Trees of manifolds and boundaries of systolic groups

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Abstract

In this paper we prove that the Pontriagin sphere and the Pontriagin nonorientable
surface occurs as the Gromov boundary of a 7-systolic group acting geometrically on
7-systolic normal pseudomanifold of dimension 3.

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manifolds

1 Introduction

k-systolic simplicial complexes ($k \geq 6$ is a natural number) were introduced by T. Januszkiewicz
and J. Świątkowski in [JS] and independently by F. Haglund in [H]. These are simplicial
analogues of metric spaces of nonpositive curvature. The idea of systolicity leads to an answer
to the question posed by M. Gromov about simple easy checkable combinatorial condition for a
simplicial complex implying hyperbolicity of this complex for the standard piecewise euclidean
metric on it. In [JS] Januszkiewicz and Świątkowski have shown that a 7-systolic simplicial
complex is hyperbolic.

Gromov boundaries of 7-systolic complexes were investigated by D. Osajda in [O]. He
showed that the ideal boundary $\partial G X$ of a 7-systolic simplicial complex $X$ is a strongly hered-
tarily aspherical compactum. He also showed that the Gromov boundary of a normal 7-systolic
pseudomanifold of finite dimension at least 3 is connected and has no local cutpoints. In this
paper we study in detail the case of such pseudomanifolds in dimension 3.

Trees of manifolds are inverse limits of certain inverse systems of manifolds. The most
common examples of such spaces are the Pontriagin sphere and the nonorientable Pontriagin
surface. Trees of manifolds were defined and investigated by W. Jakobsche (see [J]) and by P.R.
Stallings (see [S]). These spaces occur as CAT(0) boundaries of right-angled Coxeter groups
(see [F]). In the case when these groups are hyperbolic their CAT(0) boundaries coincide with
their Gromov boundaries.

The main result of this paper is:

Main Theorem. Let $X$ be a 7-systolic normal pseudomanifold of dimension 3. Let a group
$G$ act geometrically on $X$. Then:

a) (Theorem 7.2 in the text) if $X$ is orientable, then $\partial G X$ is homeomorphic to the Pontriagin
sphere,

b) (Theorem 9.5 in the text) if $X$ is nonorientable, then $\partial G X$ is homeomorphic to the nonori-
entable Pontriagin surface.
This paper is organized as follows. In Section 2 we recall some terminology related to simplicial complexes and systolic complexes. We also recall some facts about systolic complexes. In Section 3 we thoroughly examine properties of combinatorial spheres $S_n$ in 3-dimensional 7-systolic normal pseudomanifolds and properties of natural projections $\Pi_n : S_n \to S_{n-1}$ between them. D. Osajda showed that in the case of a locally finite 7-systolic simplicial complex $X$ of finite dimension the inverse limit $\lim\limits_{\leftarrow} (S_n, \Pi_n)$ of the system of these spheres and projections is homeomorphic to the Gromov boundary $\partial_G X$. In our case, we show that every sphere $S_n$ is a surface. Moreover, we show that up to a homeomorphism the sphere $S_{n+1}$ is a connected sum of $S_n$ and links of vertices $w \in S_n$. In Section 4 we recall results of Jakobsche from [J] on inverse systems of compact orientable manifolds. The proof of the first statement of Main Theorem is contained in Sections 5, 6 and 7. In Section 5 we modify the maps $\Pi_n : S_n \to S_{n-1}$ (without changing the inverse limit $\lim\limits_{\leftarrow} (S_n, \Pi_n)$). These maps become injective on some appropriate parts of domains, which is one of the conditions in the definition of a Jakobsche inverse system (which is in turn an object used to define a tree of manifolds). Properties of such modified maps $\Pi'_n : S_n \to S_{n-1}$ allow us, in Section 6, to further modify the inverse system $(S_n, \Pi'_n)$. We call this modifications a refinement. Every element of the refined system is a connected sum of its predecessor and some finite number of tori. This is one of the conditions in the definition of the Pontriagin sphere. In Section 7 we define families $D_{n,k}$ of pairwise disjoint discs in surfaces $S_{n,k}$, which turns the refined system $(S_{n,k}, \Pi'_{n,k})$ into a Jakobsche inverse system of tori, thus finishing the proof of part a) of Main Theorem. In Section 8 we examine properties of trees of nonorientable surfaces. In Section 9 we prove the second statement of Main Theorem.

2 Definitions and properties of systolic complexes

In this section we recall the notion of a systolic complex and some of its basic properties.

Let $X$ be a simplicial complex and let $\sigma \subset X$ be a simplex. The link of $X$ at $\sigma$ (denoted by $X_\sigma$) is the subcomplex of $X$ consisting of all simplices disjoint with $\sigma$ and spanning together with $\sigma$ a simplex in $X$. The residuum of $\sigma$ in $X$ (denoted by $\text{Res}(\sigma, X)$) is the union of all simplices in $X$ that contain $\sigma$.

For simplices $\sigma_1$ and $\sigma_2$ in $X$ we denote by $\sigma_1 * \sigma_2$ the simplicial join of $\sigma_1$ and $\sigma_2$ (if it exists); this means that $\sigma_1$ and $\sigma_2$ are disjoint and $\sigma_1 * \sigma_2$ is the smallest simplex in $X$ containing both of them. $X$ is flag if every set of vertices $v_1, v_2, \ldots, v_n \in X$ pairwise connected by edges in $X$ spans a simplex $v_1 * v_2 * \ldots * v_n$ in $X$. A subcomplex $K \subset X$ is full if for every set of vertices $v_1, v_2, \ldots, v_n \in K$ spanning a simplex $v_1 * v_2 * \ldots * v_n$ in $X$ this simplex is a simplex in $K$.

A simplicial complex $X$ is a pseudomanifold of dimension $n$ if it is locally finite, it is a union of its $n$-simplices and each $(n-1)$-simplex is contained in exactly two $n$-simplices. A pseudomanifold is orientable if it admits a choice of orientations on top-dimensional simplices in a consistent way, i.e. such that the orientations on each simplex of codimension 1 inherited from two top-dimensional simplices containing it are opposite. An $n$-dimensional pseudomanifold is normal if for every nonempty simplex $\sigma$ in $X$ of dimension $\dim(\sigma) < n - 1$ the link $X_\sigma$ is connected.

**Remark 2.1.** Note that if a pseudomanifold is orientable then all its links are also orientable. The converse is not true in general. However for a simply-connected normal pseudomanifolds of dimension 3 its orientability is equivalent to the orientability of its vertex links.
A cycle in $X$ is a subcomplex $\gamma \subset X$ isomorphic to some triangulation of the circle $S^1$. The length of a cycle $\gamma$ (denoted by $|\gamma|$) is the number of its 1-simplices.

**Definition 2.2.**

1. Let $X$ be a flag simplicial complex and let $k \geq 4$ be a natural number.
   - $X$ is $k$-large if every cycle $\gamma$ in $X$ of length $3 < |\gamma| < k$ is not full in $X$.
   - $X$ is locally $k$-large if for every nonempty simplex $\sigma$ in $X$ the link $X_\sigma$ is $k$-large.
   - $X$ is $k$-systolic if it is connected, simply-connected and locally $k$-large.

2. A group $G$ is $k$-systolic if it acts geometrically (i.e. properly discontinuously and cocompactly) by simplicial automorphisms on some $k$-systolic simplicial complex $X$.

For a brevity a 6-systolic complex or a group is called systolic.

**Remark 2.3.** Note that a full subcomplex of a $k$-large simplicial complex is $k$-large itself.

Now we recall some basic facts about systolic complexes. For proofs see [JS] and [O]. We start with the theorem relating the notions of systolicity and Gromov hyperbolicity.

