables B that we need in our main theorem is *local injectivity* of the stratification. This is somewhat unusual condition for people working with billiards, but it is very natural from the group theoretic perspective. To define this condition, for any stratum σ in B consider the natural map i_σ from the set of faces of the link polytope B_σ to the set of strata of B. Stratification of a billiard table is *locally injective* if the maps i_σ, for all strata σ, are injective. This condition is clearly satisfied by polytopal billiard tables, but likewise for many others. It is also not difficult to find examples of billiard tables B with convex polytopal boundary and with *not* locally injective stratification, for which no development is nonpositively curved (see Example 5.1).

We say that, given an integer $q \geq 2$, a development D of a billiard table B has *uniform thickness* q if thickness of D at any codimension 1 stratum equals q. A slightly more general result than Main Theorem of the introduction is the following.

4.4 Theorem. Let B be a compact Euclidean (respectively, hyperbolic) billiard table with convex polytopal boundary and with locally injective stratification. Then, for any integer $q \geq 2$ there exists a finite nonpositively curved (respectively, with curvature ≤ -1) development (with respect to a group) D for B, with uniform thickness q.

Note that in case $q = 2$ above theorem provides developments that are pseudomanifolds. In fact, in view of Corollary 4.2, Theorem 4.4 is a special case of the following result, which gives also developments with variable thickness.

4.5 Theorem. Let B be a Euclidean (respectively, hyperbolic) compact billiard table with convex polytopal boundary and with locally injective stratification. Suppose also that for any stratum s of codimension 1 in B we are given a finite group A_s. Then there exists a finite group G and a table of groups $\mathcal{G}^B = \{G^B_\sigma\}$ over B equipped with a morphism $m^B : \mathcal{G}^B \to G$, such that $G^B_B = \{1\}$, $G^B_s = A_s$ for any codimension 1 stratum s, and the development $D(\mathcal{G}^B, m^B)$ is nonpositively curved (respectively, has curvature ≤ -1).

Proof of Theorem 4.5: Consider links B_σ of the billiard table B, and recall that they are spherical convex polytopes. It follows from Theorem 2.2 that for each stratum σ in ∂B there is an integer $k_\sigma \geq 4$ such that if X is a k_σ-large development of B_σ then X is $CAT(1)$. Put

$$k := \max\{k_\sigma : \ \sigma \text{ is a stratum of } \partial B\}.$$

Let Δ_B be a simplex whose codimension 1 faces are in a (fixed) bijective correspondence with codimension 1 strata of B. For a stratum σ of B, let $s_1, \dots, s_m$ be the codimension 1 strata of B that contain σ. Put $\hat{\sigma}$ to be the intersection of the codimension 1 faces of Δ_B corresponding to $s_1, \dots, s_m$ (we also declare $\hat{B} = \Delta_B$).

Let $\mathcal{G} = \{G_\sigma\}$ be a simplex of groups over Δ_π equipped with a morphism $m : \mathcal{G} \to G$, as prescribed by Theorem A.5, with groups $A_{\hat{s}} := A_s$ for codimension 1 strata s in B, and for k as above. Define a table of groups $\mathcal{G}^B = \{G^B_\sigma\}$ over B by putting $G^B_\sigma := G_{\hat{\sigma}}$ for any stratum σ of B. Denote also by $m^B : \mathcal{G}^B \to G$ the corresponding morphism.

We claim that these $\mathcal{G}^B$ and m^B are as required. The fact that $G^B_B = \{1\}$ and $G^B_s = A_s$ follows directly from the definition of $\mathcal{G}^B$. In view of Criterion 4.3, it remains to show that spherical links in the development $D(\mathcal{G}^B, m^B)$ are CAT(1).

To prove the latter, we need two lemmas. First lemma below expresses links in the developments of tables of groups as developments of certain polytopes of groups (so called *local developments* in [BH, II.12]). Note that this lemma applies also to simplices of groups, as special cases of tables of groups.

