Trees of manifolds and boundaries of systolic groups

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Abstract

In this paper we prove that the Pontriagin sphere and the Pontriagin nonorientable surface occurs as the Gromov boundary of a 7-systolic group acting geometrically on 7-systolic normal pseudomanifold of dimension 3.

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1 Introduction

k-systolic simplicial complexes ($k \geq 6$ is a natural number) were introduced by T. Januszkiewicz and J. Świątkowski in [JS] and independently by F. Haglund in [H]. These are simplicial analogues of metric spaces of nonpositive curvature. The idea of systolicity leads to an answer to the question posed by M. Gromov about simple easy checkable combinatorial condition for a simplicial complex implying hyperbolicity of this complex for the standard piecewise euclidean metric on it. In [JS] Januszkiewicz and Świątkowski have shown that a 7-systolic simplicial complex is hyperbolic.

Gromov boundaries of 7-systolic complexes were investigated by D. Osajda in [O]. He showed that the ideal boundary $\partial_G X$ of a 7-systolic simplicial complex X is a strongly hereditarily aspherical compactum. He also showed that the Gromov boundary of a normal 7-systolic pseudomanifold of finite dimension at least 3 is connected and has no local cutpoints. In this paper we study in detail the case of such pseudomanifolds in dimension 3.

Trees of manifolds are inverse limits of certain inverse systems of manifolds. The most common examples of such spaces are the Pontriagin sphere and the nonorientable Pontriagin surface. Trees of manifolds were defined and investigated by W. Jakobsche (see [J]) and by P.R. Stallings (see [S]). These spaces occur as CAT(0) boundaries of right-angled Coxeter groups (see [F]). In the case when these groups are hyperbolic their CAT(0) boundaries coincide with their Gromov boundaries.

The main result of this paper is:

Main Theorem. *Let X be a 7-systolic normal pseudomanifold of dimension 3. Let a group G act geometrically on X. Then:*

a) (Theorem 7.2 in the text) if X is orientable, then $\partial_G G$ is homeomorphic to the Pontriagin sphere,

b) (Theorem 9.5 in the text) if X is nonorientable, then $\partial_G G$ is homeomorphic to the nonorientable Pontriagin surface.

This paper is organized as follows. In Section 2 we recall some terminology related to simplicial complexes and systolic complexes. We also recall some facts about systolic complexes. In Section 3 we thoroughly examine properties of combinatorial spheres S_n in 3-dimensional 7-systolic normal pseudomanifolds and properties of natural projections $\Pi_n : S_n \to S_{n-1}$ between them. D. Osajda showed that in the case of a locally finite 7-systolic simplicial complex X of finite dimension the inverse limit $\varprojlim(S_n, \Pi_n)$ of the system of these spheres and projections is homeomorphic to the Gromov boundary $\partial_G X$. In our case, we show that every sphere S_n is a surface. Moreover, we show that up to a homeomorphism the sphere S_{n+1} is a connected sum of S_n and links of vertices $w \in S_n$. In Section 4 we recall results of Jakobsche from [J] on inverse systems of compact orientable manifolds. The proof of the first statement of Main Theorem is contained in Sections 5, 6 and 7. In Section 5 we modify the maps $\Pi_n : S_n \to S_{n-1}$ (without changing the inverse limit $\varprojlim(S_n, \Pi_n)$). These maps become injective on some appropriate parts of domains, which is one of the conditions in the definition of a Jakobsche inverse system (which is in turn an object used to define a tree of manifolds). Properties of such modified maps $\Pi'_n : S_n \to S_{n-1}$ allow us, in Section 6, to further modify the inverse system (S_n, Π'_n). We call this modifications a refinement. Every element of the refined system is a connected sum of its predecessor and some finite number of tori. This is one of the conditions in the definition of the Pontriagin sphere. In Section 7 we define families $\mathcal{D}_{n,k}$ of pairwise disjoint discs in surfaces $S_{n,k}$, which turns the refined system $(S_{n,k}, \Pi'_{n,k})$ into a Jakobsche inverse system of tori, thus finishing the proof of part a) of Main Theorem. In Section 8 we examine properties of trees of nonorientable surfaces. In Section 9 we prove the second statement of Main Theorem.

2 Definitions and properties of systolic complexes

In this section we recall the notion of a systolic complex and some of its basic properties.

Let X be a simplicial complex and let $\sigma \subset X$ be a simplex. The *link* of X at σ (denoted by X_σ) is the subcomplex of X consisting of all simplices disjoint with σ and spanning together with σ a simplex in X. The *residuum* of σ in X (denoted by $\mathrm{Res}(\sigma, X)$) is the union of all simplices in X that contain σ.

For simplices σ_1 and σ_2 in X we denote by $\sigma_1 * \sigma_2$ the *simplicial join* of σ_1 and σ_2 (if it exists); this means that σ_1 and σ_2 are disjoint and $\sigma_1 * \sigma_2$ is the smallest simplex in X containing both of them. X is *flag* if every set of vertices $v_1, v_2, \ldots, v_n \in X$ pairwise connected by edges in X spans a simplex $v_1 * v_2 * \ldots * v_n$ in X. A subcomplex $K \subset X$ is *full* if for every set of vertices $v_1, v_2, \ldots, v_n \in K$ spanning a simplex $v_1 * v_2 * \ldots * v_n$ in X this simplex is a simplex in K.

A simplicial complex X is a *pseudomanifold* of dimension n if it is locally finite, it is a union of its n-simplices and each $(n-1)$-simplex is contained in exactly two n-simplices. A pseudomanifold is *orientable* if it admits a choice of orientations on top-dimensional simplices in a consistent way, i.e. such that the orientations on each simplex of codimension 1 inherited from two top-dimensional simplices containing it are opposite. An n-dimensional pseudomanifold is *normal* if for every nonempty simplex σ in X of dimension $\dim(\sigma) < n-1$ the link X_σ is connected.

Remark 2.1. Note that if a pseudomanifold is orientable then all its links are also orientable. The converse is not true in general. However for a simply-connected normal pseudomanifolds of dimension 3 its orientability is equivalent to the orientability of its vertex links.

A *cycle* in X is a subcomplex $\gamma \subset X$ isomorphic to some triangulation of the circle S^1. The *length* of a cycle γ (denoted by $|\gamma|$) is the number of its 1-simplices.

Definition 2.2. 1. Let X be a flag simplicial complex and let $k \geq 4$ be a natural number.

- X is *k-large* if every cycle γ in X of length $3 < |\gamma| < k$ is not full in X.
- X is *locally k-large* if for every nonempty simplex σ in X the link X_σ is k-large.
- X is *k-systolic* if it is connected, simply-connected and locally k-large.

2. A group G is *k-systolic* if it acts geometrically (i.e. properly discontinuously and cocompactly) by simplicial automorphisms on some k-systolic simplicial complex X.

For a brevity a 6-systolic complex or a group is called *systolic*.

Remark 2.3. Note that a full subcomplex of a k-large simplicial complex is k-large itself.

Now we recall some basic facts about systolic complexes. For proofs see [JS] and [O]. We start with the theorem relating the notions of systolicity and Gromov hyperbolicity.

Theorem 2.4. *[JS, Theorem 2.1] The 1-skeleton of a 7-systolic simplicial complex is hyperbolic.*

For a subset $A \subset X$ which is a union of some simplices in X we denote by $\mathrm{span}_X(A)$ the full subcomplex of X spanned on A (i.e. the intersection of all full subcomplexes of X containing A). Now we recall the definition of combinatorial *balls* and *spheres* in a simplicial complex X centred at a simplex $\sigma \subset X$:

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  - $B_0(\sigma, X) = \sigma$,
  - $B_{n+1}(\sigma, X) = \mathrm{span}_X\Big(\{\tau \subset X : \tau \cap B_n(\sigma, X) \neq \emptyset\}\Big)$,
- $S_n(\sigma, X) = \mathrm{span}_X\Big(\{w \in X^{(0)} : \mathrm{d}(w, \sigma) = n\}\Big)$, where $d(w, \sigma)$ denote the distance in the 1-skeleton $X^{(1)}$.

In the following proposition we recall some natural properties of balls and spheres in systolic complexes.

Fact 2.5. *[JS, Lemma 7.7] Let X be a systolic simplicial complex and let $v \in X$ be a vertex. Then for every natural number $n > 0$ and for every simplex $\tau \subset S_n(v, X)$ the intersection $\rho = B_{n-1}(v, X) \cap X_\tau$ is a single simplex. Moreover, the intersection $X_\tau \cap B_n(v, X)$ is equal to the ball $B_1(\rho, X_\tau)$ and the intersection $X_\tau \cap S_n(v, X)$ is equal to the sphere $S_1(\rho, X_\tau)$.*

Let b_τ denote the barycenter of a simplex τ and let X' denote the first barycentric subdivision of a simplicial complex X. We view the barycenters b_τ of simplices $\tau \subset X$ as the vertices of X'. The combinatorial properties of balls and spheres mentioned in Fact 2.5 are crucial in the definition of projections

$$\Pi_n : S_n(\sigma, X) \to [S_{n-1}(\sigma, X)]'$$

that we recall now.

For a systolic complex X and a simplex $\sigma \subset X$ let S_n denote the sphere $S_n(\sigma, X)$ and let B_n denote the ball $B_n(\sigma, X)$. Let $Y^{(0)}$ denote the 0-skeleton of Y, i.e. the vetrex set of a simplicial complex Y. Define the map

$$\Pi_n : S_n^{(0)} \to (S'_{n-1})^{(0)}$$

by the equalities $\Pi_n(v) = b_\tau$ for all vertices $v \in S_n^{(0)}$ (where the simplex τ is the intersection $B_{n-1} \cap X_v$).

Spheres and balls in 7-systolic complexes have stronger properties than these recalled above. The following fact allows to extend the map $\Pi_n : S_n^{(0)} \to (S'_{n-1})^{(0)}$ to a simplicial map

$$\Pi_n : S_n \to S'_{n-1}$$

Fact 2.6. *[O, Lemma 3.1] If X is 7-systolic then, for any vertices $v_1, v_2 \in S_n$ connected by an edge in S_n, their images $\Pi_n(v_1)$ and $\Pi_n(v_2)$ are contained in one simplex in S'_{n-1}, i.e. $\Pi_n(v_1)$ and $\Pi_n(v_2)$ are equal or they span a 1-simplex in S'_{n-1}.*

Define the map $\Pi_n : S_n \to S'_{n-1}$ as a simplicial extension of the map $\Pi_n : S_n^{(0)} \to (S'_{n-1})^{(0)}$ defined above.

We make now a comment about the notation used in this paper. We use the same symbol Π_n for the simplicial map $\Pi_n : S_n \to S'_{n-1}$ and for the related continuous map $\Pi_n : S_n \to S_{n-1}$ (when we forget the simplicial structure and treat the complexes S_n and S'_{n-1} just as metric spaces). For example this is the case in the following fact describing metric properties of the maps Π_n. We denote by d_X the standard piecewise euclidean metric on X.

Fact 2.7. *[O, Lemma 3.3] Let X be a 7-systolic complex with finite dimension. Then there is a positive constant $C < 1$ depending only on the dimension $dim(X)$ such that for all natural numbers n and for all points $x, y \in S_n$ it holds $d_{S_{n-1}}(\Pi_n(x), \Pi_n(y)) \leq C \cdot d_{S_n}(x, y)$.*

In the case of 7-sytolic 3-dimensional pseudomanifolds, combinatorial properties of the inverse system (S_n, Π_n) of spheres and projections will be thoroughly examined and described more precisely in Section 3. The next theorem shows that this system can be used to describe the Gromov boundary of a 7-systolic complex X.

Theorem 2.8. *[O, Lemma 4.1] Let X be a 7-systolic locally finite simplicial complex of finite dimension. For a vertex $v \in X$ let S_n denote the sphere $S_n(v, X)$ and let the maps $\Pi_n : S_n \to S_{n-1}$ be defined as before. Then the inverse limit $\varprojlim(S_n, \Pi_n)$ is homeomorphic to the Gromov boundary of X.*

3 Spheres and projections in 7-systolic normal pseudomanifolds of dimension 3

In this section X is a 7-systolic, normal pseudomanifold of dimension 3. We thoroughly examine properties of combinatorial spheres S_n in such pseudomanifolds and of the projections Π_n defined in Section 2.

In Lemmas 3.1, 3.2 and 3.3 we describe links of X at simplices σ of the dimensions 2, 1 and 0 respectively.

Lemma 3.1. *Let $\sigma \subset X$ be a 2-simplex. Then the link X_σ consists of two vertices.*

Proof: The simplex σ is contained in exactly two simplices of dimension 3. $\square$

Lemma 3.2. *Let $\varepsilon \subset X$ be a 1-simplex (i.e. an edge). Then the link X_ε is a 7-large triangulation of the circle S^1 (i.e. a triangulation of the circle consisting of at least 7 edges).*

Proof: Let ε be a join $v_1 * v_2$. Let $v \in X_\varepsilon$ be a vertex. There are exactly two vertices $u, w \in X_\varepsilon$ adjacent to v (since the join $v * v_1 * v_2$ is a 2-simplex lying in exactly two 3-simplices). Thus, since X is locally finite, it follows that the link X_ε is a disjoint union of copies of triangulated circles. But since X is normal it follows that the link X_ε is connected, so there must be exactly one copy. Since X is locally 7-large, it follows that this triangulation of the link X_ε must be 7-large. $\square$

Lemma 3.3. *Let $v \in X$ be a 0-simplex (i.e. a vertex). Then the link X_v is topologically a closed connected surface triangulated in a 7-large way. Moreover, if X is orientable then the link X_v is also orientable.*

Proof: For a vertex $w \in X_v$ we have the equality $(X_v)_w = X_{v*w}$. Thus the link $(X_v)_w$ is a triangulated circle (see Lemma 3.2). A simplicial complex whose every vertex link is a triangulated circle is itself a triangulated surface. Since X is 7-systolic, it follows that this triangulation is 7-large. The connectedness of the link X_v follows from the normality of X. The last assertion follows from Remark 2.1.