**Theorem 2.4.** [JS, Theorem 2.1] The 1-skeleton of a 7-systolic simplicial complex is hyperbolic.

For a subset $A \subset X$ which is a union of some simplices in $X$ we denote by $\text{span}_X(A)$ the full subcomplex of $X$ spanned on $A$ (i.e. the intersection of all full subcomplexes of $X$ containing $A$). Now we recall the definition of combinatorial balls and spheres in a simplicial complex $X$ centred at a simplex $\sigma \subset X$:

- $B_0(\sigma, X) = \sigma$, $B_{n+1}(\sigma, X) = \text{span}_X\left(\{\tau \subset X : \tau \cap B_n(\sigma, X) \neq \emptyset\}\right)$,
- $S_n(\sigma, X) = \text{span}_X\left(\{w \in X^{(0)} : d(w, \sigma) = n\}\right)$, where $d(w, \sigma)$ denote the distance in the 1-skeleton $X^{(1)}$.

In the following proposition we recall some natural properties of balls and spheres in systolic complexes.

**Fact 2.5.** [JS, Lemma 7.7] Let $X$ be a systolic simplicial complex and let $v \in X$ be a vertex. Then for every natural number $n > 0$ and for every simplex $\tau \subset S_n(v, X)$ the intersection $\rho = B_{n-1}(v, X) \cap X_\tau$ is a single simplex. Moreover, the intersection $X_\tau \cap B_n(v, X)$ is equal to the ball $B_1(\rho, X_\tau)$ and the intersection $X_\tau \cap S_n(v, X)$ is equal to the sphere $S_1(\rho, X_\tau)$.

Let $b_\tau$ denote the barycenter of a simplex $\tau$ and let $X'$ denote the first barycentric subdivision of a simplicial complex $X$. We view the barycenters $b_\tau$ of simplices $\tau \subset X$ as the vertices of $X'$. The combinatorial properties of balls and spheres mentioned in Fact 2.5 are crucial in the definition of projections

$$\Pi_n : S_n(\sigma, X) \to [S_{n-1}(\sigma, X)]'$$

that we recall now.

For a systolic complex $X$ and a simplex $\sigma \subset X$ let $S_n$ denote the sphere $S_n(\sigma, X)$ and let $B_n$ denote the ball $B_n(\sigma, X)$. Let $Y^{(0)}$ denote the 0-skeleton of $Y$, i.e. the vertex set of a simplicial complex $Y$. Define the map

$$\Pi_n : S_n^{(0)} \to (S_{n-1})^{(0)}$$
by the equalities \( \Pi_n(v) = b_\tau \) for all vertices \( v \in S_n^{(0)} \) (where the simplex \( \tau \) is the intersection \( B_{n-1} \cap X_v \)).

Spheres and balls in 7-systolic complexes have stronger properties than these recalled above. The following fact allows to extend the map \( \Pi_n : S_n^{(0)} \to (S'_{n-1})^{(0)} \) to a simplicial map

\[
\Pi_n : S_n \to S'_{n-1}
\]

**Fact 2.6.** [O, Lemma 3.1] If \( X \) is 7-systolic then, for any vertices \( v_1, v_2 \in S_n \) connected by an edge in \( S_n \), their images \( \Pi_n(v_1) \) and \( \Pi_n(v_2) \) are contained in one simplex in \( S'_{n-1} \), i.e. \( \Pi_n(v_1) \) and \( \Pi_n(v_2) \) are equal or they span a 1-simplex in \( S'_{n-1} \).

Define the map \( \Pi_n : S_n \to S'_{n-1} \) as a simplicial extension of the map \( \Pi_n : S_n^{(0)} \to (S'_{n-1})^{(0)} \) defined above.

We make now a comment about the notation used in this paper. We use the same symbol \( \Pi_n \) for the simplicial map \( \Pi_n : S_n \to S'_{n-1} \) and for the related continuous map \( \Pi_n : S_n \to S_{n-1} \) (when we forget the simplicial structure and treat the complexes \( S_n \) and \( S'_{n-1} \) just as metric spaces). For example this is the case in the following fact describing metric properties of the maps \( \Pi_n \). We denote by \( d_X \) the standard piecewise euclidean metric on \( X \).

**Fact 2.7.** [O, Lemma 3.3] Let \( X \) be a 7-systolic complex with finite dimension. Then there is a positive constant \( C < 1 \) depending only on the dimension \( \dim(X) \) such that for all natural numbers \( n \) and for all points \( x, y \in S_n \) it holds \( d_{S_{n-1}}(\Pi_n(x), \Pi_n(y)) \leq C \cdot d_X(x, y) \).

In the case of 7-sytoic 3-dimensional pseudomanifolds, combinatorial properties of the inverse system \((S_n, \Pi_n)\) of spheres and projections will be thoroughly examined and described more precisely in Section 3. The next theorem shows that this system can be used to describe the Gromov boundary of a 7-systolic complex \( X \).

**Theorem 2.8.** [O, Lemma 4.1] Let \( X \) be a 7-systolic locally finite simplicial complex of finite dimension. For a vertex \( v \in X \) let \( S_n(v, X) \) denote the sphere \( S_n(v, X) \) and let the maps \( \Pi_n : S_n \to S_{n-1} \) be defined as before. Then the inverse limit \( \lim \leftarrow (S_n, \Pi_n) \) is homeomorphic to the Gromov boundary of \( X \).

### 3 Spheres and projections in 7-systolic normal pseudomanifolds of dimension 3

In this section \( X \) is a 7-systolic, normal pseudomanifold of dimension 3. We thoroughly examine properties of combinatorial spheres \( S_n \) in such pseudomanifolds and of the projections \( \Pi_n \) defined in Section 2.

In Lemmas 3.1, 3.2 and 3.3 we describe links of \( X \) at simplices \( \sigma \) of the dimensions 2, 1 and 0 respectively.

**Lemma 3.1.** Let \( \sigma \subset X \) be a 2-simplex. Then the link \( X_\sigma \) consists of two vertices.

**Proof:** The simplex \( \sigma \) is contained in exactly two simplices of dimension 3.

**Lemma 3.2.** Let \( \epsilon \subset X \) be a 1-simplex (i.e. an edge). Then the link \( X_\epsilon \) is a 7-large triangulation of the circle \( S^1 \) (i.e. a triangulation of the circle consisting of at least 7 edges).
Proof: Let $\varepsilon$ be a join $v_1 \ast v_2$. Let $v \in X_\varepsilon$ be a vertex. There are exactly two vertices $u, w \in X_\varepsilon$ adjacent to $v$ (since the join $v \ast v_1 \ast v_2$ is a 2-simplex lying in exactly two 3-simplices). Thus, since $X$ is locally finite, it follows that the link $X_\varepsilon$ is a disjoint union of copies of triangulated circles. But since $X$ is normal it follows that the link $X_\varepsilon$ is connected, so there must be exactly one copy. Since $X$ is locally 7-large, it follows that this triangulation of the link $X_\varepsilon$ must be 7-large. \hfill \Box

Lemma 3.3. Let $v \in X$ be a 0-simplex (i.e. a vertex). Then the link $X_v$ is topologically a closed connected surface triangulated in a 7-large way. Moreover, if $X$ is orientable then the link $X_v$ is also orientable.

Proof: For a vertex $w \in X_v$ we have the equality $(X_v)_w = X_v \ast w$. Thus the link $(X_v)_w$ is a triangulated circle (see Lemma 3.2). A simplicial complex whose every vertex link is a triangulated circle is itself a triangulated surface. Since $X$ is 7-systolic, it follows that this triangulation is 7-large. The connectedness of the link $X_v$ follows from the normality of $X$. The last assertion follows from Remark 2.1. \hfill \Box

Combinatorial properties of 7-large complexes imply the following:

Remark 3.4. Let $\Sigma$ be a 7-large triangulated closed surface and let $\sigma \subset \Sigma$ be a simplex. Then:

1. the balls $B_1(\sigma, \Sigma)$ and $B_2(\sigma, \Sigma)$ are triangulated 2-discs; moreover, topological boundaries of these balls in $\Sigma$ are spheres $S_1(\sigma, \Sigma)$ and $S_2(\sigma, \Sigma)$ respectively,

2. the ball $B_3(\sigma, \Sigma)$ can contain a loop in the 1-skeleton $\Sigma^{(1)}$ homotopically nontrivial in $\Sigma$ (if the simplex $\sigma$ has dimension greater than 0).

In the next lemma we describe combinatorial and topological properties of spheres $S_n(v, X)$.