Lemma 1 (compare [BH, Proposition 4.11, p.558] or [SNPC, Proposition 19.3]). Let $\mathcal{G} = \{G_\sigma\}$ be a table of groups over B equipped with a morphism $m : \mathcal{G} \to G$. Given a stratum σ in ∂B, consider the link B_σ, which is a convex spherical polytope. Define a polytope of groups $\mathcal{G}_\sigma = \{G_{\sigma,\tau_\sigma}\}$ over B_σ by putting $G_{\sigma,\tau_\sigma} := G_\tau$ for all strata τ containing σ. Consider also the corresponding morphism $m_\sigma : \mathcal{G}_\sigma \to G_\sigma$. Then, for any $g \in G$, the link of $D(\mathcal{G}, m)$ at the stratum $[\sigma, g]$ is isomorphic to the development $D(\mathcal{G}_\sigma, m_\sigma)$.

Next result is a straightforward consequence of definitions of the objects involved, and we omit its proof.

Lemma 2. Under notation as in the proof of Theorem 4.5 above, for any stratum σ in ∂B the polytope of groups $(\mathcal{G}^B)_\sigma$ is induced from the simplex of groups $\mathcal{G}_{\hat{\sigma}}$ (in the sense defined in Section 3, just before Proposition 3.1). More precisely, the map $s_\sigma \to \hat{s}_{\hat{\sigma}}$ is a bijective correspondence between the codimension 1 faces in the links B_σ and $(\Delta_B)_{\hat{\sigma}}$, so that we may view $(\Delta_B)_{\hat{\sigma}}$ as the simplex Δ_{B_σ} associated to the polytope B_σ. Under this interpretation, we have equalities $\widehat{\tau_\sigma} = \hat{\tau}_{\hat{\sigma}}$, and the corresponding equalities for groups.

Returning to the proof of Theorem 4.5, take any stratum $[\sigma, g]$ in the development $D(\mathcal{G}^B, m^B)$. By Lemma 1 above, the link at this stratum is isomorphic and isometric to the development $D(\mathcal{G}^B_\sigma, m^B_\sigma)$. We need to show that this development is $CAT(1)$. Note that, in view of the choice of k as above, it is sufficient to show that the development $D(\mathcal{G}^B_\sigma, m^B_\sigma)$ is k-large.

Note that, again by Lemma 1 above, the development $D(\mathcal{G}_{\hat{\sigma}}, m_{\hat{\sigma}})$ is a simplicial complex isomorphic to the link of the development $D(\mathcal{G}, m)$ at the simplex $[\hat{\sigma}, 1]$. It follows from Lemma A.2(2) that the development $D(\mathcal{G}_{\hat{\sigma}}, m_{\hat{\sigma}})$ is k-large. In view of Lemma 2 above and Proposition 3.1, this implies that the development $D(\mathcal{G}_\sigma^B, m_\sigma^B)$ is k-large, hence the theorem.

5. Examples, remarks, speculations.

In this section we discuss sharpness of the assumptions and possible generalizations of the main results of this paper, i.e. Theorems 4.4 and 4.5. However, it should be clear that the most interesting class of examples is that of convex polytopal billiards, and it is covered by our results.

First, we describe a simple example showing that our assumption about local injectivity of stratification is in general necessary.

5.1 Example. Let B be the unit Euclidean 3-dimensional cube with the top face identified with the bottom one via vertical translation followed by the rotation through angle $\pi/2$ around the center of the face. Then B is a Euclidean billiard table with convex polytopal boundary. However, its boundary consists of just two strata, one in each of the dimensions 1 and 2. In particular, if σ is the 1-dimensional stratum (of the boundary) of B then the stratification of B is not locally injective at σ, i.e. the associated map i_σ is clearly not injective.

On the other hand, since B has only one stratum of codimension 1, which we denote by s, the only possible developments of B are made of a family of copies of B glued together via identities of the corresponding strata s. Clearly, all these developments are not nonpositively curved.

A striking feature of the above example is that B is not simply connected. The reader should note that the connected double cover B' of B has locally injective stratification, and that it admits finite nonpositively curved developments. Such developments D of B' can be viewed as some kind of "non-simple" developments of B (in the sense that lifts of the whole of B to D do not exist). However, in this paper we only deal with "simple" developments, i.e. developments D as defined in Section 4. In case of simply connected billiard tables "non-simple" developments do not exist.