$\square$

Combinatorial properties of 7-large complexes imply the following:

Remark 3.4. Let Σ be a 7-large triangulated closed surface and let $\sigma \subset \Sigma$ be a simplex. Then:

1. the balls $B_1(\sigma, \Sigma)$ and $B_2(\sigma, \Sigma)$ are triangulated 2-discs; moreover, topological boundaries of these balls in Σ are spheres $S_1(\sigma, \Sigma)$ and $S_2(\sigma, \Sigma)$ respectively,
2. the ball $B_3(\sigma, \Sigma)$ can contain a loop in the 1-skeleton $\Sigma^{(1)}$ homotopically nontrivial in Σ (if the simplex σ has dimension greater than 0).

In the next lemma we describe combinatorial and topological properties of spheres $S_n(v, X)$.

Lemma 3.5. *Let $v \in X$ be a vertex and let $S_n = S_n(v, X)$ be the combinatorial sphere of radius n centered at v. Then S_n is a connected surface triangulated in a 7-large way.*

Proof: For $n = 1$ we have the equality $S_1 = X_v$. Thus the assertion follows from Lemma 3.3.

Let $w \in S_n$ be a vertex and let ρ denote the intersection $X_w \cap S_{n-1}$. Since X_w is a 7-large surface and ρ is a simplex (see Fact 2.5), by Remark 3.4 it follows that the ball $B_1(\rho, X_w)$ is a triangulated 2-disc. By the equalities

$$(S_n)_w = X_w \cap S_n = S_1(\rho, X_w) = \mathrm{bd}(B_1(\rho, X_w)) = S^1$$

(see Fact 2.5) it follows that vertex links of S_n are triangulated circles. Thus S_n is a triangulated surface.

Since S_n is full in X (by definition) and X is 7-large, it follows that this triangulation of S_n is 7-large (see Remark 2.3).

Connectedness of S_n can be shown using inductive argument and Corollary 3.18 below. $\square$

Lemmas 3.6, 3.7, 3.8 and 3.9 describe local properties of the projections $\Pi_{n+1} : S_{n+1} \to S_n$.

Lemma 3.6. *For a 2-simplex $\sigma \subset S_n$ there is exactly one vertex $w_\sigma \in S_{n+1}$ such that the join $w_\sigma * \sigma$ is a simplex in X. This vertex coincides with the preimage $\Pi_{n+1}^{-1}(b_\sigma)$.*

Proof: By Fact 2.5, the intersection $X_\sigma \cap S_{n-1}$ is a single simplex. Thus, for dimensional reasons, it is a vertex. By Lemma 3.1, the link X_σ consists of two vertices. Moreover, the intersection $X_\sigma \cap S_n$ is empty, since S_n is a surface and a full subcomplex. It follows that the intersection $X_\sigma \cap S_{n+1}$ must be equal to the other vertex of X_σ. Denote this vertex by w_σ. From definition of projections it is easy to see that $\Pi_{n+1}^{-1}[b_\sigma] = X_\sigma \cap S_{n+1}$ (σ is a 2-simplex). It follows that $\Pi_{n+1}^{-1}[b_\sigma] = w_\sigma$.

□

Lemma 3.7. *For an edge $\varepsilon \subset S_n$ the intersection $\alpha_\varepsilon = X_\varepsilon \cap S_{n+1}$ is an arc (triangulated). If σ_1 and σ_2 are two 2-simplices in S_n containing ε, then the endpoints of this arc coincide with the preimage vertices $\Pi_{n+1}^{-1}(b_{\sigma_1})$ and $\Pi_{n+1}^{-1}(b_{\sigma_2})$.*

Proof: Since S_n is a surface, it follows that there are exactly two 2-simplices in S_n (say $\sigma_1 = v_1 * \varepsilon$ and $\sigma_2 = v_2 * \varepsilon$ that contain ε. For these two simplices there are two vertices w_{σ_1} and w_{σ_2} in S_{n+1} such that for $i = 1, 2$ the joins $w_{\sigma_i} * \sigma_i$ are simplices in X.

First we show that w_{σ_1} and w_{σ_2} do not lie in a common simplex in X. To see this suppose that the join $w_{\sigma_1} * w_{\sigma_2}$ is a simplex in X. By Fact 2.6 it follows that the images $\Pi_{n+1}(w_{\sigma_1})$ and $\Pi_{n+1}(w_{\sigma_2})$ lie in a common simplex in the barycentric subdivision S'_n. Now Π_{n+1} maps the vertex w_{σ_i} to the barycenter b_{σ_i} for $i = 1, 2$. But the barycenters b_{σ_1} and b_{σ_2} do not span a simplex in S'_n, a contradiction.

Now for $i = 1, 2$ let a vertex u_i be the intersection $X_{\sigma_i} \cap S_{n-1}$. Note that since u_1 and u_2 belong to the intersection $X_\varepsilon \cap S_{n-1}$, they are equal or span a simplex in S_{n-1} (see Fact 2.5). Since the link X_ε is a triangulated circle, and u_1, u_2, v_1, v_2 are all vertices of the link X_ε lying in the ball $B_n(v, X)$, it follows that the vertices w_{σ_1} and w_{σ_2} are connected by an arc $\alpha_\varepsilon = (w_{\sigma_1} = w_0, w_1, \ldots, w_m = w_{\sigma_2})$ in S_{n+1} (for some $m > 1$). Lemma 3.6 implies that the vertices w_{σ_i} are exactly the preimages $\Pi_{n+1}^{-1}(b_{\sigma_i})$ for $i = 1, 2$. This finishes the proof. □

Lemma 3.8. *Let $\varepsilon \subset S_n$ be an edge, let σ_1 and σ_2 be two different 2-simplices in S_n containig ε and let $\alpha_\epsilon = (w_0, w_1, \ldots, w_m)$ be the arc in S_{n+1} given by Lemma 3.7. Then the projection Π_{n+1} maps edges $w_0 * w_1$ and $w_{m-1} * w_m$ homeomorphically onto edges $b_{\sigma_1} * b_\varepsilon$ and $b_{\sigma_2} * b_\varepsilon$ in S'_n respectively, and collapses the subarc $(w_1, w_2, \ldots, w_{m-2}, w_{m-1})$ to the barycenter b_ε.*

Proof: By Lemma 3.7, the projection Π_{n+1} maps w_0 to the barycenter b_{σ_1} and w_m to the barycenter b_{σ_2}. We show that Π_{n+1} maps w_i to the barycenter b_ε for $i = 1, 2, \ldots, m-1$. It is enough to show that the intersection $X_{w_i} \cap S_n$ is exactly equal to the edge ε.

For this note that w_i and ε span a simplex in X. Thus ε is a simplex in the intersection $X_{w_i} \cap S_n$. If this intersection contains a vertex u not contained in ε, it follows that u and ε span a simplex in S_n. But 2-simplices in S_n containing ε are exactly σ_1 and σ_2. It follows that w_i is equal to w_0 or to w_m, a contradiction.

□

Lemma 3.9. *Let $w \in S_n$ be a vertex. Then there exists a cycle (i.e. a triangulated circle) α_w in the 1-skeleton of $X_w \cap S_{n+1}$ such that the image $\Pi_{n+1}[\alpha_w]$ is equal to the sphere $S_1(w, S'_n)$ (which is a cycle in the barycentric subdivision S'_n) and the preimage $\Pi_{n+1}^{-1}[S_1(w, S'_n)]$ is equal to α_w.*

Proof: The sphere S_n is a triangulated surface, so the residuum $\mathrm{Res}(w, S_n)$ is a triangulated 2-disc. Let this residuum consist of 2-simplices $\sigma_i = w * w_i * w_{i+1}$ for $i = 0, 1, \ldots, k-1$, where $k = |X_w \cap S_n| \geq 7$ is the length of the link $(S_n)_w$ (indices taken modulo k). Let $\alpha_i = X_{w*w_i} \cap S_{n+1}$ be the arc in S_{n+1} given by Lemma 3.7. Let α_w be the union $\alpha_0 \cup \alpha_1 \cup \ldots \cup \alpha_{k-1}$. We claim that α_w is a cycle.

It is enough to show that the intersection $\alpha_i \cap \alpha_j$ is not empty only for $|i - j| \leq 1$ and moreover, for $|i - j| = 1$ it consists of one point. For this suppose that the intersection $\alpha_i \cap \alpha_j$ is nonempty for some $i < j \in \{0, 1, \ldots, k-1\}$ and let $u \in \alpha_i \cap \alpha_j$ be a vertex. Since the arcs α_i and α_j are contained in the links X_{w*w_i} and X_{w*w_j} respectively, it follows that simplices $w * w_i$ and $w * w_j$ are contained in the intersection $X_u \cap S_n$. By Fact 2.5 the join $w_i * w_j * w$ is a simplex in $S_n \cap X_u$. It follows that j is equal to $i + 1$. Since α_w is connected (the intersection $\alpha_i \cap \alpha_{i+1}$ is exactly the single vertex equal to the intersection $X_{\sigma_i} \cap S_{n+1}$), it must be a cycle.

By Lemma 3.7 and the definition of the cycle α_w it follows that Π_{n+1} maps α_w onto $S_1(w, S'_n)$. Since the preimage $\Pi^{-1}_{n+1}[B_1(w, S'_n)]$ is contained in the intersection $X_w \cap S_{n+1}$, it is enough to show that for all vertices $u \in X_w \cap S_{n+1}$ not contained in the cycle α_w the projection Π_{n+1} maps u to w. We show that the intersection $X_u \cap S_n$ is equal exactly to w. For this suppose that there is another vertex, say w', lying in the intersection $X_u \cap S_n$. It follows that w' is equal to a vertex w_i for some $i = 0, 1, \ldots, k-1$. Thus u lies in the arc α_i, a contradiction. This finishes the proof. $\square$

From the proof of Lemma 3.9 we get the following additional information:

Fact 3.10. *Let $w \in S_n$ be a vertex and let $\{\varepsilon_i : i = 1, 2, \ldots, k\}$ be the set of all edges in S_n that contain w. Then the cycle α_w is equal to the union $\bigcup_{i=1}^{k} \alpha_{\varepsilon_i}$.*

In the next lemma we show that the cycle α_w given by Lemma 3.9 bounds some 2-disc $D_w \subset X_w \cap B_{n+1}$.

Lemma 3.11. *Each cycle α_w bounds a 2-disc $D_w = B_2(\sigma_w, X_w)$ in the intersection $X_w \cap B_{n+1}$, for some simplex $\sigma_w \subset X_w$.*

Proof: For a vertex $w \in S_n$ giving the arc α_w let σ_w be the intersection $X_w \cap S_{n-1}$ (this intersection is a single simplex). We show that the cycle α_w is equal to the sphere $S_2(\sigma_w, X_w)$. It is obvious that α_w is contained in $S_2(\sigma_w, X_w)$. For the opposite inclusion let $u \in S_2(\sigma_w, X_w)$ be a vertex. There is a vertex $u' \in S_n \cap X_w$ connected by edges with u and with some vertex of σ_w. It follows that u is a vertex in the arc $\alpha_{w*u'}$. Thus, by Fact 3.10, u is a vertex in α_w.

By Remark 3.4, the cycle α_w is the boundary of the ball $B_2(\sigma_w, X_w)$. Since the link X_w is a surface (triangulated in a 7-large way), it follows that the ball $B_2(\sigma_w, X_w)$ is a 2-disc (see Remark 3.4 again). This finishes the proof. $\square$

For a vertex $w \in S_n$ let P_w denote the closure $cl(X_w \setminus D_w)$. Clearly, we have the following:

Fact 3.12. *The set P_w is a subcomplex of S_{n+1}. Topologically it is a connected surface with the boundary α_w.*

The next lemma describes the map Π_{n+1} restricted to the subcomplex $P_w \subset S_{n+1}$.

Lemma 3.13. *For every vertex $w \in S_n$ the projection $\Pi_{n+1} : S_{n+1} \to S_n$ maps the subcomplex P_w onto the ball $B_1(w, S'_n)$. Moreover, the preimage $\Pi^{-1}_{n+1}[w]$ is the union of simplices in P_w disjoint with the cycle α_w.*

Before proving Lemma 3.13 note the following:

Remark 3.14.
- The ball $B_1(w, S_n')$ is topologically a 2-disc with the boundary $S_1(w, S_n')$.
- Lemma 3.13 together with previous results (Lemmas and Facts 3.8-3.12) fully describe the restricted map $\Pi_{n+1}\lceil_{P_w}$.

Proof of Lemma 3.13: By Lemma 3.9, the preimage $\Pi_{n+1}^{-1}[S_1(w, S_n')]$ is equal to the cycle α_w. Let $u \in P_w$ be a vertex not contained in the cycle α_w. Since P_w is a subcomplex of the link X_w, it follows that the vertices w and u span an edge in X. We show that the intersection $X_u \cap S_n$ is equal exactly to the vertex w. It follows that Π_{n+1} maps u to w. It is enough to show that the dimension $\dim(X_u \cap S_n)$ is equal to 0 (since w a vertex in this intersection, which is a single simplex).