Lemma 3.5. Let $v \in X$ be a vertex and let $S_n = S_n(v, X)$ be the combinatorial sphere of radius $n$ centered at $v$. Then $S_n$ is a connected surface triangulated in a 7-large way.

Proof: For $n = 1$ we have the equality $S_1 = X_v$. Thus the assertion follows from Lemma 3.3.

Let $w \in S_n$ be a vertex and let $\rho$ denote the intersection $X_w \cap S_{n-1}$. Since $X_w$ is a 7-large surface and $\rho$ is a simplex (see Fact 2.5), by Remark 3.4 it follows that the ball $B_1(\rho, X_w)$ is a triangulated 2-disc. By the equalities $(S_n)_w = X_w \cap S_n = S_1(\rho, X_w) = \text{bd}(B_1(\rho, X_w)) = S^1$

(see Fact 2.5) it follows that vertex links of $S_n$ are triangulated circles. Thus $S_n$ is a triangulated surface.

Since $S_n$ is full in $X$ (by definition) and $X$ is 7-large, it follows that this triangulation of $S_n$ is 7-large (see Remark 2.3).

Connectedness of $S_n$ can be shown using inductive argument and Corollary 3.18 below. \hfill \Box

Lemmas 3.6, 3.7, 3.8 and 3.9 describe local properties of the projections $\Pi_{n+1} : S_{n+1} \to S_n$.

Lemma 3.6. For a 2-simplex $\sigma \subset S_n$ there is exactly one vertex $w_\sigma \in S_{n+1}$ such that the join $w_\sigma \ast \sigma$ is a simplex in $X$. This vertex coincides with the preimage $\Pi_{n+1}^{-1}(b_\sigma)$. 

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Proof: By Fact 2.5, the intersection $X_\sigma \cap S_{n-1}$ is a single simplex. Thus, for dimensional reasons, it is a vertex. By Lemma 3.1, the link $X_\sigma$ consists of two vertices. Moreover, the intersection $X_\sigma \cap S_n$ is empty, since $S_n$ is a surface and a full subcomplex. It follows that the intersection $X_\sigma \cap S_{n+1}$ must be equal to the other vertex of $X_\sigma$. Denote this vertex by $w_\sigma$. From definition of projections it is easy to see that $\Pi^{-1}_{n+1}[b_\sigma] = X_\sigma \cap S_{n+1}$ ($\sigma$ is a 2-simplex). It follows that $\Pi^{-1}_{n+1}[b_\sigma] = w_\sigma$. 

\[ \square \]

**Lemma 3.7.** For an edge $\varepsilon \subset S_n$ the intersection $\alpha_\varepsilon = X_\varepsilon \cap S_{n+1}$ is an arc (triangulated). If $\sigma_1$ and $\sigma_2$ are two $2$-simplices in $S_n$ containing $\varepsilon$, then the endpoints of this arc coincide with the preimage vertices $\Pi^{-1}_{n+1}(b_{\sigma_1})$ and $\Pi^{-1}_{n+1}(b_{\sigma_2})$.

Proof: Since $S_n$ is a surface, it follows that there are exactly two 2-simplices in $S_n$ (say $\sigma_1 = v_1^* \varepsilon$ and $\sigma_2 = v_2^* \varepsilon$ that contain $\varepsilon$. For these two simplices there are two vertices $w_{\sigma_1}$ and $w_{\sigma_2}$ in $S_{n+1}$ such that for $i = 1, 2$ the joins $w_{\sigma_i} \star \varepsilon_i$ are simplices in $X$.

First we show that $w_{\sigma_1}$ and $w_{\sigma_2}$ do not lie in a common simplex in $X$. To see this suppose that the join $w_{\sigma_1} \star w_{\sigma_2}$ is a simplex in $X$. By Fact 2.6 it follows that the images $\Pi_{n+1}(w_{\sigma_1})$ and $\Pi_{n+1}(w_{\sigma_2})$ lie in a common simplex in the barycentric subdivision $S'_n$. Now $\Pi_{n+1}$ maps the vertex $w_{\sigma_i}$ to the barycenter $b_{\sigma_i}$ for $i = 1, 2$. But the barycenters $b_{\sigma_1}$ and $b_{\sigma_2}$ do not span a simplex in $S'_n$, a contradiction.

Now for $i = 1, 2$ let a vertex $u_i$ be the intersection $X_{\sigma_i} \cap S_{n-1}$. Note that since $u_1$ and $u_2$ belong to the intersection $X_\varepsilon \cap S_{n-1}$, they are equal or span a simplex in $S_{n-1}$ (see Fact 2.5). Since the link $X_\varepsilon$ is a triangulated circle, and $u_1, u_2, v_1, v_2$ are all vertices of the link $X_\varepsilon$ lying in the ball $B_n(v, X)$, it follows that the vertices $w_{\sigma_1}$ and $w_{\sigma_2}$ are connected by an arc $\alpha_\varepsilon = (w_{\sigma_1} = w_0, w_1, \ldots, w_m = w_{\sigma_2})$ in $S_{n+1}$ (for some $m > 1$). Lemma 3.6 implies that the vertices $w_{\sigma_i}$ are exactly the preimages $\Pi^{-1}_{n+1}(b_{\sigma_i})$ for $i = 1, 2$. This finishes the proof. 

\[ \square \]

**Lemma 3.8.** Let $\varepsilon \subset S_n$ be an edge, let $\sigma_1$ and $\sigma_2$ be two different 2-simplices in $S_n$ containing $\varepsilon$ and let $\alpha_\varepsilon = (w_0, w_1, \ldots, w_m)$ be the arc in $S_{n+1}$ given by Lemma 3.7. Then the projection $\Pi_{n+1}$ maps edges $w_0 \star w_1$ and $w_{m-1} \star w_m$ homeomorphically onto edges $b_{\sigma_1} \star \varepsilon$ and $b_{\sigma_2} \star \varepsilon$ in $S'_n$ respectively, and collapses the subarc $(w_1, w_2, \ldots, w_{m-2}, w_{m-1})$ to the barycenter $b_{\varepsilon}$.

Proof: By Lemma 3.7, the projection $\Pi_{n+1}$ maps $w_0$ to the barycenter $b_{\sigma_1}$ and $w_m$ to the barycenter $b_{\sigma_2}$. We show that $\Pi_{n+1}$ maps $w_i$ to the barycenter $b_{\varepsilon}$ for $i = 1, 2, \ldots, m - 1$. It is enough to show that the intersection $X_{w_i} \cap S_n$ is exactly equal to the edge $\varepsilon$.

For this note that $w_i$ and $\varepsilon$ span a simplex in $X$. Thus $\varepsilon$ is a simplex in the intersection $X_{w_i} \cap S_n$. If this intersection contains a vertex $u$ not contained in $\varepsilon$, it follows that $u$ and $\varepsilon$ span a simplex in $S_n$. But 2-simplices in $S_n$ containing $\varepsilon$ are exactly $\sigma_1$ and $\sigma_2$. It follows that $w_i$ is equal to $w_0$ or to $w_m$, a contradiction. 

\[ \square \]

**Lemma 3.9.** Let $w \in S_n$ be a vertex. Then there exists a cycle (i.e. a triangulated circle) $\alpha_w$ in the 1-skeleton of $X_w \cap S_{n+1}$ such that the image $\Pi_{n+1}[\alpha_w]$ is equal to the sphere $S_1(w, S'_n)$ (which is a cycle in the barycentric subdivision $S'_n$) and the preimage $\Pi^{-1}_{n+1}[S_1(w, S'_n)]$ is equal to $\alpha_w$. 

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Proof: The sphere $S_n$ is a triangulated surface, so the residuum $\text{Res}(w, S_n)$ is a triangulated 2-disc. Let this residuum consist of 2-simplices $\sigma_i = w \ast w_i \ast w_{i+1}$ for $i = 0, 1, \ldots, k - 1$, where $k = |X_w \cap S_n| \geq 7$ is the length of the link $(S_n \ast w)$ (indices taken modulo $k$). Let $\alpha_i = X_{w_i \ast w_{i+1}} \cap S_{n+1}$ be the arc in $S_{n+1}$ given by Lemma 3.7. Let $\alpha_w$ be the union $\alpha_0 \cup \alpha_1 \cup \ldots \cup \alpha_{k-1}$. We claim that $\alpha_w$ is a cycle.