Next example shows that in some cases neither convexity of the boundary nor local injectivity of the stratification is necessary to have a nonpositively curved development.

5.2 Example. Let B be a topological annulus equipped with Euclidean metric, with piecewise geodesic boundary, and such that
(1) first component of the boundary consists of a single vertex v_0 and a single 1-dimensional stratum s_0 meeting v_0 twice, so that together with v_0 it forms a loop;
(2) B has (non convex) angle $3\pi/2$ at v_0;
(3) second component of the boundary consists of two vertices v_1, v_2 and two 1-dimensional strata s_1, s_2 forming together a circle;
(4) B has angles $3\pi/8$ at v_1 and v_2.
We leave it for the reader to verify that such annuli exist.

Now, the boundary of B is clearly not convex and the stratification is not locally injective (both things fail at the vertex v_0).

Consider the table of groups $\mathcal{G}$ over B defined by putting $G_s = Z_2$ at all 1-dimensional strata s, $G_{v_i} = Z_2 \oplus Z_2$ for $i = 1, 2$, with the obvious structure homomorphisms sending the adjacent groups G_s onto distinct factors of G_{v_i}, and finally $G_{v_0} = Z_2$ with both structure homomorphisms $G_{s_0} \to G_{v_0}$ identical. Consider also the morphism $m : \mathcal{G} \to Z_2 \oplus Z_2 \oplus Z_2$ sending the three groups $G_{s_i} = Z_2$ onto distinct factors. Then the development $D(\mathcal{G}, m)$ is easily seen to be nonpositively curved.

We now turn to discussing how one can relax the assumption that links of a billiard table B are convex spherical polytopes. The next lemma and corollary show that links cannot be arbitrary. Recall that a subset A in a geodesic metric space X is *r-convex* (for some $r > 0$) if any geodesic in X of length less than r that connects some points from A is contained in A. X is *r-uniquely geodesic* if for any two points at distance smaller than r in X a geodesic connecting these points is unique.

5.3 Lemma. If a spherical billiard table B has a $CAT(1)$ development D then B is π-convex in D. Moreover, B is itself $CAT(1)$.

Proof: Note first that for any points $x, y \in B$ we have $d_B(x, y) \leq d_D(x, y)$, where d_B, d_D are the geodesic metrics in B and D respectively. This follows from existence of the *folding map* $f : D \to B$ that maps each copy of B in D identically on B. A geodesic in D connecting x, y can be mapped by f into B, hence the inequality.

Now, we prove π-convexity by induction with respect to $\dim B$. The case $\dim B = 1$ is clear. Suppose $\dim B > 1$. For any stratum σ in ∂B the link B_σ is a spherical billiard table, and the corresponding link D_σ is its $CAT(1)$ development. Thus, by inductive assumption, B_σ is π-convex in D_σ. Consider points $x, y \in B$ such that $d_D(x, y) < \pi$. Then $d_B(x, y) < \pi$. Let γ be a geodesic in B (of length $d_B(x, y)$) connecting x and y. By local π-convexity of B in D, γ is a local geodesic in D (see [CD, Lemma 1.6.5]). Since local geodesics of length less than π in $CAT(1)$ spaces are geodesics ([BH, Proposition 1.4(2), p. 160]), γ is a geodesic in D. Moreover, this is the only geodesic in D between x and y, because $CAT(1)$ spaces are π-uniquely geodesic ([BH, Theorem 5.4, p. 206]). Since γ is contained in B, this proves B is π-convex in D.

We now prove that B is $CAT(1)$. By Theorem 5.4 on p. 206 in [BH], a piecewise spherical complex is $CAT(1)$ iff it is π-uniquely geodesic. By the above argument, any geodesic in B of length less than π is a geodesic in D. Since D is $CAT(1)$, it is π-uniquely geodesic, and this implies B is π-uniquely geodesic too. Hence the lemma.

5.4 Corollary. Let B be a Euclidean (or hyperbolic) billiard table. A necessary condition for B to have a nonpositively (or negatively) curved development is that B is nonpositively (respectively, negatively) curved.

Condition $CAT(1)$ is not sufficient for a spherical table B to have a $CAT(1)$ development, as the following example shows.