Assume the opposite and let σ be an intersection $X_u \cap S_n$. It follows that Π_{n+1} maps u to the barycenter b_σ. Since σ contain w and has the dimension at least 1, it follows that the barycenter b_σ is contained in the sphere $S_1(w, S_n')$. Thus u lies in the preimage $\Pi_{n+1}^{-1}[S_1(w, S_n')]$. This contradicts the equality $\Pi_{n+1}^{-1}[S_1(w, S_n')] = \alpha_w$.

□

For better understanding of the map Π_{n+1} we introduce another cell structure on the sphere S_n. We call this cell structure *dual*.

- The set of dual 0-cells (denoted by e_σ^0) consists of the barycenters b_σ of all 2-simplices $\sigma \subset S_n$.
- The set of dual 1-cells (denoted by e_ε^1) consists of the unions $b_{\sigma_1} * b_\varepsilon \cup b_{\sigma_2} * b_\varepsilon$, where ε is an edge in S_n while σ_1 and σ_2 are the two 2-simplices in S_n containing ε.
- The set of dual 2-cells (denoted by e_w^2) consists of the balls $B_1(w, S_n')$ around all vertices $w \in S_n$.

We denote by S_n^d the cell complex related to this cell structure, and by $(S_n^d)^{(k)}$ its k-skeleton, i.e. a cell subcomplex consisting of all cells of dimension at most k.

Using this dual cell structure, as a consequence of previous lemmas we get:

Lemma 3.15.
1. *The preimage $\Pi_{n+1}^{-1}[e_\sigma^0]$ is the vertex $w_\sigma = X_\sigma \cap S_{n+1}$.*
2. *The preimage $\Pi_{n+1}^{-1}[e_\varepsilon^1]$ is equal to the arc α_ε.*
3. *The preimage $\Pi_{n+1}^{-1}[e_w^2]$ is equal to the subcomplex P_w.*

Proof: Assertion 1 follows from Lemma 3.6.

By Lemma 3.8, the projection Π_{n+1} maps the arc α_ε onto the dual 1-cell e_ε^1. By Lemma 3.7, the preimages of endpoints of the dual 1-cell e_ε^1 are exactly the endpoints of the arc α_ε. By Lemma 3.9, the preimage $\Pi_{n+1}^{-1}[e_\varepsilon^1]$ is contained in the cycle α_u for every endpoint u of ε. Let u and u' be two endpoints of the edge ε. Since the intersection $\alpha_u \cap \alpha_{u'}$ is equal to the arc α_ε, we get Assertion 2.

Assertion 3 follows from Lemma 3.13. □

The next lemma describes the relationship between the 1-skeleton $(S_n^d)^{(1)}$ of the dual cell structure on the sphere S_n and its preimage by the map Π_{n+1}.

Lemma 3.16. *The preimage $\Pi_{n+1}^{-1}[(S_n^d)^{(1)}]$ of the 1-skeleton of the dual cell structure is naturally homeomorphic to this 1-skeleton.*

Proof: The 1-skeleton $(S_n^d)^{(1)}$ of the dual cell structure on the sphere S_n is the union $\bigcup e^1_\varepsilon$ of 1-cells. By Lemma 3.8 and Lemma 3.15, the map Π_{n+1} gives one-to-one correspondence between the arcs α_ε and the dual 1-cells e^1_ε. Namely, arcs α_ε are mapped onto dual 1-cells e^1_ε. Moreover, this correspondence is consistent with the incidence relation, i.e. the intersection $e^2_u \cap e^2_{u'}$ is nonempty if and only if the intersetion $\alpha_u \cap \alpha_{u'}$ is not empty, and the same holds for triples of vertices. This finishes the proof.

□

Remark 3.17.
- Note that the restriction of the map Π_{n+1} to the preimage $\Pi_{n+1}^{-1}[(S_n^d)^{(1)}]$ is not a homeomorphism onto $(S_n^d)^{(1)}$. However, it can be approximated by homeomorphisms of the form described later in Lemma 5.2. More precisely, the map $w_\sigma \to e^0_\sigma$ can be extended to a map $S_{n+1} \to S_n$ such that every arc α_ε is homeomorphically mapped onto the dual 1-cell e^1_ε. As a consequence, the cycle α_w is mapped homeomorphically onto the boundary $\mathrm{bd}(e^2_w)$ of the dual 2-cell e^2_w.
- The sphere S_{n+1}, up to homeomorphism, can be thought of as obtained from the sphere S_n by cutting the interiors of all dual 2-cells e^2_w and replacing these interiors by surfaces P_w such that each boundary $\mathrm{bd}(P_w) = \alpha_w$ is glued homeomorphically to the boundary $\mathrm{bd}(e^2_w)$.

Recall that a *connected sum of the manifolds M and N of dimension n (with or without boundaries) along n-discs $D \subset$* $\mathrm{int}(M)$ and $D' \subset \mathrm{int}(N)$ is the quotient space

$$\Big(\big(M \setminus \mathrm{int}(D)\big) \cup \big(N \setminus \mathrm{int}(D')\big)\Big)/_{x\sim f(x)}$$

where $f : \mathrm{bd}(D) \to \mathrm{bd}(D')$ is a homeomorphism.

As a consequence of the second part of Remark 3.17 we have the following:

Corollary 3.18. *The sphere S_{n+1} is topologically a connected sum of the sphere S_n and the links X_w of vertices $w \in S_n$ along discs $D_w \subset X_w$ and $e^2_w \subset S_n$.*

4 Inverse limits, Jakobsche spaces and outline of the proof of Main Theorem

In this section we recall the result of Jakobsche from [J] concerning inverse systems of appropriately iterated connected sums of compact orientable manifolds. We use this result in the next section.

Recall, that a family $\mathcal{A}$ of subsets of a metric space X is a *null family* if for every positive number $\epsilon > 0$ only finitely many elements $A \in \mathcal{A}$ have diameter greater than ϵ. The family $\mathcal{A}$ is *dense* if the union $\bigcup \mathcal{A}$ is a dense subset of X.

Theorem 4.1. *[J, Theorem 4.6] Let $(L_0 \xleftarrow{\alpha_1} L_1 \xleftarrow{\alpha_2} L_2 \leftarrow \dots)$ be an inverse system of connected closed orientable m-manifolds ($m \geq 2$) and for each $k \geq 0$ let $\mathcal{D}_k$ be a finite collection of pairwise disjoint discs in L_k such that:*

1. *each L_k is a connected sum of finitely many copies of L_0,*

2. *every map α_{k+1} restricted to the preimage*

$$\alpha_{k+1}^{-1}\Big[L_k \setminus \bigcup\{int(D) : D \in \mathcal{D}_k\}\Big]$$

is a homeomorphism onto the set

$$L_k \setminus \bigcup\{int(D) : D \in \mathcal{D}_k\}$$

3. *every preimage $\alpha_{k+1}^{-1}[D]$ (for $D \in \mathcal{D}_k$) is homeomorphic to a copy of L_0 with the interior of a disc removed,*

4. *the family $\{\alpha_{j,i}[D] : i \geq j, D \in \mathcal{D}_i\}$ [1] is null and dense in L_j for all j,*

5. *the intersection $\alpha_{j,i}[D] \cap bd(D')$ is empty for all discs $D \in \mathcal{D}_i$, $D' \in \mathcal{D}_j$ and for all $i > j$.*

Then the inverse limit $\varprojlim(L_0 \xleftarrow{\alpha_1} L_1 \xleftarrow{\alpha_2} L_2 \leftarrow \dots)$ depends only on L_0.

We denote this inverse limit by $X(L_0)$ and call it the *Jakobsche space* for L_0, or the *Jakobsche tree of manifolds* L_0. We call a system $(L_k, \alpha_k, \mathcal{D}_k)_{k\geq 0}$ satisfying assumptions 1-5 of Theorem 4.1 a *Jakobsche inverse system* for L_0. If a system $(L_k, \alpha_k, \mathcal{D}_k)_{k\geq 0}$ satisfies assumptions 2, 4, 5 and the condition:

3a. every preimage $\alpha_{k+1}^{-1}[D]$ (for $D \in \mathcal{D}_k$) is homeomorphic to a connected closed (orientable) m-manifold with the interior of a disc removed,

than we call it a *Jakobsche inverse system* of (orientable) m-manifolds.

Remark 4.2. 1. Note that we did not state the result of Jakobsche in its full generality.

2. For $L_0 = \mathbb{T}^2$, the 2-dimensional torus, the space $X(\mathbb{T}^2)$ is known as the Pontriagin sphere and denoted by Π_P.

3. For $m = 2$ and $L_0 = \Sigma_g$, the orientable surface of genus $g > 1$, the space $X(\Sigma_g)$ is homeomorphic to the Pontriagin sphere. Actually the tree of orientable surfaces is homeomorphic to Π_P. We sketch some details of this in Section 8 (see Remark 8.6 (2)).

If X is a locally finite 7-systolic simplicial complex of finite dimension, then by Theorem 2.8, the Gromov boundary $\partial_G X$ is homeomorphic to the inverse limit $\varprojlim(S_n, \Pi_n)$. The results of Section 3 imply that the inverse system (S_n, Π_n) of spheres and projections in a 7-systolic orientable normal pseudomanifold X of dimension 3 is close to satisfy assumptions 1-5 of the Jakobsche theorem. In the next remark we make this observation more precise.

Remark 4.3. The maps Π_k are natural candidates for projections α_k and the families $\mathcal{D}_k = \left\{e_w^2 : w \in S_k^{(0)}\right\}$ of dual 2-cells in the spheres S_k are natural candidates for families of discs as in a Jakobsche inverse system. More precisely, Fact 2.7 implies that for such a choice of families $\mathcal{D}_k$ the family $\{\Pi_{j,i}[D] : i \geq j, D \in \mathcal{D}_i\}$ is null in every sphere S_j. Moreover, since the union $\bigcup \mathcal{D}_j$ covers the sphere S_j, it follows that the families $\{\Pi_{j,i}[D] : i \geq j, D \in \mathcal{D}_i\}$ are dense in

[1] For $i > j$ we denote by $\alpha_{j,i}$ the composition $\alpha_{j+1} \circ \ldots \circ \alpha_i$, whereas $\alpha_{i,i}$ denotes the identity on L_i.

every sphere S_j. If the links of all vertices of X are triangulations of the same surface Σ_0, then assumptions 1 and 3 are satisfied with $L_0 = \Sigma_0$ by Lemma 3.15 (3).
On the other hand, the maps Π_k and the families $\mathcal{D}_k$ defined as above fail to satisfy some other assumptions of the Jakobsche theorem. In particular:

- elements of so defined families $\mathcal{D}_k$ are not pairwise disjoint,
- even though the projection Π_{k+1} maps the preimage
$$\Pi_{k+1}^{-1}\Big[S_k \setminus \bigcup \{\mathrm{int}(e_w^2) : w \in S_k^{(0)}\}\Big]$$
onto the set
$$S_k \setminus \bigcup \Big\{\mathrm{int}(e_w^2) : w \in S_k^{(0)}\Big\}$$
the restriction of Π_{k+1} to this preimage is not a homeomorphism, and
- assumption 5 of Theorem 4.1 fails.

The strategy of the proof of part a) of Main Theorem is as follows. In Section 5 we modify the inverse system (S_n, Π_n), without affecting the inverse limit, by changing appropriately the bonding maps. This modification will make the inverse system satisfy assumptions 2, 4 and 5 of Theorem 4.1 (after choosing appropriately the families of discs). The modified inverse system (S_n, Π'_n) (with families of discs choosen appropraitely) will be a Jakobsche inverse system of orientable surfaces. In Section 6 we refine this new system without changing the inverse limit either. The refined system will consist of orientable surfaces $S_{n,k}$ for $k = 0, 1, \dots, g_n$ (for some natural numbers g_n) and maps $\Pi'_{n,k+1\to k} : S_{n,k+1} \to S_{n,k}$ satisfying $S_{n,0} = S_n$, $S_{n,g_n} = S_{n+1}$ and $\Pi'_{n,1\to 0} \circ \Pi'_{n,2\to 1} \circ \ldots \circ \Pi'_{n,g_n\to g_n-1} = \Pi'_{n+1}$. The refinement is necessary to get the connected sum with tori, rather than with higher genera surfaces. In Section 7 we define the family of discs in every surface of the refined system to match all the assumptions of Theorem 4.1.

5 Modification of the inverse system

In this section we modify the inverse system (S_n, Π_n) described in Section 3. Actually, we modify only the projections $\Pi_n : S_n \to S_{n-1}$ leaving the spaces S_n unchanged. This modification will be small enough so that it does not change the inverse limit. The new inverse system (S_n, Π'_n) will satisfy the following conditions:

- each of the modified projections Π'_{n+1} maps the preimage $\Pi_{n+1}^{-1}[(S_n^d)^{(1)}]$ homeomorphically onto the 1-skeleton $(S_n^d)^{(1)}$ of the dual cell structure on the sphere S_n,
- for every vertex $w \in S_n$ the projection Π'_{n+1} maps some canonical open neighbourhood U_w of the cycle α_w in P_w homeomorphically onto the 2-cell e_w^2 with the point w removed, and collapses the complement $P_w \setminus U_w$ to w.

We denote by d_{sup} the uniform metric on the set of continuous maps between two compact spaces. We perform small (with respect to the uniform distance) modifications of the maps Π_n keeping the inverse limit unchanged. To do this we use the following result due to M. Brown.