It is enough to show that the intersection $\alpha_i \cap \alpha_j$ is nonempty only for $|i - j| \leq 1$ and moreover, for $|i - j| = 1$ it consists of one point. For this suppose that the intersection $\alpha_i \cap \alpha_j$ is nonempty for some $i < j \in \{0, 1, \ldots, k-1\}$ and let $u \in \alpha_i \cap \alpha_j$ be a vertex. Since the arcs $\alpha_i$ and $\alpha_j$ are contained in the links $X_{w_i \ast w_j}$ and $X_{w_{i+1} \ast w_{j+1}}$ respectively, it follows that simplices $w \ast w_i$ and $w \ast w_{j+1}$ are contained in the intersection $X_u \ast S_n$. By Fact 2.5 the join $w \ast w_i \ast w_{j+1}$ is a 2-simplex in $S_n \ast X_u$. It follows that $j$ is equal to $i + 1$. Since $\alpha_w$ is connected (the intersection $\alpha_i \cap \alpha_{i+1}$ is exactly the single vertex equal to the intersection $X_{\sigma_i} \cap S_{n+1}$), it must be a cycle.

By Lemma 3.7 and the definition of the cycle $\alpha_w$, it follows that $\Pi_{n+1}$ maps $\alpha_w$ onto $S_1(\alpha, \alpha')$. Since the preimage $\Pi_{n+1}^{-1}[B_1(\alpha, \alpha')]$ is contained in the intersection $X_\alpha \ast S_n$, it is enough to show that for all vertices $u \in X_\alpha \ast S_n$ not contained in the cycle $\alpha_w$ the projection $\Pi_{n+1}$ maps $u$ to $w$. We show that the intersection $X_u \ast S_n$ is equal to $w$. For this suppose that there is another vertex, say $w'$, lying in the intersection $X_u \ast S_n$. It follows that $w'$ is equal to a vertex $w_i$ for some $i = 0, 1, \ldots, k-1$. Thus $u$ lies in the arc $\alpha_i$, a contradiction. This finishes the proof.

From the proof of Lemma 3.9 we get the following additional information:

**Fact 3.10.** Let $w \in S_n$ be a vertex and let $\{\epsilon_i : i = 1, 2, \ldots, k\}$ be the set of all edges in $S_n$ that contain $w$. Then the cycle $\alpha_w$ is equal to the union $\bigcup_{i=1}^k \alpha_{\epsilon_i}$.

In the next lemma we show that the cycle $\alpha_w$ given by Lemma 3.9 bounds some 2-disc $D_w \subset X_w \ast B_{n+1}$.

**Lemma 3.11.** Each cycle $\alpha_w$ bounds a 2-disc $D_w = B_2(\sigma_w, X_w)$ in the intersection $X_w \ast B_{n+1}$, for some simplex $\sigma_w \subset X_w$.

**Proof:** For a vertex $w \in S_n$ giving the arc $\alpha_w$ let $\sigma_w$ be the intersection $X_w \ast S_{n-1}$ (this intersection is a single simplex). We show that the cycle $\alpha_w$ is equal to the sphere $S_2(\sigma_w, X_w)$. It is obvious that $\alpha_w$ is contained in $S_2(\sigma_w, X_w)$. For the opposite inclusion let $u \in S_2(\sigma_w, X_w)$ be a vertex. There is a vertex $u' \in S_n \ast X_w$ connected by edges with $u$ and with some vertex of $\sigma_w$. It follows that $u$ is a vertex in the arc $\alpha_{w \ast u'}$. Thus, by Fact 3.10, $u$ is a vertex in $\alpha_w$.

By Remark 3.4, the cycle $\alpha_w$ is the boundary of the ball $B_2(\sigma_w, X_w)$. Since the link $X_w$ is a surface (triangulated in a 7-large way), it follows that the ball $B_2(\sigma_w, X_w)$ is a 2-disc (see Remark 3.4 again). This finishes the proof.

For a vertex $w \in S_n$ let $P_w$ denote the closure $cl(X_w \setminus D_w)$. Clearly, we have the following:

**Fact 3.12.** The set $P_w$ is a subcomplex of $S_{n+1}$. Topologically it is a connected surface with the boundary $\alpha_w$.

The next lemma describes the map $\Pi_{n+1}$ restricted to the subcomplex $P_w \subset S_{n+1}$.

**Lemma 3.13.** For every vertex $w \in S_n$ the projection $\Pi_{n+1} : S_{n+1} \rightarrow S_n$ maps the subcomplex $P_w$ onto the ball $B_1(\alpha, \alpha')$. Moreover, the preimage $\Pi_{n+1}^{-1}[w]$ is the union of simplices in $P_w$ disjoint with the cycle $\alpha_w$. 

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Before proving Lemma 3.13 note the following:

Remark 3.14.  

- The ball $B_1(w, S'_n)$ is topologically a 2-disc with the boundary $S_1(w, S'_n)$.
- Lemma 3.13 together with previous results (Lemmas and Facts 3.8-3.12) fully describe the restricted map $\Pi_{n+1}[P_w]$.

Proof of Lemma 3.13: By Lemma 3.9, the preimage $\Pi_{n+1}^{-1}[S_1(w, S'_n)]$ is equal to the cycle $\alpha_w$. Let $u \in P_w$ be a vertex not contained in the cycle $\alpha_w$. Since $P_w$ is a subcomplex of the link $X_u$, it follows that the vertices $w$ and $u$ span an edge in $X$. We show that the intersection $X_u \cap S_n$ is equal exactly to the vertex $w$. It follows that $\Pi_{n+1}$ maps $u$ to $w$. It is enough to show that the dimension $\dim(X_u \cap S_n)$ is equal to 0 (since $w$ a vertex in this intersection, which is a single simplex).

Assume the opposite and let $\sigma$ be an intersection $X_u \cap S_n$. It follows that $\Pi_{n+1}$ maps $u$ to the barycenter $b_\sigma$. Since $\sigma$ contain $w$ and has the dimension at least 1, it follows that the barycenter $b_\sigma$ is contained in the sphere $S_1(w, S'_n)$. Thus $u$ lies in the preimage $\Pi_{n+1}^{-1}[S_1(w, S'_n)]$. This contradicts the equality $\Pi_{n+1}^{-1}[S_1(w, S'_n)] = \alpha_w$.

For better understanding of the map $\Pi_{n+1}$ we introduce another cell structure on the sphere $S_n$. We call this cell structure dual.

- The set of dual 0-cells (denoted by $e^0_\sigma$) consists of the barycenters $b_\sigma$ of all 2-simplices $\sigma \subset S_n$.
- The set of dual 1-cells (denoted by $e^1_\sigma$) consists of the unions $b_{\sigma_1} \ast b_\varepsilon \cup b_{\sigma_2} \ast b_\varepsilon$, where $\varepsilon$ is an edge in $S_n$ while $\sigma_1$ and $\sigma_2$ are the two 2-simplices in $S_n$ containing $\varepsilon$.
- The set of dual 2-cells (denoted by $e^2_w$) consists of the balls $B_1(w, S'_n)$ around all vertices $w \in S_n$.

We denote by $S_n^d$ the cell complex related to this cell structure, and by $(S_n^d)^{(k)}$ its $k$-skeleton, i.e. a cell subcomplex consisting of all cells of dimension at most $k$.

Using this dual cell structure, as a consequence of previous lemmas we get:

Lemma 3.15.  

1. The preimage $\Pi_{n+1}^{-1}[e^0_\sigma]$ is the vertex $w_\sigma = X_\sigma \cap S_{n+1}$.

2. The preimage $\Pi_{n+1}^{-1}[e^1_\varepsilon]$ is equal to the arc $\alpha_\varepsilon$.

3. The preimage $\Pi_{n+1}^{-1}[e^2_w]$ is equal to the subcomplex $P_w$.

Proof: Assertion 1 follows from Lemma 3.6.