5.5 Example. Let B be the a 2-dimensional unit half-sphere H with removed some small spherical acute angled triangle T having one side contained in ∂H. By requiring that T is small we mean for example that its perimeter is less than π. B is then easily seen to be $CAT(1)$.

Any development D of B has to contain the union of two copies of B glued along the side contained in ∂H. This union however contains a closed geodesic γ of length less than 2π, namely the one composed of four edges of the deleted triangles T in both copies of B. It is not difficult to realize that γ is also a geodesic in the whole development D, and thus the latter cannot be $CAT(1)$.

Let us focus attention on the case of simply connected Euclidean (or hyperbolic) billiard tables B. It follows from Corollary 5.4 that, in order to have nonpositively (respectively, negatively) curved development, B has to be $CAT(0)$ (respectively, $CAT(-1)$). Next example shows that even in this restricted class the stratification of B may not be locally injective, and that this may prevent B from having an appropriate development.

5.6 Example. Let B be the union of the unit 3-dimensional cube C and a small regular tetrahedron P, glued together through an isometric embedding of a face T of P into a face S of C, such that the image of T intersects ∂S only at a single vertex v of S. B is easily seen to be $CAT(0)$ and to have not locally injective stratification at v. Moreover, the union of two copies of B along the face contained in S is not nonpositively curved, since its link at v contains closed geodesic γ of length $2\pi/3$. It is not hard to see that γ is also a geodesic in the link at v of any development of B. Hence no development of B is nonpositively curved.

It turns out that property $CAT(0)$ together with local injectivity of stratification is also not sufficient for a Euclidean table B to have nonpositively curved development.

5.7 Example. Let B be the unit 3-dimensional cube C with removed some small tetrahedron P such that two vertices of P are contained in the interior of an edge E of C and two other vertices of P, say v and w, are contained respectively in the interiors of the two square faces of C adjacent to E. B is easily seen to be $CAT(0)$ and to have locally injective stratification. However, the links of B at vertices v and w have the form as spherical 2-dimensional table of Example 5.5. It follows that B has no nonpositively curved development.

Above examples show that, in order to have appropriate developments of B, it may be necessary to put some assumption on each union of two copies of B glued identically along their (single) stratum of codimension 1. We will call such unions *doubles* of B. It is also worth noting that our method of constructing developments D in this paper (at least pseudo-manifold ones), has the following feature: for any double B' of B development D is also a development of B'. (This is so because we make use of Theorem A.5, and its proof in [SNPC] uses the notion of *extra-tilability*, which is closely related to the above property; we do not want to go to further details concerning that issue.) In view of Lemma 5.3, a necessary condition for our method to give a $CAT(1)$ development is that any double of a spherical table B is $CAT(1)$. It is quite possible (and we leave it as a conjecture) that:

(1) condition that any double of B is $CAT(1)$ is necessary for a spherical table B to have any $CAT(1)$ development, regardless of the method of constructing it;
(2) the same condition is sufficient for a spherical $CAT(1)$ table B to have a finite $CAT(1)$ development.

There is also a weaker version of the above conjecture, dealing with the *free development* of B. The latter is the infinite pseudomanifold development, whose dual graph is a tree. Here by the dual graph we mean the graph whose vertex set is the set of copies of B, and whose edges correspond to codimension one strata in the development.

Now, if the free development of a spherical table B is CAT(1) then, since it is also a development of any double of B, it follows from Corollary 5.4 that any double of B is CAT(1). Thus, requiring that the free development is CAT(1) is stronger than requiring that any double is CAT(1). So the weaker conjecture reads: If the free development of B is CAT(1) then B has also a finite CAT(1) development.

Appendix A. Large simplicial complexes.

In this appendix we recall definitions and some useful properties of k-large simplicial complexes. The reader is referred to [SNPC] for more details. We also present (in Lemma A.3) certain new criterion for k-largeness.

A *cycle* in a simplicial complex is a subcomplex homeomorphic to the circle S^1. *Length* of a cycle is the number of its edges. The following definition of k-largeness appears in [SNPC], as a characterization, in Fact 1.2(3)(4). The case $k = \infty$ occurs in Section 15 of [SNPC], just before Proposition 15.2.