Theorem 5.1. *[B, Theorem 2] There is an assignment of positive real numbers*

$$a(s_1, s_2, \ldots, s_{k-1}, t_1, t_2, \ldots, t_{k-1}, t_k)$$

to pairs of finite sequences

$$(X_0 \xleftarrow{s_1} X_1 \xleftarrow{s_2} \ldots \xleftarrow{s_{k-1}} X_{k-1}) \text{ and } (X_0 \xleftarrow{t_1} X_1 \xleftarrow{t_2} \ldots \xleftarrow{t_{k-1}} X_{k-1} \xleftarrow{t_k} X_k)$$

of continuous maps between compact metric spaces, for all integer k, such that the following holds: if two inverse systems $(Y_0 \xleftarrow{\alpha_1} Y_1 \xleftarrow{\alpha_2} \ldots)$ and $(Y_0 \xleftarrow{\beta_1} Y_1 \xleftarrow{\beta_2} \ldots)$ satisfy the inequalities

$$d_{sup}(\alpha_k, \beta_k) < a(\alpha_1, \alpha_2, \ldots, \alpha_{k-1}, \beta_1, \beta_2, \ldots, \beta_{k-1}, \beta_k)$$

for all k, then the inverse limits $\varprojlim(Y_0 \xleftarrow{\alpha_1} Y_1 \xleftarrow{\alpha_2} \ldots)$ and $\varprojlim(Y_0 \xleftarrow{\beta_1} Y_1 \xleftarrow{\beta_2} \ldots)$ are homeomorphic.

The next lemma shows that it is possible to approximate the projections $\Pi_{n+1} : S_{n+1} \longrightarrow S_n$ arbitrarily close by maps $\Pi_{n+1,\epsilon} : S_{n+1} \longrightarrow S_n$ having much better properties (from the point of view of fullfilling the requirements of Jakobsche inverse system).

Lemma 5.2. *For any number $\epsilon > 0$ and any integer $n > 0$ there is a continous map $\Pi_{n+1,\epsilon} : S_{n+1} \longrightarrow S_n$ satisfying the following:*

1. *$d_{sup}(\Pi_{n+1}, \Pi_{n+1,\epsilon}) < \epsilon$,*
2. *$\Pi_{n+1}^{-1}[w] = (\Pi_{n+1,\epsilon})^{-1}[w] = \Big(X_w \setminus B_3(\sigma_w, X_w)\Big) \cup S_3(\sigma_w, X_w)$ for all vertices $w \in S_n^{(0)}$ (where σ_w is the intersection $X_w \cap S_{n-1}$),*
3. *the restriction of the map $\Pi_{n+1,\epsilon}$ to the set*

   $$S_{n+1} \setminus \bigcup \{(\Pi_{n+1,\epsilon})^{-1}[w] : w \in S_n^{(0)}\}$$

   is a homeomorphism onto the set

   $$S_n \setminus \{w : w \in S_n^{(0)}\},$$

4. *$\Pi_{n+1}[S_{n+1}^{(0)}] \subseteq \Pi_{n+1,\epsilon}[S_{n+1}^{(0)}]$.*

Proof:

Let $w \in S_n$ be a vertex. Let l_w denote the number of 2-simplices in S_n that contain w. For $i = 0, 1, \ldots, l_w - 1$ let $\sigma_i = w * w_i * w_{i+1}$ be all these 2-simplices (indices taken modulo l_w).

Consider the cycle $\alpha_w \subset S_{n+1}$ as described in Lemma 3.9. Denote vertices of α_w in the following way:

$$w_{0,0}, w_{0,1}, \ldots, w_{0,k_0} = w_{1,0}, w_{1,1}, \ldots, w_{1,k_1} = w_{2,0}, \ldots, w_{l_w-1,0}, \ldots, w_{l_w-1,k_{l_w-1}} = w_{0,0}$$

(for some natural numbers $k_0 > 1, \ldots, k_{l_w-1} > 1$). We choose the indices in such a way that $\Pi_{n+1}(w_{i,0}) = b_{\sigma_i}$, $\Pi_{n+1}(w_{i,j}) = b_{\sigma_i \cap \sigma_{i+1}}$ for $0 < j < k_i$ and succesive vertices are connected by an edge.

Consider 2-simplices in P_w intersecting α_w. Denote these 2-simplices in the following way (see Figure 5, note that this figure does not exhibit the geometry of the subcomplex P_w, in fact all simplices have sizes of length 1):

$$w_{0,0}*w_{0,0,1}*w_{0,0,2}\ ,\ w_{0,0}*w_{0,0,2}*w_{0,0,3}\ ,\ \dots\ ,\ w_{0,0}*w_{0,0,m_{0,0}-1}*w_{0,0,m_{0,0}}\ ,\ w_{0,0}*w_{0,1}*w_{0,0,m_{0,0}}\ ,$$
$$w_{0,1}*w_{0,1,1}*w_{0,1,2}\text{ (where }w_{0,1,1}=w_{0,0,m_{0,0}}),\ \dots\ ,\ w_{0,k_0-1}*w_{0,k_0}*w_{0,k_0-1,m_{0,k_0-1}}\ ,$$
$$w_{1,0}*w_{1,0,1}*w_{1,0,2}\text{ (where }w_{1,0}=w_{0,k_0}\text{ and }w_{1,0,1}=w_{0,k_0-1,m_{0,k_0-1}}),\ \dots\ ,$$
$$w_{l_w-1,k_{(l_w-1)}-1}*w_{l_w-1,k_{(l_w-1)}-1,0}*w_{l_w-1,k_{(l_w-1)}-1,1}\ ,\ \dots\ ,$$
$$w_{l_w-1,k_{(l_w-1)}-1}*w_{l_w-1,k_{(l_w-1)}-1,m_{l_w-1,k_{(l_w-1)}-1}-1}*w_{l_w-1,k_{(l_w-1)}-1,m_{[l_w-1,k_{(l_w-1)}-1]}}\ ,$$
$$w_{l_w-1,k_{(l_w-1)}}*w_{0,0}*w_{0,0,1}\text{ (where }w_{l_w,k_{l_w}}=w_{0,0}\text{ and }w_{l_w-1,k_{(l_w-1)}-1,m_{[l_w-1,k_{(l_w-1)}-1]}}=w_{0,0,1}\ ).$$

For two points x and y lying in a single simplex we denote by $[x,y]$ the interval connecting them.

For $i=0,1,\dots,l_w-1$ choose points a_i,b_i,c_i,d_i in the following way: $a_i\in[w_{i,0},w_{i-1,k_{(i-1)}-1}]$ with $d(a_i,w_{i-1,k_{(i-1)}-1})=\epsilon$, $b_i\in[w_{i,0},w_{i-1,k_{(i-1)}-1}]$ with $d(a_i,w_{i,0})=\epsilon$, $c_i\in[w_{i,0},w_{i,1}]$ with $d(c_i,w_{i,0})=\epsilon$, $d_i\in[w_{i,0},w_{i,1}]$ with $d(d_i,w_{i,1})=\epsilon$. For $s\in[0,\frac{\sqrt{3}}{2}]$ and $i=0,1,\dots,l_w-1$ choose points a_i^s,b_i^s,c_i^s,d_i^s in the following way: $a_i^s\in[w_{i,0,1},a_i]$ with $d(a_i^s,[w_{i,0},w_{i-1,k_{i-1}-1}])=s$, $b_i^s\in[w_{i,0,1},b_i]$ with $d(b_i^s,[w_{i,0},w_{i-1,k_{i-1}-1}])=s$, $c_i^s\in[w_{i,0,m_{i,0}},c_i]$ with $d(c_i^s,[w_{i,0},w_{i,1}])=s$, $d_i^s\in[w_{i,0,m_{i,0}},d_i]$ with $d(d_i^s,[w_{i,0},w_{i,1}])=s$. For $s\in[0,\frac{\sqrt{3}}{2}]$, $i=0,1,\dots,l_w-1$, $j=0,1,\dots,k_i-1$ and $k=1,\dots,m_{i,j}$ choose points $e_{i,j,k}^s\in[w_{i,j},w_{i,j,k}]$ with $d(e_{i,j,k}^s,[w_{i,j,k},w_{i,j,k+1}])=\frac{\sqrt{3}}{2}-s$.

For $i=0,1,\dots,l_w-1$ let $a_i'=\Pi_{n+1}(a_i)$, $b_i'=\Pi_{n+1}(b_i)$, $c_i'=\Pi_{n+1}(c_i)$ and $d_i'=\Pi_{n+1}(d_i)$. For $i=0,1,\dots,l_w-1$ and $s\in[0,\frac{\sqrt{3}}{2}]$ let $a_i'^s=\Pi_{n+1}(a_i^s)$, $b_i'^s=\Pi_{n+1}(b_i^s)$, $c_i'^s=\Pi_{n+1}(c_i^s)$, $d_i'^s=\Pi_{n+1}(d_i^s)$, $e_i'^s=\Pi_{n+1}(e_{i,0,k}^s)$ and ${e'_{i,i+1}}^s=\Pi_{n+1}(e_{i,j,k}^s)$ (for $j>0$).

For each $i=0,1,\dots,l_w-1$ choose and fix vertices $w_{i_i,j_i}\in\{w_{i,1},\dots,w_{i,k_i-1}\}$ and $w_{i_i,j_i,k_i}\in\{w_{i_i,j_i,1},\dots,w_{i_i,j_i,m_{i_i,j_i}}\}$ (vertices w_{i_i,j_i} will be mapped onto the barycenters $b_{w*w_{i+1}}$ in order to fulfill the condition 4).

Define the map $\Pi_{n+1}^{w,\epsilon}:P_w\to S_n$ as follows:

- $\Pi_{n+1}^{w,\epsilon}(x)=\Pi_{n+1}(w)$ for $w\in[w_{i,0,1},a_i,b_i]$, $w\in[w_{i,0,m_{i,0}},c_i,d_i]$ and for $x\in\Pi_{n+1}^{-1}[w]$,
- for $s\in[0,\frac{\sqrt{3}}{2}]$ and $i=0,1,\dots,l_w-1$ let $\Pi_{n+1}^{w,\epsilon}:[b_i^s,e_{i,0,1}^s]\cup[e_{i,0,1}^s,e_{i,0,2}^s]\cup\dots,\cup[e_{i,0,m_{i,0}}^s,c_i^s]\to[b_i'^s,e_i'^s]\cup[e_i'^s,c_i'^s]$ be linear (with respect to the length of segments),
- for $s\in[0,\frac{\sqrt{3}}{2}]$ and $i=0,1,\dots,l_w-1$ let $\Pi_{n+1}^{w,\epsilon}:[d_i^s,e_{i,1,1}^s]\cup[e_{i,1,1}^s,e_{i,1,2}^s]\cup\dots\cup[e_{i_i,j_i,k_i-1}^s,e_{i_i,j_i,k_i}^s]\to[d_i'^s,e_{i,i+1}'^s]$ and $\Pi_{n+1}^{v,\epsilon}:[e_{i_i,j_i,k_i}^s,e_{i_i,j_i,k_i+1}^s]\cup\dots\cup[e_{i,k_i-1,m_{i,k_i-1}}^s,a_{i+1}^s]\to[e_{i,i+1}'^s,a_{i+1}'^s]$ be linear.

Note that $\Pi_{n+1}^{w,\epsilon}$ is a well defined continuous map. Note also that with vertices $w_{i_{i_w},j_{i_w}}$ chosen in a coherent way (i.e. for two adjancent wertices $w,w'\in S_n$ the chosen wertices $w_{i_{i_w},j_{i_w}}$ and $w_{i_{i_{w'}},j_{i_{w'}}}$ lying on the arc $\alpha_{w*w'}$ must coincide) the map $\Pi_{n+1}^{\epsilon}=\bigcup_{w\in S_n^{(0)}}\Pi_{n+1}^{w,\epsilon}:S_{n+1}\to S_n$ is well defined and satisfies the required condition. We omit further details.