By Lemma 3.8, the projection $\Pi_{n+1}$ maps the arc $\alpha_\varepsilon$ onto the dual 1-cell $e^1_\varepsilon$. By Lemma 3.7, the preimages of endpoints of the dual 1-cell $e^1_\varepsilon$ are exactly the endpoints of the arc $\alpha_\varepsilon$. By Lemma 3.9, the preimage $\Pi_{n+1}^{-1}[e^1_\varepsilon]$ is contained in the cycle $\alpha_u$ for every endpoint $u$ of $\varepsilon$. Let $u$ and $u'$ be two endpoints of the edge $\varepsilon$. Since the intersection $\alpha_u \cap \alpha_{u'}$ is equal to the arc $\alpha_\varepsilon$, we get Assertion 2.

Assertion 3 follows from Lemma 3.13.

The next lemma describes the relationship between the 1-skeleton $(S_n^d)^{(1)}$ of the dual cell structure on the sphere $S_n$ and its preimage by the map $\Pi_{n+1}$.
Lemma 3.16. The preimage \( \Pi_{n+1}^{-1}[(S_n^{d})^{(1)}] \) of the 1-skeleton of the dual cell structure is naturally homeomorphic to this 1-skeleton.

Proof: The 1-skeleton \((S_n^{d})^{(1)}\) of the dual cell structure on the sphere \(S_n\) is the union \( \bigcup e_{\varepsilon}^1 \) of 1-cells. By Lemma 3.8 and Lemma 3.15, the map \( \Pi_{n+1} \) gives one-to-one correspondence between the arcs \( \alpha_{\varepsilon} \) and the dual 1-cells \( e_{\varepsilon}^1 \). Namely, arcs \( \alpha_{\varepsilon} \) are mapped onto dual 1-cells \( e_{\varepsilon}^1 \). Moreover, this correspondence is consistent with the incidence relation, i.e. the intersection \( e_{2}^2 \cap e_{2}^w \) is nonempty if and only if the intersection \( \alpha_{u} \cap \alpha_{u}^' \) is not empty, and the same holds for triples of vertices. This finishes the proof.

\[ \square \]

Remark 3.17. • Note that the restriction of the map \( \Pi_{n+1} \) to the preimage \( \Pi_{n+1}^{-1}[(S_n^{d})^{(1)}] \) is not a homeomorphism onto \((S_n^{d})^{(1)}\). However, it can be approximated by homeomorphisms of the form described later in Lemma 5.2. More precisely, the map \( w_{\sigma} \to e_{0}^\sigma \) can be extended to a map \( S_{n+1} \to S_n \) such that every arc \( \alpha_{\varepsilon} \) is homeomorphically mapped onto the dual 1-cell \( e_{\varepsilon}^1 \). As a consequence, the cycle \( \alpha_{w} \) is mapped homeomorphically onto the boundary \( \text{bd}(e_{2}^2) \) of the dual 2-cell \( e_{2}^w \).

• The sphere \( S_{n+1} \), up to homeomorphism, can be thought of as obtained from the sphere \( S_n \) by cutting the interiors of all dual 2-cells \( e_{2}^w \) and replacing these interiors by surfaces \( P_w \) such that each boundary \( \text{bd}(P_w) = \alpha_{w} \) is glued homeomorphically to the boundary \( \text{bd}(e_{2}^2) \).

Recall that a connected sum of the manifolds \( M \) and \( N \) of dimension \( n \) (with or without boundaries) along \( n \)-discs \( D \subset \text{int}(M) \) and \( D' \subset \text{int}(N) \) is the quotient space

\[
\left( \left( M \setminus \text{int}(D) \right) \cup \left( N \setminus \text{int}(D') \right) \right) / x \sim f(x)
\]

where \( f : \text{bd}(D) \to \text{bd}(D') \) is a homeomorphism.

As a consequence of the second part of Remark 3.17 we have the following:

Corollary 3.18. The sphere \( S_{n+1} \) is topologically a connected sum of the sphere \( S_n \) and the links \( X_w \) of vertices \( w \in S_n \) along discs \( D_w \subset X_w \) and \( e_{2}^w \subset S_n \).

4 Inverse limits, Jakobsche spaces and outline of the proof of Main Theorem

In this section we recall the result of Jakobsche from [J] concerning inverse systems of appropriately iterated connected sums of compact orientable manifolds. We use this result in the next section.

Recall, that a family \( \mathcal{A} \) of subsets of a metric space \( X \) is a null family if for every positive number \( \epsilon > 0 \) only finitely many elements \( A \in \mathcal{A} \) have diameter greater than \( \epsilon \). The family \( \mathcal{A} \) is dense if the union \( \bigcup \mathcal{A} \) is a dense subset of \( X \).

Theorem 4.1. [J, Theorem 4.6] Let \( (L_0 \xrightarrow{\alpha_1} L_1 \xrightarrow{\alpha_2} L_2 \leftarrow \ldots) \) be an inverse system of connected closed orientable \( m \)-manifolds \( (m \geq 2) \) and for each \( k \geq 0 \) let \( D_k \) be a finite collection of pairwise disjoint discs in \( L_k \) such that:
1. each \( L_k \) is a connected sum of finitely many copies of \( L_0 \),

2. every map \( \alpha_{k+1} \) restricted to the preimage

\[
\alpha_{k+1}^{-1} \left[ L_k \setminus \bigcup \{ \text{int}(D) : D \in \mathcal{D}_k \} \right]
\]

is a homeomorphism onto the set

\[
L_k \setminus \bigcup \{ \text{int}(D) : D \in \mathcal{D}_k \}
\]

3. every preimage \( \alpha_{k+1}^{-1}[D] \) (for \( D \in \mathcal{D}_k \)) is homeomorphic to a copy of \( L_0 \) with the interior of a disc removed,

4. the family \( \{ \alpha_{j,i}[D] : i \geq j, D \in \mathcal{D}_i \} \) is null and dense in \( L_j \) for all \( j \),

5. the intersection \( \alpha_{j,i}[D] \cap \text{bd}(D') \) is empty for all discs \( D \in \mathcal{D}_i, D' \in \mathcal{D}_j \) and for all \( i > j \).

Then the inverse limit \( \lim_{\rightarrow} (L_0 \xleftarrow{\alpha_1} L_1 \xleftarrow{\alpha_2} L_2 \leftarrow \ldots) \) depends only on \( L_0 \).

We denote this inverse limit by \( X(L_0) \) and call it the \textit{Jakobsche space} for \( L_0 \), or the \textit{Jakobsche tree of manifolds} \( L_0 \). We call a system \((L_k, \alpha_k, \mathcal{D}_k)\) \( k \geq 0 \) satisfying assumptions 1-5 of Theorem 4.1 a \textit{Jakobsche inverse system} for \( L_0 \). If a system \((L_k, \alpha_k, \mathcal{D}_k)\) \( k \geq 0 \) satisfies assumptions 2, 4, 5 and the condition:

3a. every preimage \( \alpha_{k+1}^{-1}[D] \) (for \( D \in \mathcal{D}_k \)) is homeomorphic to a connected closed (orientable) \( m \)-manifold with the interior of a disc removed,

than we call it a \textit{Jakobsche inverse system} of (orientable) \( m \)-manifolds.

**Remark 4.2.**

1. Note that we did not state the result of Jakobsche in its full generality.

2. For \( L_0 = \mathbb{T}^2 \), the 2-dimensional torus, the space \( X(\mathbb{T}^2) \) is known as the Pontriagin sphere and denoted by \( \Pi_P \).

3. For \( m = 2 \) and \( L_0 = \Sigma_g \), the orientable surface of genus \( g > 1 \), the space \( X(\Sigma_g) \) is homeomorphic to the Pontriagin sphere. Actually the tree of orientable surfaces is homeomorphic to \( \Pi_P \). We sketch some details of this in Section 8 (see Remark 8.6 (2)).

If \( X \) is a locally finite 7-systolic simplicial complex of finite dimension, then by Theorem 2.8, the Gromov boundary \( \partial_G X \) is homeomorphic to the inverse limit \( \lim_{\rightarrow} (S_n, \Pi_n) \). The results of Section 3 imply that the inverse system \((S_n, \Pi_n)\) of spheres and projections in a 7-systolic orientable normal pseudomanifold \( X \) of dimension 3 is close to satisfy assumptions 1-5 of the Jakobsche theorem. In the next remark we make this observation more precise.