A.1 Definition. Let $k \geq 4$ be an integer. A simplicial complex Z is *k-large* if it is flag [1] and if every full [2] cycle in Z has length at least k. It is *∞-large* if it is flag and contains no full cycle.

The following properties, follow directly from the definition.

A.2 Lemma. Let Z be a k-large simplicial complex, for some $k \geq 4$ or $k = \infty$. Then
(1) every full subcomplex of Z is k-large;
(2) links Z_σ of Z, at all its simplices σ, are k-large.

We now turn to a new characterization of k-largeness. An *immersed cycle* in a simplicial complex Z is a non-degenerate [3] simplicial map from a 1-dimensional complex S homeomorphic to the circle S^1 to Z, such that any two consecutive edges in S are mapped to different edges. An immersed cycle in a flag simplicial complex Z is 2-*geodesic* if no two consecutive edges in this cycle belong to a simplex of Z. If Z is flag, this is equivalent to the property that any two consecutive edges in the cycle form a geodesic in Z (for the polygonal distance in the 1-skeleton). This justifies the name of the property. Note that any full cycle in Z is clearly 2-geodesic, but not vice versa.

A.3 Lemma. For any $k \geq 4$ or $k = \infty$, a simplicial complex Z is k-large iff it is flag and there is no 2-geodesic immersed cycle in Z of length less than k.

Proof: Since any full cycle is 2-geodesic, one implication follows. To prove the remaining one, suppose that Z contains no full cycle of length less than k. We need to show that Z

[1] A simplicial complex is *flag* if any set of its vertices pairwise connected by edges spans a simplex.

[2] A subcomplex K in a simplicial complex X is *full* if any set of vertices from K spanning a simplex in X spans also a simplex in K.

[3] A simplicial map is *non-degenerate* if it is injective on every simplex.

contains no 2-geodesic immersed cycle of length less than k. Let γ be an immersed cycle of length less than k in Z. It clearly provides an embedded cycle γ_0 in Z, of length less than k, whose all edges are edges of γ (if γ is embedded we take $\gamma_0 = \gamma$ and if γ self-intersects then γ_0 is a part of γ). Since γ_0 has length less than k, it is not full. It means that some two non-consecutive vertices in γ_0 are connected in Z with an edge, say e. We can choose a shorter cycle γ_1 in Z containing edge e, with the remaining part consisting of several consecutive edges of γ. Repeating this finitely many times we end up with certain cycle γ_m of length 3, with two of its consecutive edges coinciding with certain two consecutive edges of γ. But this shows γ is not 2-geodesic, which finishes the proof.

We turn to a result (Theorem A.5 below) concerning existence of certain k-large simplicial complexes. For this we need some terminology related to simplices of groups. The standard reference is [BH, II.12]. Notation that we use agrees with that introduced in Section 19 of [SNPC], and is slightly different from that in [BH].

Let G be a group of automorphisms of a simplicial complex X. We say that a simplex Δ of X is a *strict fundamental domain* for the action of G if the restricted quotient map $\Delta \to G\backslash X$ is a bijection.

A pair (X, G) as above, with $\Delta \subset X$ a strict fundamental domain, defines a *simplex of groups* $\mathcal{G}$ and a *morphism* $m : \mathcal{G} \to G$, as follows. For every face σ of Δ put G_σ to be the subgroup of G fixing σ pointwise. The *simplex of groups associated to* (X, G) is the family $\mathcal{G} = \{G_\sigma : \ \sigma \text{ is a face of } \Delta\}$. The *morphism associated to* (X, G) is the family $m = \{m_\sigma : \ \sigma \text{ is a face of } \Delta\}$ of inclusion homomorphisms $m_\sigma : G_\sigma \to G$.

Note that $\mathcal{G}$ above satisfies the following property:

$$(*) \qquad \text{if } \sigma \subset \tau \text{ then } G_\tau < G_\sigma.$$

The above motivates the following general definition. A *simplex of groups* over a simplex Δ, equipped with a *morphism* to a group G, is a family $\mathcal{G} = \{G_\sigma : \ \sigma \text{ is a face of } \Delta\}$ of subgroups of G satisfying property (*) above, and the family $m = \{m_\sigma : \ \sigma \text{ is a face of } \Delta\}$ of inclusion homomorphisms $m_\sigma : G_\sigma \to G$.