□

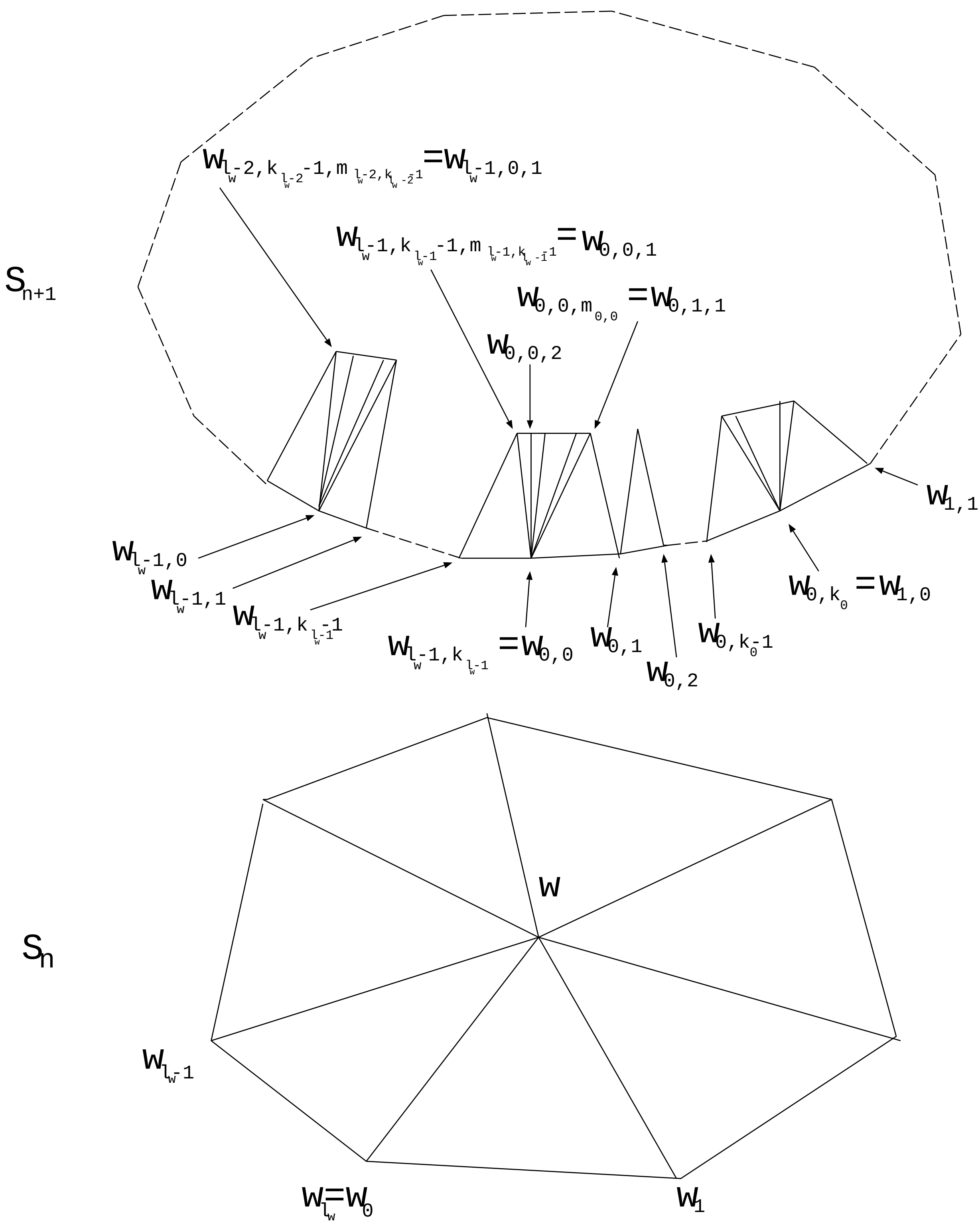

Figure 1: Proof of Lemma 5.2

In the next lemma we define a sequence of maps $(\Pi'_{n+1} : S_{n+1} \to S_n)_{n\geq 1}$ such that the inverse limits $\varprojlim(S_1 \xleftarrow{\Pi_2} S_2 \xleftarrow{\Pi_3} S_3 \ldots)$ and $\varprojlim(S_1 \xleftarrow{\Pi'_2} S_2 \xleftarrow{\Pi'_3} S_3 \ldots)$ are homeomorphic. The new inverse sytem (S_n, Π'_n) satisfies the conditions mentioned at the beginning of this section. As we will show later, after refinement of this new system, we will be able to define the families of discs $\mathcal{D}_{n,k}$ such that the refined system $(S_{n,k}, \Pi'_{n,k}, \mathcal{D}_{n,k})$ will become a Jakobsche inverse system for the torus.

Lemma 5.3. *There is a sequence of continuous maps $(\Pi'_{n+1} : S_{n+1} \to S_n)_{n\geq 1}$ such that:*

1. *the inverse limits $\varprojlim(S_1 \xleftarrow{\Pi_2} S_2 \xleftarrow{\Pi_3} S_3 \ldots)$ and $\varprojlim(S_1 \xleftarrow{\Pi'_2} S_2 \xleftarrow{\Pi'_3} S_3 \ldots)$ are homeomorphic,*
2. *$\Pi_{n+1}^{-1}[w] = (\Pi'_{n+1})^{-1}[w] = \Big(X_w \setminus B_3(\sigma_w, X_w)\Big) \cup S_3(\sigma_w, X_w)$ for all vertices $w \in S_n^{(0)}$ (where σ_w is the intersection $X_w \cap S_{n-1}$) ,*
3. *the restriction of the map Π'_{n+1} to the set*
$$S_{n+1} \setminus \bigcup \{(\Pi'_{n+1})^{-1}[w] : w \in S_n^{(0)}\}$$
is a homeomorphism onto the set
$$S_n \setminus \{w : w \in S_n^{(0)}\}$$
4. *$\Pi_{n+1}[S_{n+1}^{(0)}] \subset \Pi'_{n+1}[S_{n+1}^{(0)}]$*

Proof: Inductively we define a sequence of maps $(\Pi'_n : S_n \to S_{n-1})_{n\geq 2}$ and a decreasing sequence of positive numbers $(\epsilon_n)_{n\geq 2}$ such that:

- for each natural number $n > 1$ the map Π'_n satisfies conditions 2, 3 and 4,
- $d_{sup}(\Pi_n, \Pi'_n) < \epsilon_n$, with $\epsilon_n < a(\Pi'_2, \Pi'_3, \ldots, \Pi'_{n-1}, \Pi_2, \Pi_3, \ldots, \Pi_n)$, where the latter are the positive numbers given by the Brown theorem.

Let ϵ_2 be a positive number satisfying $0 < \epsilon_2 < a(\Pi_2)$ and let a map $\Pi'_2 : S_2 \to S_1$ satisfy conditions 2, 3 and 4 and the inequality $d_{sup}(\Pi_2, \Pi'_2) < \epsilon_2$. Such a map exists due to Lemma 5.2. Note that we can additionally assume that $\epsilon_2 \frac{1}{1-C} < 1$ where $C < 1$ is a positive constant given by Fact 2.7. This property will be used in the proof of Lemma 9.4.

Suppose now that we have defined positive numbers $\epsilon_2 > \ldots > \epsilon_n$ satisfying the inequalities
$$\epsilon_k < a(\Pi'_2, \Pi'_3, \ldots, \Pi'_{k-1}, \Pi_2, \Pi_3, \ldots, \Pi_k)$$
for $k = 2, 3, \ldots, n$ and maps $\Pi'_2, \Pi'_3, \ldots, \Pi'_n$ satisfying the required conditions and the inequalities $d_{sup}(\Pi_k, \Pi'_k) < \epsilon_k$ for $k = 2, 3, \ldots, n$. Let $\epsilon_{n+1} < \epsilon_n$ be a number satisfying
$$0 < \epsilon_{n+1} < a(\Pi'_2, \Pi'_3, \ldots, \Pi'_n, \Pi_2, \Pi_3, \ldots, \Pi_{n+1})$$
and let $\Pi'_{n+1} : S_{n+1} \to S_n$ be a map satisfying conditions 2, 3 and 4 and the inequality $d_{sup}(\Pi_{n+1}, \Pi'_{n+1}) < \epsilon_{n+1}$. Such a map exists again due to Lemma 5.2.

Now the sequence $(\Pi'_n)_{n>1}$ satisfies conditions 2, 3 and 4. Moreover, by the Brown theorem, the inverse limits $\varprojlim(S_1 \xleftarrow{\Pi_2} S_2 \xleftarrow{\Pi_3} S_3 \ldots)$ and $\varprojlim(S_1 \xleftarrow{\Pi'_2} S_2 \xleftarrow{\Pi'_3} S_3 \ldots)$ are homeomorphic.

$\square$

6 Refinement of the inverse system

In this section X is orientable. Recall that it follows that vertex links X_u are orientable surfaces (see Remark 2.1). We refine the inverse system (S_n, Π'_n). The refinement does not change the inverse limit, and the refined system $(S_{n,k}, \Pi'_{n,k})$ will have the property that every surface $S_{n,k+1}$ will be a connected sum of its predecessor $S_{n,k}$ and a finite number of tori.

As it will be made clear in Section 7, the inverse system (S_n, Π'_n), after appropriate choice of families $\mathcal{D}_n$ of discs in S_n, fulfills assumptions 2, 4 and 5 of Theorem 4.1. Preimages of the chosen discs under the bonding maps Π'_{n+1} will correspond to surfaces that are links of X at vertices of S_n.

Two phenomena may appear that prevent the system (S_n, Π'_n) from satisfying assumptions 1 and 3 of the Jakobsche theorem for $L_0 = \mathbb{T}^2$. The first one is that links at vertices do not have to be surfaces of the same genus. The second phenomenom is that even since all vertex links are homeomorphic, they may be surfaces of genus greater than 1.

Using the following two lemmas we will be able to refine the system (S_n, Π'_n) to overcome these difficulties. We start with some terminology.

Definition 6.1. Let $f : \Sigma \to \Sigma'$ be a map between compact orientable surfaces and let $\mathcal{D} = \{D_1, D_2, \ldots, D_l\}$ be a family of pairwise disjoint discs $D_i \subset \Sigma'$. We say that f *collapses* Σ to Σ' along the family $\mathcal{D}$ if:

- Σ is a connected sum of Σ' and a finite number of surfaces $\Sigma_{g_1}, \Sigma_{g_2}, \ldots, \Sigma_{g_l}$ of genera $g_1 > 0, g_2 > 0, \ldots, g_l > 0$ respectively (for some $l > 0$) along discs $D_i \subset \Sigma'$ and $D'_i \subset \Sigma_{g_i}$ for $i = 1, 2, \ldots, l$,
- $f(x) = x$ for all $x \in \Sigma' \setminus \Big(\bigcup_{i=1}^{l} \mathrm{int}(D_i)\Big)$,
- there are open neighbourhoods U_i of $\mathrm{bd}(D'_i)$ in $\Sigma_{g_i} \setminus \mathrm{int}(D'_i)$ and points $x_i \in \mathrm{int}(D_i)$ such that f maps homeomorphically U_i onto $D_i \setminus \{x_i\}$ and collapses $(\Sigma_{g_i} \setminus \mathrm{int}(D'_i)) \setminus U_i$ to x_i.

We call such a map a *collapsing map.* If it is clear which family $\mathcal{D}$ we mean, we say that f *collapses* Σ to Σ'.

Note that the maps $\Pi_{n,\epsilon}$ from Lemma 5.2, and hence the maps Π'_n from Lemma 5.3, are examples of collapsing maps.

We state without the proofs two obvious lemmas which we use in the refinement procedure.

Lemma 6.2. *Let Σ be an orientable surface of genus $g > 1$. Then there exist orientable surfaces $\Sigma_1, \Sigma_2, \ldots, \Sigma_g = \Sigma$, discs $D_i \subset \Sigma_i$ (for $i = 1, 2, \ldots, g-1$) and maps $f_i : \Sigma_i \to \Sigma_{i-1}$ (for $i = 2, 3, \ldots, g$) such that:*

- *Σ_i is an orientable surface of genus i for $i = 1, 2, \ldots, g$,*
- *Σ_i is a connected sum of Σ_{i-1} and a torus T^2_{i-1} along the disc D_{i-1} and some disc $D'_{i-1} \subset T^2_{i-1}$ such that the disc D_i is contained in the complement $T^2_{i-1} \setminus D'_{i-1}$ and the map f_i collapses Σ_i to Σ_{i-1} along a one-element family $\mathcal{D}_{i-1} = \{D_{i-1}\}$.*

Lemma 6.3. *Let $f : \Sigma \to \Sigma'$ collapse an orientable surface Σ to an orientable surface Σ' along a family $\mathcal{D} = \{D_1, \ldots, D_l\}$. Let $\Sigma_{g_1}, \Sigma_{g_2}, \ldots, \Sigma_{g_l}$ be orientable surfaces as in Definition 6.1. For $i = 1, 2, \ldots, l$ let $D'_i \subset \Sigma_{g_i}$ be discs as in Definition 6.1. Let the genus g_j of the surface Σ_{g_j} be greater than 1 for some $j \in \{1, 2, \ldots, l\}$. Then there exist:*

- *a decomposition of Σ_{g_j} to a connected sum of two orientable surfaces $\Sigma_{g'_j}$ and $\Sigma_{g''_j}$ of genera $g'_j = 1$ and $g''_j = g_j - 1$ respectively (i.e. $\Sigma_{g'_j}$ is a torus) along discs $D''_j \subset \Sigma_{g'_j}$ and $D'''_j \subset \Sigma_{g''_j}$ such that the disc D'_j is contained in the surface $\Sigma_{g'_j}$ and the intersection $D''_j \cap D'_j$ is empty,*
- *a surface Σ'', which is a connected sum of Σ' and $\Sigma_{g_1}, \ldots, \Sigma_{g_{j-1}}, \Sigma_{g'_j}, \Sigma_{g_{j+1}}, \ldots, \Sigma_{g_l}$ along discs D_i and D'_i respectively,*
- *maps $f_1 : \Sigma \to \Sigma''$ and $f_2 : \Sigma'' \to \Sigma'$*

such that f_1 collapses Σ to Σ'' along the family $\{D''_j\}$, f_2 collapses Σ'' to Σ' along the family $\mathcal{D}$, and $f = f_2 \circ f_1$.

Remark 6.4. Note that Lemma 6.2 and Lemma 6.3 are also true for nonorientable surfaces. The only difference is that collapsing maps are related to connected sums with projective planes rather than with tori.