**Remark 4.3.** The maps \( \Pi_k \) are natural candidates for projections \( \alpha_k \) and the families \( \mathcal{D}_k = \left\{ e^2_w : w \in S_k^{(0)} \right\} \) of dual 2-cells in the spheres \( S_k \) are natural candidates for families of discs as in a Jakobsche inverse system. More precisely, Fact 2.7 implies that for such a choice of families \( \mathcal{D}_k \) the family \( \{ \Pi_{j,i}[D] : i \geq j, D \in \mathcal{D}_i \} \) is null in every sphere \( S_j \). Moreover, since the union \( \bigcup \mathcal{D}_j \) covers the sphere \( S_j \), it follows that the families \( \{ \Pi_{j,i}[D] : i \geq j, D \in \mathcal{D}_i \} \) are dense in

\footnote{For \( i > j \) we denote by \( \alpha_{j,j} \) the composition \( \alpha_{j+1} \circ \ldots \circ \alpha_1 \), whereas \( \alpha_{i,i} \) denotes the identity on \( L_i \).}
every sphere \( S_j \). If the links of all vertices of \( X \) are triangulations of the same surface \( \Sigma_0 \), then assumptions 1 and 3 are satisfied with \( L_0 = \Sigma_0 \) by Lemma 3.15 (3).

On the other hand, the maps \( \Pi_k \) and the families \( \mathcal{D}_k \) defined as above fail to satisfy some other assumptions of the Jakobsche theorem. In particular:

- elements of so defined families \( \mathcal{D}_k \) are not pairwise disjoint,
- even though the projection \( \Pi_{k+1} \) maps the preimage
  \[
  \Pi_{k+1}^{-1} \left[ S_k \setminus \bigcup \{ \text{int}(e^2_w) : w \in S_k^{(0)} \} \right]
  \]
  onto the set
  \[
  S_k \setminus \bigcup \{ \text{int}(e^2_w) : w \in S_k^{(0)} \}
  \]
  the restriction of \( \Pi_{k+1} \) to this preimage is not a homeomorphism, and
- assumption 5 of Theorem 4.1 fails.

The strategy of the proof of part a) of Main Theorem is as follows. In Section 5 we modify the inverse system \((S_n, \Pi_n)\), without affecting the inverse limit, by changing appropriately the bonding maps. This modification will make the inverse system satisfy assumptions 2, 4 and 5 of Theorem 4.1 (after choosing appropriately the families of discs). The modified inverse system \((S_n, \Pi'_n)\) (with families of discs chosen appropriately) will be a Jakobsche inverse system of orientable surfaces. In Section 6 we refine this new system without changing the inverse limit either. The refined system will consist of orientable surfaces \( S_{n,k} \) for \( k = 0, 1, \ldots, g_n \) (for some natural numbers \( g_n \)) and maps \( \Pi'_{n,k+1-k} : S_{n,k+1} \to S_{n,k} \) satisfying \( S_n, 0 = S_n, S_{n,g_n} = S_{n+1} \) and \( \Pi'_{n,1-0} \circ \Pi'_{n,2-1} \circ \ldots \circ \Pi'_{n,g_n-1} = \Pi'_{n+1}. \) The refinement is necessary to get the connected sum with tori, rather than with higher genera surfaces. In Section 7 we define the family of discs in every surface of the refined system to match all the assumptions of Theorem 4.1.

5 Modification of the inverse system

In this section we modify the inverse system \((S_n, \Pi_n)\) described in Section 3. Actually, we modify only the projections \( \Pi_n : S_n \to S_{n-1} \) leaving the spaces \( S_n \) unchanged. This modification will be small enough so that it does not change the inverse limit. The new inverse system \((S_n, \Pi'_n)\) will satisfy the following conditions:

- each of the modified projections \( \Pi'_{n+1} \) maps the preimage \( \Pi_{n+1}^{-1}[(S_n^{d}(1))] \) homeomorphically onto the 1-skeleton \( (S_n^{d}(1)) \) of the dual cell structure on the sphere \( S_n \),
- for every vertex \( w \in S_n \) the projection \( \Pi'_{n+1} \) maps some canonical open neighbourhood \( U_w \) of the cycle \( \alpha_w \) in \( P_w \) homeomorphically onto the 2-cell \( e^2_w \) with the point \( w \) removed, and collapses the complement \( P_w \setminus U_w \) to \( w \).

We denote by \( d_{sup} \) the uniform metric on the set of continuous maps between two compact spaces. We perform small (with respect to the uniform distance) modifications of the maps \( \Pi_n \) keeping the inverse limit unchanged. To do this we use the following result due to M. Brown.
Theorem 5.1. [B, Theorem 2] There is an assignment of positive real numbers
\[ a(s_1, s_2, \ldots, s_{k-1}, t_1, t_2, \ldots, t_{k-1}, t_k) \]
to pairs of finite sequences
\[(X_0 \leftarrow s_1, X_1 \leftarrow s_2 \ldots \leftarrow s_{k-1}, X_{k-1}) \text{ and } (X_0 \leftarrow t_1, X_1 \leftarrow t_2 \ldots \leftarrow t_{k-1}, X_k)\]
of continuous maps between compact metric spaces, for all integer \( k \), such that the following holds: if two inverse systems \( (Y_0 \xrightarrow{\alpha_1} Y_1 \xrightarrow{\alpha_2} \ldots) \) and \( (Y_0 \xrightarrow{\beta_1} Y_1 \xrightarrow{\beta_2} \ldots) \) satisfy the inequalities
\[ d_{\sup}(\alpha_k, \beta_k) < a(\alpha_1, \alpha_2, \ldots, \alpha_{k-1}, \beta_1, \beta_2, \ldots, \beta_{k-1}, \beta_k) \]
for all \( k \), then the inverse limits \( \lim_{\sup} (Y_0 \xrightarrow{\alpha_1} Y_1 \xrightarrow{\alpha_2} \ldots) \) and \( \lim_{\sup} (Y_0 \xrightarrow{\beta_1} Y_1 \xrightarrow{\beta_2} \ldots) \) are homeomorphic.

The next lemma shows that it is possible to approximate the projections \( \Pi_{n+1} : S_{n+1} \to S_n \) arbitrarily close by maps \( \Pi_{n+1, \epsilon} : S_{n+1} \to S_n \) having much better properties (from the point of view of fulfilling the requirements of Jakobsche inverse system).

Lemma 5.2. For any number \( \epsilon > 0 \) and any integer \( n > 0 \) there is a continuous map \( \Pi_{n+1, \epsilon} : S_{n+1} \to S_n \) satisfying the following:

1. \( d_{\sup}(\Pi_{n+1}, \Pi_{n+1, \epsilon}) < \epsilon \),
2. \( \Pi_{n+1}^{-1}[w] = (\Pi_{n+1, \epsilon})^{-1}[w] = (X_w \setminus B_3(\sigma_w, X_w)) \cup S_3(\sigma_w, X_w) \) for all vertices \( w \in S_n^{(0)} \) (where \( \sigma_w \) is the intersection \( X_w \cap S_{n-1} \)),
3. the restriction of the map \( \Pi_{n+1, \epsilon} \) to the set
   \[ S_{n+1} \setminus \bigcup \{(\Pi_{n+1, \epsilon})^{-1}[w] : w \in S_n^{(0)}\} \]
is a homeomorphism onto the set
   \[ S_n \setminus \{w : w \in S_n^{(0)}\} \],
4. \( \Pi_{n+1}[S_n^{(0)}] \subseteq \Pi_{n+1, \epsilon}[S_n^{(0)}] \).

Proof:
Let \( w \in S_n \) be a vertex. Let \( l_w \) denote the number of 2-simplices in \( S_n \) that contain \( w \). For
\[ i = 0, 1, \ldots, l_w - 1 \]
let \( \sigma_i = w * w_i * w_{i+1} \) be all these 2-simplices (indices taken modulo \( l_w \)).