For a simplex of groups $\mathcal{G} = \{G_\sigma\}$ over Δ equipped with a morphism $m : \mathcal{G} \to G$ define the *development* of $\mathcal{G}$ with respect to m as the quotient

$$D(\mathcal{G}, m) := \Delta \times G / \sim,$$

where $\sim$ is the equivalence relation defined by

$$(x, g) = (y, h) \quad \text{iff} \quad x = y \in \sigma \text{ and } g^{-1}h \in G_\sigma \text{ for some face } \sigma \text{ of } \Delta.$$

Denote by $[x, g]$ the equivalence class of (x, g), and put $[\sigma, g] := \{[x, g] : x \in \sigma\}$. $D(\mathcal{G}, m)$ is then a multi-simplicial complex with the faces $[\sigma, g]$ (being injective images of $\sigma \times \{g\}$ through the quotient map of $\sim$). It is multi-simplicial (and not just simplicial) since the intersection of its faces may be a union of faces (and not just a single face). However, developments of simplices of groups considered in this paper will always be genuine simplicial complexes.

The above description of the development $D(\mathcal{G}, m)$ can be found in [BH, II.12], where it is called Basic Construction. We recall few more properties of developments.

A.4 Properties. Let $\mathcal{G} = \{G_\sigma\}$ be a simplex of groups over Δ equipped with a morphism $m : \mathcal{G} \to G$.

(1) The formula $h \cdot [x, g] = [x, hg]$ defines an action of the group G on $D(\mathcal{G}, m)$ by automorphisms.

(2) (see [BH], Proposition 12.20(1), p.385) If $\mathcal{G}$ and m are induced by an action of G on X, with $\Delta \subset X$ a strict fundamental domain, then $D(\mathcal{G}, m)$ is equivariantly isomorphic to X.

(3) $D(\mathcal{G}, m)$ is finite iff the index $(G : G_\Delta)$ is finite.

Crucial for our arguments in this paper is the following substantial result.

A.5 Theorem ([SNPC, Theorem H]). Let Δ be a simplex (of arbitrary dimension) and suppose that for any codimension 1 face s of Δ we are given a finite group A_s. Then for any $k \geq 4$ there exist a finite group G, and a simplex of groups $\mathcal{G} = \{G_\sigma\}$ equipped with a morphism $m : \mathcal{G} \to G$, such that $G_\Delta = \{1\}$, $G_s = A_s$ for any codimension 1 face s of Δ, and the development $D(\mathcal{G}, m)$ is a finite k-large simplicial complex.

Note that, in view of Property A.4(3), finiteness of $D(\mathcal{G}, m)$ follows from finiteness of G.

Appendix B: Face complex of a simplicial complex.

In this Appendix we present proof of the following result noticed by Frederic Haglund.

B.1 Proposition. The face complex of a simplicial complex X is k-large if and only if X is k-large.

Proof: Denote by $\Phi_0(X)$ the subcomplex in the face complex $\Phi(X)$ spanned by all vertices $[v]$ corresponding to vertices v in X. By definition, $\Phi_0(X)$ is a full subcomplex in $\Phi(X)$. Moreover, since X is simplicial, there is a canonical simplicial map $X \to \Phi_0(X)$ which is easily seen to be a simplicial isomorphism. Thus, if $\Phi(X)$ is k-large, it follows from Lemma A.2(1) that $\Phi_0(X)$ is k-large, and then X is also k-large. This proves one implication of the proposition.

To prove the converse implication, we first show that if X is flag then $\Phi(X)$ is also flag. Let $[\tau_1], \ldots, [\tau_m]$ be a set of vertices that are pairwise connected with edges in $\Phi(X)$. Consider the set T of all vertices in all simplices $\tau_1, \ldots, \tau_m$. Note that, by our assumptions, any two vertices in T are connected with an edge in X. Since X is flag, T spans a simplex of X, and we denote it σ. The simplices $\tau_1, \ldots, \tau_m$ are clearly the faces of σ and thus they form a joinable set of faces. Consequently, there is a simplex in $\Phi(X)$ containing the vertices $[\tau_1], \ldots, [\tau_m]$, hence flagness of $\Phi(X)$.