As an immediate consequence we get the following:

Corollary 6.5. *For $n \geq 1$ let the map $\Pi'_{n+1} : S_{n+1} \to S_n$ be defined as before. For each vertex $w \in (S_n)^{(0)}$ denote by g_w the genus of the link X_w (which is a closed orientable surface). Let $g_n = max\{g_w : w \in (S_n)^{(0)}\}$. Then there exist surfaces*

$$S_n = S_{n,0},\ S_{n,1},\ \ldots,\ S_{n,g_n} = S_{n+1}$$

and maps

$$S_{n,0} \xleftarrow{\Pi'_{n,1\to 0}} S_{n,1} \xleftarrow{\Pi'_{n,2\to 1}} S_{n,2} \xleftarrow{\Pi'_{n,3\to 2}} \ldots \xleftarrow{\Pi'_{n,g_n-1\to g_n-2}} S_{n,g_n-1} \xleftarrow{\Pi'_{n,g_n\to g_n-1}} S_{n,g_n}$$

such that:

- *$S_{n,k}$ is a connected sum of $S_{n,k-1}$ and some tori $T_{n,k,1}$, $T_{n,k,2}$, $\ldots$, $T_{n,k,m_{n,k}}$ (for some natural number $m_{n,k} \geq 1$) along pairwise disjoint discs*

  $$D_{n,k,1} \subset S_{n,k-1}\ ,\ D_{n,k,2} \subset S_{n,k-1}\ ,\ \ldots\ ,\ D_{n,k,m_{n,k}} \subset S_{n,k-1}$$

  and

  $$D'_{n,k,1} \subset T_{n,k,1}\ ,\ D'_{n,k,2} \subset T_{n,k,2}\ ,\ \ldots\ ,\ D'_{n,k,m_{n,k}} \subset T_{n,k,m_{n,k}}$$

  respectively,
- *every disc $D_{n,k+1,i}$ is contained in some torus $T_{n,k,j}$ and is disjoint with the disc $D'_{n,k,j}$,*
- *for each $n > 0$ and each $k = 1, 2, \ldots, g_n$ the map $\Pi'_{n+1,k\to k-1}$ collapses $S_{n,k}$ to $S_{n,k-1}$ along the family $\{D_{n,k,i} : i = 1, 2, \ldots, m_{n,k}\}$,*
- *$\Pi'_{n+1} = \Pi'_{n,g_n\to 0}$ (where $\Pi'_{n,g_n\to 0}$ denote the composition $\Pi'_{n,1\to 0} \circ \Pi'_{n,2\to 1} \circ \ldots \circ \Pi'_{n,g_n\to g_n-1}$).*

Corollary 6.5 gives the refined inverse system of orientable surfaces

$$(S_1 \xleftarrow{\Pi'_{1,1\to 0}} S_{1,1} \xleftarrow{\Pi'_{1,2\to 1}} \dots \xleftarrow{\Pi'_{1,g_1\to g_1-1}} S_2 \xleftarrow{\Pi'_{2,1\to 0}} \dots)$$

In this inverse system every surface is a connected sum of its predecessor and a finite number of tori (possibly only one). If the genus of the sphere S_1 is greater than 1, we use Lemma 6.2 for the sphere S_1 to get the inverse system

$$(S_0 \xleftarrow{\Pi'_{0,1\to 0}} S_{0,1} \xleftarrow{\Pi'_{0,2\to 1}} \dots \xleftarrow{\Pi'_{0,g_0\to g_0-1}} S_1 \xleftarrow{\Pi'_{1,1\to 0}} \dots)$$

with S_0 a torus. We don't do this if S_1 is a torus.

The last condition of Corollary 6.5 implies that the refining of the inverse system does not change the inverse limit. Thus we get the following:

Corollary 6.6. *Suppose that X is a 7-systolic normal orientable pseudomanifold of dimension 3. Then the Gromov boundary $\partial_G X$ is homeomorphic to the inverse limit of the refined inverse system* $\varprojlim(S_0 \xleftarrow{\Pi'_{0,1\to 0}} S_{0,1} \xleftarrow{\Pi'_{0,2\to 1}} \dots \xleftarrow{\Pi'_{0,g_0\to g_0-1}} S_1 \xleftarrow{\Pi'_{1,1\to 0}} S_{1,1} \xleftarrow{\Pi'_{1,2\to 1}} \dots)$

($\varprojlim(S_1 \xleftarrow{\Pi'_{1,1\to 0}} S_{1,1} \xleftarrow{\Pi'_{1,2\to 0}} \dots)$ if S_1 is a torus).

7 Getting the structure of a Jakobsche inverse system

In this section we continue considerations of the previous one, under the same assumption that X is orientable. We define some finite families $\mathcal{D}_{n,k}$ of pairwise disjoint discs in every surface $S_{n,k}$. The inverse system $(S_{n,k}, \Pi'_{n,k})$ with families $\mathcal{D}_{n,k}$ will satisfy all assumptions of the Jakobsche theorem, with $L_0 = \mathbb{T}^2$.

To define these families we need some preparations. For $n = 0, 1, \dots$ let $A_n \subset S_n$ be defined as $A_n = \{\Pi'_{n,l}(w) : l \geq n, w \in S_l^{(0)}\}$ and let $A_{n,k}$ be the image $\Pi'_{n,g_n\to k}[A_{n+1}] \subset S_{n,k}$ of the set A_{n+1}, where $\Pi'_{n,l} : S_n \to S_l$ and $\Pi'_{n,g_n\to k} : S_{n+1} \to S_{n,k}$ are the compositions $\Pi'_{l+1} \circ \dots \circ \Pi'_n$ and $\Pi'_{n,k+1\to k} \circ \dots \circ \Pi'_{n,g_n\to g_n-1}$ respectively.

Lemma 7.1. *A_n is a countable dense subset of S_n for all $n \geq 1$, and $A_{n,k}$ is a countable dense subset of $S_{n,k}$ for all $n \geq 0$ and all $k = 0, 1, \dots, g_n$.*

Proof: Recall that every map Π_i is a C-contraction and the 0-skeleton $S_i^{(0)}$ is a finite 1-net in the sphere S_i for all numbers i. It follows that the set $\{\Pi_{n,l}(v) : v \in S_l^{(0)}\}$ is a finite C^{l-n}-net in S_n. Since the maps Π'_i satisfy the condition $\Pi_i[S_i^{(0)}] \subset \Pi'_i[S_i^{(0)}]$ (see assertion 4 of Lemma 5.3) it follows that the assertion holds for the sets A_n. Since the set $A_{n,k}$ is the image of the countable dense set A_{n+1} by a surjection, it is itself countable and dense. $\square$

Now we define inductively families of discs $\mathcal{D}_n$ and $\mathcal{D}_{n,k}$ in the spheres S_n and $S_{n,k}$ respectively to match all assumptions of Theorem 4.1.

Suppose the genus of the sphere S_1 is equal to 1 (i.e. the sphere S_1 is a torus). Let

$$\mathcal{D}_1 = \{D_w : w \in (S_1)^{(0)}\}$$

be a family of pairwise disjoint discs such that $w \in \mathrm{int}(D_w)$ with diameters $\mathrm{diam}(D_w) < \frac{1}{2}$ and such that the intersections $\mathrm{bd}(D_w) \cap A_1$ are empty. Note that since every disc is a union of uncountable family of disjoint circles and a point (just take a homeomorphism to the unit plane disc and circles of radius $r \in (0,1]$ centred at 0), it follows that such discs exist.

If the sphere S_1 has genus greater then 1, then as in Corollary 6.6 we start with the surface S_0. Let $\mathcal{D}_0$ be a family consisting of one small 2-disc D in S_0 (contained in the disc $D_{0,0}$ as in Lemma 6.2 and satisfying the inequality $\operatorname{diam}(D) < 1$), with $x \in \operatorname{int}(D)$ (where a point $x \in \operatorname{int}(D_{0,0})$ and a disc $D_{0,0}$ are given by the fact that the map $\Pi'_{0,1\to 0} : S_{0,1} \to S_0$ is a collapsing map). Again we can assume that the intersection $\operatorname{bd}(D) \cap A_0$ is empty.

Now suppose we have defined the families $\mathcal{D}_0, \ldots, \mathcal{D}_{n-1}$ and $\mathcal{D}_{i,j}$ for all $i \leq n-1$ and $j = 0, 1, \ldots, g_i$. We define the family $\mathcal{D}_n$ as follows. For every vertex $u \in S_n^{(0)}$ we choose a small 2-disc D_u containing u in its interior such that:

- the discs D_u are pairwise disjoint,
- the intersection $\operatorname{bd}(D_u) \cap A_n$ is empty for all vertices $u \in S_n^{(0)}$,
- the intersection $\Pi'_{i,n}[D_u] \cap \operatorname{bd}(D')$ is empty for all $i < n$ and for all discs $D' \in \mathcal{D}_i$,
- the intersection $\Pi'_{i,g_i\to j} \circ \Pi'_{i+1,n}[D_u] \cap \operatorname{bd}(D')$ is empty for all $i < n$, all $j = 0, 1, \ldots, g_i$ and for all discs $D' \in \mathcal{D}_{i,j}$,
- $\operatorname{diam}_{S_i}(\Pi'_{i,n}[D_u]) < \frac{1}{2^n}$ for all $i < n$,
- $\operatorname{diam}_{S_{i,j}}(\Pi'_{i,g_i\to j} \circ \Pi'_{i+1,n}[D_u]) < \frac{1}{2^n}$ for all $i < n$ and all $j = 1, 2, \ldots, g_i$.

It is possible to choose such a family $\mathcal{D}_n$. To see this consider a point $a = \Pi'_{i,n}(u) \in A_i$. This point is not contained in the boundary $\operatorname{bd}(D')$ of any disc $D' \in \mathcal{D}_i$. Analogously, any point $a = \Pi'_{i,g_i\to j} \circ \Pi'_{i+1,n}(y) \in A_{i,j}$ is not contained in the boundary $\operatorname{bd}(D')$ of any disc $D' \in \mathcal{D}_{i,j}$. Thus for small enough $\epsilon > 0$ the intersection $\Pi'_{i,n}[B_{S_n}(u,\epsilon)] \cap \operatorname{bd}(D')$ is empty for all discs $D' \in \mathcal{D}_i$ and the intersection $\Pi'_{i,g_i\to j} \circ \Pi'_{i+1,n}[B_{S_n}(u,\epsilon)] \cap \operatorname{bd}(D')$ is empty for all discs $D' \in \mathcal{D}_{i,j}$ (there are only finitely many such discs D'). Since S_n is a surface, the metric ball $B_{S_n}(u,\epsilon)$ contains a 2-disc D_u containing u in its interior. Again we can assume that the intersection $\operatorname{bd}(D_u) \cap A_n$ is empty.

Now suppose we have defined the families $\mathcal{D}_0, \mathcal{D}_1, \ldots, \mathcal{D}_n$, the families $\mathcal{D}_{i,j}$ for all $i < n$ and $j = 0, 1, \ldots, g_i$ and the families $\mathcal{D}_{n,j}$ for all $j < k$. We define the family $\mathcal{D}_{n,k}$ as follows. For all points $x_{n,k,l}$ given by the fact that the map $\Pi'_{n,k+1\to k} : S_{n,k+1} \to S_{n,k}$ is a collapsing map let $D_{n,k,l}$ be a small disc in $S_{n,k}$ containing $x_{n,k,l}$ in its interior such that:

- the discs $D_{n,k,l}$ are pairwise disjoint,
- the intersection $\operatorname{bd}(D_{n,k,l}) \cap A_{n,k}$ is empty for all l,
- the intersection $\Pi'_{i,n} \circ \Pi'_{n,k\to 0}[D_{n,k,l}] \cap \operatorname{bd}(D')$ is empty for all $i < n$ and for all discs $D' \in \mathcal{D}_i$,
- the intersection $\Pi'_{i,g_i\to j} \circ \Pi'_{i+1,n} \circ \Pi'_{n,k\to 0}[D_{n,k,l}] \cap \operatorname{bd}(D')$ is empty for all $i < n$, all $j = 0, 1, \ldots, g_i$ and all discs $D' \in \mathcal{D}_{i,j}$,
- $\operatorname{diam}_{S_i}(\Pi'_{i,n} \circ \Pi'_{n,k\to 0}[D_{n,k,l}]) < \frac{1}{2^n}$ for all $i < n$,
- $\operatorname{diam}_{S_{i,j}}(\Pi'_{i,g_i\to j} \circ \Pi'_{i+1,n} \circ \Pi'_{n,k\to 0}[D_{n,k,l}]) < \frac{1}{2^n}$ for all $i < n$ and all $j = 0, 1, \ldots, g_i$.

Choosing such discs is possible and we do it analogously as we have defined the elements of the families $\mathcal{D}_n$.

Now the families of discs $\mathcal{D}_n$ and $\mathcal{D}_{n,k}$ together with the maps Π'_n and $\Pi'_{n,k\to k-1}$ satisfy assumptions of Theorem 4.1 (note that A_n is dense in S_n and $A_{n,k}$ is dense in $S_{n,k}$). Since the maps Π'_n were chosen to be close enough to the maps Π_n to preserve the inverse limit, as a corollary we get:

Theorem 7.2. *The Gromov boundary of a 7-systolic normal orientable pseudomanifold of dimension 3 is a Jakobsche tree of tori, i.e. the Pontriagin sphere.*

8 Nonorientable trees of surfaces

In this section we examine properties of Jakobsche inverse systems of non-orientable surfaces. An extension of the Jacobsche's construction for nonorientable case was considered in [S]. In dimension 2, i.e. for non-orientable surfaces, it is possible and more convenient to follow rather Jakobsche's approach then that of Stallings. We sketch here some details of this.

We call a family $\mathcal{D}$ of pairwise disjoint closed discs contained in the interior of a manifold M a *good family*, if it is null family and the family $\{\mathrm{int}(D) : D \in \mathcal{D}\}$ is a dense family in M.