Consider the cycle \( \alpha_w \subset S_{n+1} \) as described in Lemma 3.9. Denote vertices of \( \alpha_w \) in the following way:
\[ w_0, w_1, \ldots, w_{k_0}, w_1, w_2, \ldots, w_k, \ldots, w_{l_w-1}, w_{l_w-1} \]
(for some natural numbers \( k_0 > 1, \ldots, k_{l_w-1} > 1 \)). We choose the indices in such a way that
\[ \Pi_{n+1}(w_{i,0}) = b_{\sigma_i}, \Pi_{n+1}(w_{i,j}) = b_{\sigma_i \cap \sigma_{i+1}} \text{ for } 0 < j < k_i \text{ and successive vertices are connected by an edge.} \]
Consider 2-simplices in $P_w$ intersecting $\alpha_w$. Denote these 2-simplices in the following way (see Figure 5, note that this figure does not exhibit the geometry of the subcomplex $P_w$, in fact all simplices have sizes of length 1):

\[
\begin{align*}
&w_{0,0} * w_{0,0,1} * w_{0,0,2} : w_{0,0} * w_{0,0,2} * w_{0,0,3} : \ldots : w_{0,0} * w_{0,m_0-1} * w_{0,m_0,0} : w_{0,0} * w_{0,1} * w_{0,0,m_0,0} : w_{0,1} * w_{0,1,1} * w_{0,1,2} \text{ (where } w_{0,1,1} = w_{0,0,m_0} : w_{0,0} \ldots : w_{0,k_0-1} * w_{0,k_0} * w_{0,k_0-1,m_0,k_0-1} : w_{1,0} * w_{1,0,1} * w_{1,0,2} \text{ (where } w_{1,0,1} = w_{0,k_0} \text{ and } w_{1,0,1} = w_{0,k_0-1,m_0,k_0-1} : \ldots : w_{l_w-1,k_{(l_w-1)}-1} * w_{l_w-1,k_{(l_w-1)}-1,1} * w_{l_w-1,k_{(l_w-1)}-1,1} \text{ :} w_{l_w-1,k_{(l_w-1)}-1} * w_{l_w-1,k_{(l_w-1)}-1} * \ldots : w_{l_w-1,k_{(l_w-1)}-1} * w_{0,0} * w_{0,0,1} (where w_{l_w,k_{l_w}} = w_{0,0} \text{ and } w_{l_w-1,k_{(l_w-1)}-1} * m_{l_w-1,k_{(l_w-1)}-1} = w_{0,0,1}).
\end{align*}
\]

For two points $x$ and $y$ lying in a single simplex we denote by $[x,y]$ the interval connecting them.

For $i = 0, 1, \ldots , l_w - 1$ choose points $a_i, b_i, c_i, d_i$ in the following way: $a_i \in [w_{i,0}, w_{i-1,k_{(i-1)}-1}]$ with $d(a_i, w_{i-1,k_{(i-1)}-1}) = \epsilon$, $b_i \in [w_{i,0}, w_{i-1,k_{(i-1)}-1}]$ with $d(a_i, w_{i,0}) = \epsilon$, $c_i \in [w_{i,0}, w_{i,1}]$ with $d(c_i, w_{i,0}) = \epsilon$, $d_i \in [w_{i,0}, w_{i,1}]$ with $d(d_i, w_{i,1}) = \epsilon$. For $s \in [0,\frac{\alpha}{2}]$ and $i = 0, 1, \ldots , l_w - 1$ choose points $a_i^s, b_i^s, c_i^s, d_i^s$ in the following way: $a_i^s \in [w_{i,0}, a_i]$ with $d(a_i^s, [w_{i,0}, w_{i-1,k_{(i-1)}-1}]) = s$, $b_i^s \in [w_{i,0}, b_i]$ with $d(b_i^s, [w_{i,0}, w_{i-1,k_{(i-1)}-1}]) = s$, $c_i^s \in [w_{i,0}, c_i]$ with $d(c_i^s, [w_{i,0}, w_{i,1}]) = s$, $d_i^s \in [w_{i,0}, d_i]$ with $d(d_i^s, [w_{i,0}, w_{i,1}]) = s$. For $s \in [0,\frac{\alpha}{2}]$, $i = 0, 1, \ldots , l_w - 1, j = 0, 1, \ldots , k_i - 1$ and $k = 0, 1, \ldots , m_{i,j}$ choose points $e_{i,j,k}^s \in [w_{i,j}, w_{i,j,k}]$ with $d(e_{i,j,k}^s, [w_{i,j}, w_{i,j,k+1}]) = \frac{\alpha}{2} - s$.

For $i = 0, 1, \ldots , l_w - 2$ let $a_i^s \in [\Pi_{n+1}(a_i), b_i^s \in \Pi_{n+1}(b_i), c_i^s \in \Pi_{n+1}(c_i)]$ and $d_i^s \in \Pi_{n+1}(d_i)$. For $i = 0, 1, \ldots , l_w - 1$ and $s \in [0,\frac{\alpha}{2}]$ let $a_i^s \in [\Pi_{n+1}(a_i), b_i^s \in \Pi_{n+1}(b_i), c_i^s \in \Pi_{n+1}(c_i), d_i^s \in \Pi_{n+1}(d_i)]$. For each $i = 0, 1, \ldots , l_w - 1$ and fix vertices $w_{i,j} \in \{w_{i,1}, \ldots , w_{i,k_i-1}\}$ and $w_{i,j,i} \in \{w_{i,j-1}, \ldots , w_{i,j,m_{i,j}-1}\}$ (vertices $w_{i,j,i}$ will be mapped onto the barycenters $b_{w_{i,j,i+1}}$ in order to fulfill the condition 4).

Define the map $\Pi_{n+1}^{w,e} : P_w \to S_n$ as follows:

- $\Pi_{n+1}^{w,e}(x) = \Pi_{n+1}(x)$ for $w \in [w_{0,0}, a_i, b_i], w \in [w_{i,0,m_{i,0}}, c_i, d_i]$ and for $x \in \Pi_{n+1}\{w\}$,
- for $s \in [0,\frac{\alpha}{2}]$ and $i = 0, 1, \ldots , l_w - 1$ let $\Pi_{n+1}^{w,e}(w) : [b_i^s, e_i^s, c_i^s, d_i^s] \to [b_i^s, e_i^s, c_i^s]$ be linear (with respect to the length of segments),
- for $s \in [0,\frac{\alpha}{2}]$ and $i = 0, 1, \ldots , l_w - 1$ let $\Pi_{n+1}^{w,e}(a_i) : [d_i^s, e_i^s, c_i^s, d_i^s] \to [d_i^s, e_i^s, c_i^s]$ and $\Pi_{n+1}^{w,e}(e_i^s) : [e_i^s, e_i^s, j_{i,k_i+1}] \to [e_i^s, e_i^s, j_{i,k_i+1}]$ be linear.

Note that $\Pi_{n+1}^{w,e}$ is a well defined continuous map. Note also that with vertices $w_{i,j,i} \in \Pi_{n+1}$ chosen in a coherent way (i.e. for two adjacent vertices $w, w' \in S_n$ the chosen vertices $w_{i,j,i} \in \Pi_{n+1}$ and $w_{i,j,i} \in \Pi_{n+1}$ lying on the arc $\alpha_{w,w'}$ must coincide) the map $\Pi_{n+1}^{w,e} : S_n \to S_n$ is well defined and satisfies the required condition. We omit further details.
Figure 1: Proof of Lemma 5.2
In the next lemma we define a sequence of maps \((\Pi'_{n+1} : S_{n+1} \to S_n)_{n \geq 1}\) such that the inverse limits \(\varprojlim(S_1 \overset{\Pi_2}{\leftarrow} S_2 \overset{\Pi_3}{\leftarrow} S_3 \ldots)\) and \(\varprojlim(S_1 \overset{\Pi'_2}{\leftarrow} S_2 \overset{\Pi'_3}{\leftarrow} S_3 \ldots)\) are homeomorphic. The new inverse system \((S_n, \Pi'_n)\) satisfies the conditions mentioned at the beginning of this section. As we will show later, after refinement of this new system, we will be able to define the families of discs \(D_{n,k}\) such that the refined system \((S_{n,k}, \Pi'_{n,k}, D_{n,k})\) will become a Jakobsche inverse system for the torus.