Since 4-largeness is equivalent to flagness, it remains to prove (the converse implication of) the proposition for $k \geq 5$. We will do this using induction on k, with the already proved case $k = 4$ as the first inductive step.

Fix some $k \geq 5$. We need to show that any cycle

$$\gamma = ([\tau_0], [\tau_1], \ldots, [\tau_L])$$

(where $[\tau_0] = [\tau_m]$) with length satisfying $4 \leq L < k$ is not full in $\Phi(X)$. We will do this by induction with respect to the length L and the parameter $D = \dim \tau_0 + \ldots + \dim \tau_{L-1}$.

We consider first cycles γ of length $L = 4$. If $D = 0$, i.e. simplices $\tau_0, \ldots, \tau_{L-1}$ are vertices of X, the statement follows easily from k-largeness of X. Suppose the statement is true for all $0 \leq D < m$, for some natural number m. We will prove it for $D = m$. We may assume that the simplex τ_0 representing the first vertex in our cycle γ is of dimension > 0, and we fix a vertex v and its complementary face τ in τ_0. Note that $\gamma' = ([\tau], [\tau_1], [\tau_2], [\tau_3], [\tau])$ is a cycle in $\Phi(X)$ with $D = m-1$ and, by inductive assumption, it is not full in $\Phi(X)$. It means that either τ_1, τ_3 or τ, τ_2 are joinable, so that the corresponding two vertices span an edge in $\Phi(X)$. If the simplices τ_1, τ_3 are joinable, the cycle γ is clearly not full. Otherwise the simplices τ, τ_2 are joinable, and by a similar argument v, τ_2 are also joinable. But, due to flagness of X, this means that the simplices τ_0, τ_2 are joinable, and γ is not full in this case too.

Now, fix L satisfying $4 < L < k$ and suppose that the statement (saying that cycles are not full) holds for all cycles of length $< L$. Let $\gamma = ([\tau_0], [\tau_1], \ldots, [\tau_L])$ be a cycle of length L. If $D = 0$, this cycle is not full by k-largeness of X. Suppose the statement is true for all $0 \leq D < m$, for some natural number m. We will prove it for $D = m$. As before, may assume that the simplex τ_0 representing the first vertex in γ is of dimension > 0, and we fix a vertex v and its complementary face τ in τ_0. As before,

$$\gamma' = ([\tau], [\tau_1], \ldots, [\tau_{L-1}], [\tau])$$

is a cycle in $\Phi(X)$ with $D = m - 1$ and, by inductive assumption, it is not full in $\Phi(X)$. If some two non-consecutive simplices in the sequence $\tau_1, \ldots, \tau_{L-1}$ are joinable then the cycle γ is easily seen not to be full. For the remaining case we need the following.

Claim. If no two non-consecutive simplices in the sequence $\tau_1, \ldots, \tau_{L-1}$ are joinable then the simplex τ is joinable to each of the simplices $\tau_2, \ldots, \tau_{L-2}$.

Proof of Claim: τ is clearly joinable with at least one of the simplices $\tau_2, \ldots, \tau_{L-2}$, say τ_j, by the fact that the cycle γ' is not full. But then we get two shorter cycles

$$\gamma_1 = ([\tau], [\tau_1], \ldots, [\tau_j], [\tau]) \quad \text{and} \quad \gamma_2 = ([\tau], [\tau_j], \ldots, [\tau_{L-1}], [\tau])$$

in $\Phi(X)$, and those of them which have length ≥ 4 are not full by inductive assumption. This implies joinability of τ with at least one more of the simplices $\tau_2, \ldots, \tau_{L-2}$, and iteration of this argument proves the claim.

Under assumption as in Claim, the same argument shows v is joinable to each of the simplices $\tau_2, \ldots, \tau_{L-2}$. By flagness of X, this shows that the simplex τ_0 is joinable to each of the simplices $\tau_2, \ldots, \tau_{L-2}$, and thus γ is not full.

This finishes the proof of proposition.

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