The following lemma is a simple extension of Toruńczyk's lemma (see [J, Lemma 3.1]).

Lemma 8.1. *Let Σ and Σ' be nonorientable surfaces (with or without boundaries) and let $f : \Sigma \to \Sigma'$ be a homeomorphism. Let $\mathcal{Z}$ and $\mathcal{Z}'$ be two good families of closed 2-discs in Σ and Σ' respectively. Then there exists a bijective function $p : \mathcal{Z} \to \mathcal{Z}'$ and a homeomorphism*

$$f' : \Sigma \setminus \bigcup_{D\in\mathcal{Z}} int(D) \to \Sigma' \setminus \bigcup_{D'\in\mathcal{Z}'} int(D')$$

such that

$$f'\lceil_{bd(\Sigma)} = f\lceil_{bd(\Sigma)} \text{ and } f'[bd(D)] = bd(p(D)) \text{ for each } D \in \mathcal{Z}.$$

The proof of this lemma is the same as in [J], thus we omit it.

Using Lemma 8.1 and the fact that every homeomorphism of the boundary of a closed nonorientable surface with the interior of a disc removed can be extended to a homeomorphism of this surface, by the same argument as in the proof of Theorem 4.6 in [J], we get the following:

Theorem 8.2. *Let $(L_0 \xleftarrow{\alpha_1} L_1 \xleftarrow{\alpha_2} L_2 \leftarrow \ldots)$ be an inverse system of connected closed nonorientable surfaces and for each $k \geq 0$ let $\mathcal{D}_k$ be a finite collection of pairwise disjoint discs in L_k such that:*

1. *each L_k is a connected sum of finitely many copies of L_0,*
2. *every map α_{k+1} restricted to the preimage*

$$\alpha_{k+1}^{-1}\Big[L_k \setminus \bigcup\{int(D) : D \in \mathcal{D}_k\}\Big]$$

is a homeomorphism onto the set

$$L_k \setminus \bigcup\{int(D) : D \in \mathcal{D}_k\}$$

3. every preimage $\alpha_{k+1}^{-1}[D]$ (for $D \in \mathcal{D}_k$) is homeomorphic to a copy of L_0 with the interior of a disc removed,

4. the family $\{\alpha_{j,i}[D] : i \geq j, D \in \mathcal{D}_i\}$ is null and dense in L_j for all j,

5. the intersection $\alpha_{j,i}[D] \cap bd(D')$ is empty for for all $i > j$ and for all discs $D \in \mathcal{D}_i$ and $D' \in \mathcal{D}_j$.

Then the inverse limit $\varprojlim(L_0 \xleftarrow{\alpha_1} L_1 \xleftarrow{\alpha_2} L_2 \leftarrow \ldots)$ depends only on L_0.

As in the orientable case we denote this space by $X(L_0)$ and call it a *Jacobsche tree of nonorientable surfaces* L_0. Similary as in the orientable case, we call the system $(L_k, \alpha_k, \mathcal{D}_k)_{k\geq 0}$ satisfying assumptions 1-5 of Theorem 8.2 a *Jakobsche inverse system* for L_0. If the system $(L_k, \alpha_k, \mathcal{D}_k)_{k\geq 0}$ satisfies assumptions 2, 4, 5 and the condition:

3a. every preimage $\alpha_{k+1}^{-1}[D]$ (for $D \in \mathcal{D}_k$) is homeomorphic to a connected closed nonorientable surfaces with the interior of a disc removed,

then we call it a *Jakobsche inverse system* of nonorientable surfaces.

Remark 8.3. 1. For $L_0 = \mathbb{RP}^2$, the projective plane, the space $X(\mathbb{RP}^2)$ is known as the nonorientable Pontriagin surface and denoted by Σ_P.

2. For $L_0 = \Sigma_g$, the nonorientable surface of genus $g > 1$, the space $X(\Sigma_g)$ is homeomorphic to the nonorientable Pontriagin surface. Actually the tree of nonorientable surfaces is homeomorphic to the nonorientable Pontriagin surface (see Remark 8.6)

The next two lemmas show that if nonorientable surfaces occur densly enough in a tree of surfaces, than this tree is homeomorphic to the nonorientable Pontriagin surface.

Lemma 8.4. *Let $(X_0 \xleftarrow{s_1} X_1 \xleftarrow{s_2} X_2 \ldots)$ and $(Y_0 \xleftarrow{t_1} Y_1 \xleftarrow{t_2} Y_2 \ldots)$ be two inverse systems of topological spaces such that the maps s_i and t_i are continuous and onto for all natural numbers i and such that there exist:*

- *increasing sequences $\{n_k\}$, $\{m_k\}$, $\{n'_k\}$ and $\{m'_k\}$ of natural numbers satisfying $n_{k-1} \leq n'_k \leq n_k$ and $m_{k-1} \leq m'_k \leq m_k$,*
- *continuous maps $f_k : X_{n_k} \to Y_{m_k}$ and $g_k : Y_{m'_k} \to X_{n'_k}$ being onto for all k,*

such that the following diagrams are commutative:

$$\begin{tikzcd} X_{n'_k} & X_{n_k} \arrow[l, "s_{n'_k,n_k}"'] \arrow[d, "f_k"] \\ Y_{m'_k} \arrow[u, "g_k"] & Y_{m_k} \arrow[l, "t_{m'_k,m_k}"] \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} X_{n_{k-1}} \arrow[d, "f_{k-1}"'] & X_{n'_k} \arrow[l, "s_{n_{k-1},n'_k}"'] \\ Y_{m_{k-1}} & Y_{m'_k} \arrow[l, "t_{m_{k-1},m'_k}"] \arrow[u, "g_k"'] \end{tikzcd}$$

i.e. it holds $g_k \circ t_{m'_k,m_k} \circ f_k = s_{n'_k,n_k}$ and $f_{k-1} \circ s_{n_{k-1},n'_k} \circ g_k = t_{m_{k-1},m'_k}$. Then the inverse limits $\varprojlim(X_k, s_k)$ and $\varprojlim(Y_k, t_k)$ are homeomorphic.

Proof: Define maps

$$F : \varprojlim(X_{n_0} \xleftarrow{s_{n_0,n_1'}} X_{n_1'} \xleftarrow{s_{n_1',n_1}} \ldots) \to \varprojlim(Y_{m_0} \xleftarrow{t_{m_0,m_1'}} Y_{m_1'} \xleftarrow{t_{m_1',m_1}} \ldots)$$

and

$$G : \varprojlim(Y_{m_0} \xleftarrow{t_{m_0,m_1'}} Y_{m_1'} \xleftarrow{t_{m_1',m_1}} \ldots) \to \varprojlim(X_{n_0} \xleftarrow{s_{n_0,n_1'}} X_{n_1'} \xleftarrow{s_{n_1',n_1}} \ldots)$$

by formulas

$$F((x_0, x_1', x_1, x_2', \ldots)) = (f_0(x_0), t_{m_1',m_1}(f_1(x_1)), f_1(x_1), t_{m_2',m_2}(f_2(x_2)), \ldots)$$

and

$$G((y_0, y_1', y_1, y_2', \ldots)) = (s_{n_0,n_1'}(g_1(y_1')), g_1(y_1'), s_{n_1,n_2'}(g_2(y_2')), g_2(y_2'), \ldots)$$

respectively.

These maps are well defined, continuous and inverse one to the other. Thus they are both homeomorphisms. Moreover, the inverse limits

$$\varprojlim(X_k, s_k) \text{ and } \varprojlim(X_{n_0} \xleftarrow{s_{n_0,n_1'}} X_{n_1'} \xleftarrow{s_{n_1',n_1}} X_{n_1} \xleftarrow{s_{n_1,n_2'}} \ldots)$$

and similarly

$$\varprojlim(Y_k, t_k) \text{ and } \varprojlim(Y_{m_0} \xleftarrow{t_{m_0,m_1'}} Y_{m_1'} \xleftarrow{t_{m_1',m_1}} Y_{m_1} \xleftarrow{t_{m_1,m_2'}} \ldots)$$

are naturally homeomorphic. Thus the assertion holds. $\square$

Lemma 8.5. *Let $(L_0 \xleftarrow{\alpha_1} L_1 \xleftarrow{\alpha_2} L_2 \ldots)$ be an inverse system of connected closed nonorientable surfaces and for each $k \geq 0$ let $\mathcal{D}_k$ be a finite collection of pairwise disjoint discs in L_k such that:*

1. *$(L_k, \alpha_k, \mathcal{D}_k)$ is a Jakobsche inverse system of surfaces,* [2]
2. *for every natural number k and for every disc $D \in \mathcal{D}_k$ there is a natural number $l_D > k$ such that the preimage $(\alpha_{k,l_D})^{-1}[D]$ is a nonorientable surface with the interior of a disc removed,*
3. *every map α_{k+1} collapses L_{k+1} to L_k along $\mathcal{D}_k$.*

Then the inverse limit $\varprojlim(L_0 \xleftarrow{\alpha_1} L_1 \xleftarrow{\alpha_2} L_2 \leftarrow \ldots)$ is homeomorphic to the nonorientable Pontriagin surface.

Proof: We shall define the following collection of data:

- an infinite increasing sequence $\{n_k\}$ of natural numbers,
- a sequence $\{L_k'\}$ of nonorientable closed surfaces,
- a sequence $\{\mathcal{D}_k'\}$ of finite families of pairwise disjoint discs in every surface L_k',

[2] In particular we require that the preimage $\alpha_{k+1}^{-1}[D]$ (for $D \in \mathcal{D}_k$) is a closed surface (orientable or not) with the interior of a disc removed.

- sequences of maps $\{f_k : L_{n_k} \to L'_{k-1}\}$, $\{g_k : L'_k \to L_{n_k}\}$ and $\{\alpha'_k : L'_k \to L'_{k-1}\}$

satisfying the following:

a) the diagrams:

$$\begin{tikzcd} L_{n_k} & & L_{n_{k+1}} \arrow[ll, "\alpha_{n_k,n_{k+1}}"'] \arrow[dl, "f_{k+1}"] \\ & L'_k \arrow[ul, "g_k"] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & L_{n_k} \arrow[dl, "f_k"'] & \\ L'_{k-1} & & L'_k \arrow[ll, "\alpha'_k"] \arrow[ul, "g_k"'] \end{tikzcd}$$

are commutative, i.e. $g_k \circ f_{k+1} = \alpha_{n_k,n_{k+1}}$ and $f_k \circ g_k = \alpha'_k$,

b) g_k maps

$$L'_k \setminus \bigcup_{D \in \mathcal{D}'_{k-1}} (\alpha'_k)^{-1}[\mathrm{int}(D)]$$

homeomorphically onto

$$L_{n_k} \setminus \bigcup_{D \in \mathcal{D}'_{k-1}} f_k^{-1}[\mathrm{int}(D)]$$

and maps $(\alpha'_k)^{-1}[D]$ onto $f_k^{-1}[D]$ for all discs $D \in \mathcal{D}'_{k-1}$,

c) f_{k+1} maps

$$L_{n_{k+1}} \setminus \bigcup_{D \in \mathcal{D}'_k} f_{k+1}^{-1}[\mathrm{int}(D)]$$

homeomorphically onto

$$L'_k \setminus \bigcup_{D \in \mathcal{D}'_k} \mathrm{int}(D)$$

d) α'_k collapses L'_k to L'_{k-1} along $\mathcal{D}'_{k-1}$,

e) $(L'_k, \alpha'_k, \mathcal{D}'_k)$ is a Jakobsche inverse system of nonorientable surfaces.

Note that by Lemma 8.4 the inverse limits $\varprojlim(L_k, \alpha_k)$ and $\varprojlim(L'_k, \alpha'_k)$ are homeomorphic. By the nonorientable analogues of Lemma 6.2 and Lemma 6.3 the inverse limit $\varprojlim(L'_k, \alpha'_k)$ is homeomorphic to the Jakobsche tree of projective planes. Thus, by Theorem 8.2, both of these inverse limits are homeomorphic to the nonorientable Pontriagin surface.

It remains to construct the desired data. We preceed to do this inductively. Let $n_0 = 0$, $L'_0 = L_0$, $g_0 = Id_{L_0}$, $\mathcal{D}'_0 = \mathcal{D}_0$. Let $n_1 = 1$, $f_1 = \alpha_1$.

Suppose that we have defined the following:

- natural numbers n_j for $j = 0, 1, \ldots, k$ satisfying $n_j < n_{j+1}$ for $j = 0, 1, \ldots, k-1$,
- nonorientable closed surfaces L'_j for $j = 0, 1, \ldots, k-1$,
- finite families $\mathcal{D}'_j$ (for $j = 0, 1, \ldots, k-1$) of pairwise disjoint discs in every surface L'_j respectively,
- maps $f_j : L_{n_j} \to L'_{j-1}$ for $j = 0, 1, \ldots, k$, $g_j : L'_j \to L_{n_j}$ for $j = 0, 1, \ldots, k-1$ and $\alpha'_j : L'_j \to L'_{j-1}$ for $j = 0, 1, \ldots, k-1$ (if $k > 0$)

satisfying (a), (b), (c), (d) and an additional condition:

f) every preimage $(\alpha'_j)^{-1}[D]$ (for $D \in \mathcal{D}'_{j-1}$) is homeomorphic to a nonorientable closed surface with the interior of a disc removed.

Let $n_{k+1} > n_k$ be the smallest integer such that the preimage $(f_k \circ \alpha_{n_k,n_{k+1}})^{-1}[D]$ is a nonorientable surface with the interior of a disc removed for all discs $D \in \mathcal{D}'_{k-1}$ (such a number exists due to assumption 2).

Let

$$L'_k = \Big(L'_{k-1} \setminus \bigcup_{D \in \mathcal{D}'_{k-1}} \mathrm{int}(D)\Big) \cup \bigcup_{D \in \mathcal{D}'_{k-1}} (f_k \circ \alpha_{n_k,n_{k+1}})^{-1}[D]$$

where points $x \in \mathrm{bd}(D)$ are identified with their preimages $(f_k \circ \alpha_{n_k,n_{k+1}})^{-1}[x]$ due to (c) and assumption 1.

Define the maps $g_k : L'_k \to L_{n_k}$, $f_{k+1} : L_{n_{k+1}} \to L'_k$ and $\alpha'_k : L'_k \to L'_{k-1}$ as follows:

$$g_k(x) = \begin{cases} f_k^{-1}(x) & \text{if } x \in L'_{k-1} \setminus \bigcup_{D \in \mathcal{D}'_{k-1}} \mathrm{int}(D) \quad \text{(by (c) for } f_k) \\ \alpha_{n_k,n_{k+1}}(x) & \text{if } x \in \bigcup_{D \in \mathcal{D}'_{k-1}} (f_k \circ \alpha_{n_k,n_{k+1}})^{-1}[D] \end{cases}$$

$$f_{k+1}(x) = \begin{cases} x & \text{if } x \in \bigcup_{D \in \mathcal{D}'_{k-1}} (f_k \circ \alpha_{n_k,n_{k+1}})^{-1}[D] \\ f_k \circ \alpha_{n_k,n_{k+1}}(x) & \text{otherwise} \end{cases}$$

and

$$\alpha'_k(x) = \begin{cases} x & \text{if } L'_{k-1} \setminus \bigcup_{D \in \mathcal{D}'_{k-1}} \mathrm{int}(D) \\ f_k \circ \alpha_{n_k,n_{k+1}}(x) & \text{otherwise} \end{cases}$$

These maps are of course well defined and continuous. They satisfy (a), (b) and (d) in an obvious way.

To define the family $\mathcal{D}'_k$ we need some technical definition. For $n_k \leq j < n_{k+1}$ let

$$\mathcal{D}^+_j = \{D \in \mathcal{D}_j : D \cap (f_k \circ \alpha_{n_k,j})^{-1}[D'] = \emptyset \text{ for } D' \in \mathcal{D}'_{k-1} \text{ and } D \cap \alpha^{-1}_{s,j}[D''] = \emptyset \text{ for } n_k \leq s < j \text{ and } D'' \in \mathcal{D}_s\}$$

Define the family $\mathcal{D}'_k$ by

$$\mathcal{D}'_k = \big\{f_{k+1}\big[\alpha^{-1}_{j,n_{k+1}}[D]\big] : n_k \leq j < n_{k+1}, D \in \mathcal{D}^+_j\big\}$$

We skip the straightforward checking of conditions (c) and (e).

□

Remark 8.6. 1. Note that the assumption 3 in Lemma 8.5 is not necessary. Indeed, as in the proof of Theorem 4.6 in [J] it can be shown that it is possible to change the maps $\alpha_k : L_k \to L_{k-1}$ on the preimages $\alpha_k^{-1}[D]$ (for $D \in \mathcal{D}_{k-1}$), while keeping the inverse limit unchanged, to get the collapsing maps.

2. The same argument shows that the tree of orientable surfaces is homeomorphic to the Pontriagin sphere.

9 Proof of part b) of Main Theorem

In this section we extend Theorem 7.2 to the nonorientable case. We need some preparations. We start with the following property of group actions on metric spaces, the proof of which we skip.

Lemma 9.1. *Let X be a proper metric space and let a group G act on X cocompactly by isometries. Then there is a positive constant $R > 0$ such that for all points $x \in X$ translates of the metric ball $B_X(x, R)$ under elements of G cover X, i.e. it holds $G \cdot B_X(x, R) = X$.*

Consider now a 3-dimensional 7-systolic normal pseudomanifold X with a cocompact action of a group G by simplicial automorphisms. For a vertex $w \in X$ and a simplex $\sigma \subset X_w$ consider a subcomplex

$$X_{w,\sigma} = \big(X_w \setminus B_2(\sigma, X_w)\big) \cup S_2(\sigma, X_w)$$

Let $K_{w,\sigma}$ denote the diameter $\operatorname{diam}(X^{(1)}_{w,\sigma})$ (in the intrinsic metric $d_{X^{(1)}_{w,\sigma}}$). Note that the number $K = \max\{K_{w,\sigma} : w \in X^{(0)}, \sigma \subset X_w\}$ is finite.

The next lemma describes the relationship between distances in succesive spheres in a 7-systolic normal pseudomanifold of dimension 3.

Lemma 9.2. *Let X be a 7-systolic 3-dimensional normal pseudomanifold with a cocompact action of a group G by simplicial automorphisms. Let K be as above. Let p and q be two vertices in the sphere S_k and let p' and q' be two vertices in the sphere S_{k+1} connected by an edge with p and q respectively. Then*

$$d_{S^{(1)}_{k+1}}(p', q') \leq K \cdot (d_{S^{(1)}_k}(p, q) + 1)$$

Proof: Let $p = p_0, p_1, \ldots, p_n = q$ be a geodesic in the 1-skeleton $S^{(1)}_k$. For $i = 1, 2, \ldots, n$ let p'_i be a vertex in the intersection $X_{p_{i-1} * p_i} \cap S_{k+1}$. Note that $\operatorname{diam}\big((X_{p_i} \cap S_{k+1})^{(1)}\big) \leq K$, since $X_{p_i} \cap S_{k+1} = X_{p_i,\rho}$, where $\rho = \Pi_k(p_i)$. Thus

$$d_{S^{(1)}_{k+1}}(p', q') \leq d_{S^{(1)}_{k+1}}(p', p'_1) + \Sigma_{j=1}^{n-1} d_{S^{(1)}_{k+1}}(p'_i, p'_{i+1}) + d_{S^{(1)}_{k+1}}(p'_n, q') \leq K \cdot (n + 1)$$

□

The next lemma shows that if a nonorientable complex X is as in Lemma 9.2 then there are enough many vertices with nonorientable links in X, in certain precise sense.

Lemma 9.3. *Let X be a 7-systolic nonorientable pseudomanifold of dimension 3 with a cocompact action of a group G by simplicial isometries. Let $v \in X$ be a vertex. Let $w \in S_k = S_k(v, X)$ be a vertex. Then for every positive number $\epsilon > 0$ there are a number $k' > k$ and a vertex $u \in S_{k'}$ such that the link X_u is a nonorientable surface and $\Pi_{k,k'}(u) \in B_{S_k}(w, \epsilon)$.*

Proof: Let ϵ be a positive number. By Lemma 9.1 there is a positive number R such that for all points $x \in X$ translates of a metric ball $B_X(x, R)$ under elements of G cover X, i.e. it holds $G \cdot B_X(x, R) = X$. Thus there is a positive integer N such that $G \cdot B_N(w, X) = X$ for all vertices $w \in X$ (where $B_N(w, X)$ denote the combinatorial ball).

For a natural number $l > 0$ and for $i = 0, 1, \ldots, 2N$ consider the combinatorial spheres $S_{k+l+i} = S_{k+l+i}(v, X)$. Let $u_i \in S_{k+l+i}$ be such points that $\Pi_{k+l+i+1}(u_{i+1}) = u_i$ for $i =$

$0, 1, \ldots, 2N-1$ and $\Pi_{k,k+l}(u_0) = w$. There is a vertex $u \in B_N(u_N, X)$ such that the link X_u is a nonorientable surface. Since $B_N(u_N, X) \subseteq B_{k+l+2N}(v, X) \setminus B_{k+l-1}(v, X)$, it is enough to show that for l large enough for all vertices $z \in B_N(u_N, X)$ we have $d_{S_k}(\Pi_{k,k+l+i}(z), w) < \epsilon$, where $k + l + i = d_{X^{(1)}}(v, z)$ (here we use the convention that $\Pi_{k,k} = Id_{S_k}$).

For this let z be a vertex in the intersection $B_N(u_N, X) \cap S_{k+l+i}$ for some $i = 0, 1, \ldots, 2N$. Let

$$u_0 = z_{0,1}, z_{0,2}, \ldots, z_{0,j_0}, z_{1,1}, \ldots, z_{1,j_1}, \ldots, z_{i-1,1}, \ldots, z_{i-1,j_{i-1}}, z_{i,1}, z_{i,j_i} = z$$

(for some natural numbers $j_0 \geq 1, j_1 \geq 1, \ldots, j_i \geq 1$) be a geodesic in the 1-skeleton $X^{(1)}$ satisfying $z_{m,n} \in B_N(u_N, X) \cap S_{k+l+m}(v, X)$ for $m = 0, 1, \ldots, i$ and $n = 1, 2, \ldots, j_m$ (actually all geodesics between z and u_0 have this form, since combinatorial balls are convex (see [JS, Corollary 7.5]) and thus geodesically convex (see [HS, Proposition 4.9])).

Let K be a constant as in Lemma 9.2 and let $L = \max\{K, 2N+2\}$. We will show that

$$d_{S_{k+l+i}}(z, u_i) < L^{2N+3}.$$

Using this inequality we get that

$$d_{S_k}(\Pi_{k,k+l+i}(z), w) < C^{l+i} L^{2N+3} < C^l L^{2N+3}$$

where C is a constant given by Fact 2.7. Thus for l large enough the assertion holds.

To prove the inequality, we inductively show that for all $t = 0, 1, \ldots, i$ it holds

$$d_{S^{(1)}_{k+l+t}}(z_{t,j_t}, u_t) < tL^{t+1} + 2R \quad \text{and} \quad d_{S^{(1)}_{k+l+t}}(z_{t,0}, u_t) < tL^{t+1}$$

Since z_{0,j_0} and u_0 are vertices in the intersection $B_N(u_N, X) \cap B_{k+l}(v, X)$, it follows that

$$d_{S^{(1)}_{k+l}}(z_{0,j_0}, u_0) = d_{X^{(1)}}(z_{0,j_0}, u_0) \leq 2N$$

Suppose that

$$d_{S^{(1)}_{k+l+t}}(z_{t,j_t}, u_t) < tL^{t+1} + 2N$$

By Lemma 9.2 it holds

$$d_{S^{(1)}_{k+l+t+1}}(z_{t+1,0}, u_{t+1}) < K(tL^{t+1} + 2N + 1) < L(t+1)L^{t+1} = (t+1)L^{t+2}$$

And thus

$$d_{S^{(1)}_{k+l+t+1}}(z_{t+1,j_{t+1}}, u_{t+1}) < (t+1)L^{t+2} + 2N$$

It follows that

$$d_{S^{(1)}_{k+l+i}}(z, u_i) < iL^{i+1} + 2N < L^{2N+3}$$

and the lemma follows. ☐

Lemma 9.4. *Let X and G be as in Lemma 9.3. Let $v \in X$ be a vertex. Let $w \in S_k = S_k(v, X)$ be a vertex. Then there are a number $k' > k$ and a vertex $u \in S_{k'}$ such that the link X_u is a nonorientable surface and $\Pi'_{k,k'}(u) = w$.*

Proof: Let $w' \in S_{k+1}$ be a vertex such that $\Pi_{k+1}[Res(w', S_{k+1})] = w$. Let $\epsilon > 0$ be such a number that $\epsilon + \epsilon_l \frac{1}{1-C} < 1$ for $l = 2, 3, \ldots$ (where ϵ_l are numbers given by the proof of Lemma 5.3). Due to Lemma 9.3 there are a number $k' > k+1$ and a vertex $u \in S_{k'}$ such that the link X_u is a nonorientable surface and $\Pi_{k+1,k'}(u) \in B_{S_{k+1}}(w', \epsilon)$.

Note, that if $d_{S_l}(x, y) \le \delta$, then

$$d_{S_{l-1}}(\Pi'_l(x), \Pi_l(y)) \le d_{S_{l-1}}(\Pi'_l(x), \Pi_l(x)) + d_{S_{l-1}}(\Pi_l(x), \Pi_l(y)) \le C\delta + \epsilon_l$$

It follows that

$$d_{S_{k+1}}(\Pi'_{k+1,k'}(u), \Pi_{k+1,k'}(u)) \le \epsilon_{k+1} + C\epsilon_{k+2} + \ldots + C^{k'-k-1}\epsilon_{k'} < \epsilon_{k+1}\frac{1}{1-C} < 1 - \epsilon$$

Thus $\Pi'_{k+1,k'}(u) \in B_{S_{k+1}}(w', 1) \subset Res(w', S_{k+1})$, so $\Pi_{k,k'}(u) = w$. □

Now we can prove part b) of Main Theorem.

Theorem 9.5. *Let X be a 7-systolic nonorientable pseudomanifold of dimension 3. Let a group G act cocompactly on X by simplicial automorphisms. Then the Gromov boundary $\partial_G X$ is homeomorphic to the nonorientable Pontriagin surface.*

Proof: By Sections 3, 5, 6 and 7 we can assume that the Gromov boundary $\partial_G X$ is homeomorphic to the inverse limit of a system of nonorientable surfaces satisfying assumptions 1 and 3 of Lemma 8.5. By Lemma 9.4 we can assume that assumption 2 is also satisfied. Thus the assertion holds by Lemma 8.5. □

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