**Lemma 5.3.** There is a sequence of continuous maps \((\Pi'_{n+1} : S_{n+1} \to S_n)_{n \geq 1}\) such that:

1. the inverse limits \(\varprojlim(S_1 \overset{\Pi_2}{\leftarrow} S_2 \overset{\Pi_3}{\leftarrow} S_3 \ldots)\) and \(\varprojlim(S_1 \overset{\Pi'_2}{\leftarrow} S_2 \overset{\Pi'_3}{\leftarrow} S_3 \ldots)\) are homeomorphic,

2. \(\Pi'^{-1}_{n+1}[w] = (\Pi'_{n+1})^{-1}[w] = (X_w \setminus B_3(\sigma_w, X_w)) \cup S_3(\sigma_w, X_w)\) for all vertices \(w \in S'_n\) (where \(\sigma_w\) is the intersection \(X_w \cap S_{n-1}\)),

3. the restriction of the map \(\Pi'_{n+1}\) to the set \(S_{n+1} \setminus \{(\Pi'_{n+1})^{-1}[w] : w \in S'_n\}\) is a homeomorphism onto the set \(S_n \setminus \{w : w \in S'_n\}\)

4. \(\Pi'_{n+1}[S'_n] \subset \Pi'_{n+1}[S'_n]\)

**Proof:** Inductively we define a sequence of maps \((\Pi'_n : S_n \to S_{n-1})_{n \geq 2}\) and a decreasing sequence of positive numbers \((\epsilon_n)_{n \geq 2}\) such that:

- for each natural number \(n > 1\) the map \(\Pi'_n\) satisfies conditions 2, 3 and 4,

- \(d_{sup}(\Pi_n, \Pi'_{n}) < \epsilon_n\), with \(\epsilon_n < a(\Pi'_2, \Pi'_3, \ldots, \Pi'_{n-1}, \Pi_2, \Pi_3, \ldots, \Pi_n)\), where the latter are the positive numbers given by the Brown theorem.

Let \(\epsilon_2\) be a positive number satisfying \(0 < \epsilon_2 < a(\Pi_2)\) and let a map \(\Pi'_2 : S_2 \to S_1\) satisfy conditions 2, 3 and 4 and the inequality \(d_{sup}(\Pi_2, \Pi'_2) < \epsilon_2\). Such a map exists due to Lemma 5.2. Note that we can additionally assume that \(\frac{1}{C} < 1\) where \(C < 1\) is a positive constant given by Fact 2.7. This property will be used in the proof of Lemma 9.4.

Suppose now that we have defined positive numbers \(\epsilon_2 > \ldots > \epsilon_n\) satisfying the inequalities

\[\epsilon_k < a(\Pi'_2, \Pi'_3, \ldots, \Pi'_{k-1}, \Pi_2, \Pi_3, \ldots, \Pi_k)\]

for \(k = 2, 3, \ldots, n\) and maps \(\Pi'_2, \Pi'_3, \ldots, \Pi'_n\) satisfying the required conditions and the inequalities \(d_{sup}(\Pi_k, \Pi'_k) < \epsilon_k\) for \(k = 2, 3, \ldots, n\). Let \(\epsilon_{n+1} < \epsilon_n\) be a number satisfying

\[0 < \epsilon_{n+1} < a(\Pi'_2, \Pi'_3, \ldots, \Pi'_{n}, \Pi_2, \Pi_3, \ldots, \Pi_{n+1})\]

and let \(\Pi'_{n+1} : S_{n+1} \to S_n\) be a map satisfying conditions 2, 3 and 4 and the inequality \(d_{sup}(\Pi_{n+1}, \Pi'_{n+1}) < \epsilon_{n+1}\). Such a map exists again due to Lemma 5.2.

Now the sequence \((\Pi'_n)_{n \geq 1}\) satisfies conditions 2, 3 and 4. Moreover, by the Brown theorem, the inverse limits \(\varprojlim(S_1 \overset{\Pi'_2}{\leftarrow} S_2 \overset{\Pi'_3}{\leftarrow} S_3 \ldots)\) and \(\varprojlim(S_1 \overset{\Pi''_2}{\leftarrow} S_2 \overset{\Pi''_3}{\leftarrow} S_3 \ldots)\) are homeomorphic.

\[\square\]
6 Refinement of the inverse system

In this section $X$ is orientable. Recall that it follows that vertex links $X_u$ are orientable surfaces (see Remark 2.1). We refine the inverse system $(S_n, \Pi'_n)$. The refinement does not change the inverse limit, and the refined system $(S_{n,k}, \Pi'_{n,k})$ will have the property that every surface $S_{n,k+1}$ will be a connected sum of its predecessor $S_{n,k}$ and a finite number of tori.

As it will be made clear in Section 7, the inverse system $(S_n, \Pi'_n)$, after appropriate choice of families $D_n$ of discs in $S_n$, fulfills assumptions 2, 4 and 5 of Theorem 4.1. Preimages of the chosen discs under the bonding maps $\Pi'_{n+1}$ will correspond to surfaces that are links of $X$ at vertices of $S_n$.

Two phenomena may appear that prevent the system $(S_n, \Pi'_n)$ from satisfying assumptions 1 and 3 of the Jakobsche theorem for $L_0 = \mathbb{T}^2$. The first one is that links at vertices do not have to be surfaces of the same genus. The second phenomenon is that even since all vertex links are homeomorphic, they may be surfaces of genus greater than 1.

Using the following two lemmas we will be able to refine the system $(S_n, \Pi'_n)$ to overcome these difficulties. We start with some terminology.

**Definition 6.1.** Let $f : \Sigma \rightarrow \Sigma'$ be a map between compact orientable surfaces and let $\mathcal{D} = \{D_1, D_2, \ldots, D_l\}$ be a family of pairwise disjoint discs $D_i \subset \Sigma'$. We say that $f$ collapses $\Sigma$ to $\Sigma'$ along the family $\mathcal{D}$ if:

- $\Sigma$ is a connected sum of $\Sigma'$ and a finite number of surfaces $\Sigma_{g_1}, \Sigma_{g_2}, \ldots, \Sigma_{g_l}$ of genera $g_1 > 0, g_2 > 0, \ldots, g_l > 0$ respectively (for some $l > 0$) along discs $D_1 \subset \Sigma'$ and $D'_1 \subset \Sigma_{g_1}$ for $i = 1, 2, \ldots, l$,

- $f(x) = x$ for all $x \in \Sigma \setminus \left( \bigcup_{i=1}^l \text{int}(D_i) \right)$.

- there are open neighbourhoods $U_i$ of $\text{bd}(D'_i)$ in $\Sigma_{g_i} \setminus \text{int}(D'_i)$ and points $x_i \in \text{int}(D_i)$ such that $f$ maps homeomorphically $U_i$ onto $D_i \setminus \{x_i\}$ and collapses $\Sigma_{g_i} \setminus \text{int}(D'_i) \setminus U_i$ to $x_i$.

We call such a map a collapsing map. If it is clear which family $\mathcal{D}$ we mean, we say that $f$ collapses $\Sigma$ to $\Sigma'$.

Note that the maps $\Pi_{n,\epsilon}$ from Lemma 5.2, and hence the maps $\Pi'_n$ from Lemma 5.3, are examples of collapsing maps.

We state without the proofs two obvious lemmas which we use in the refinement procedure.

**Lemma 6.2.** Let $\Sigma$ be an orientable surface of genus $g > 1$. Then there exist orientable surfaces $\Sigma_1, \Sigma_2, \ldots, \Sigma_g = \Sigma$, discs $D_i \subset \Sigma_i$ (for $i = 1, 2, \ldots, g-1$) and maps $f_i : \Sigma_i \rightarrow \Sigma_{i+1}$ (for $i = 2, 3, \ldots, g$) such that:

- $\Sigma_i$ is an orientable surface of genus $i$ for $i = 1, 2, \ldots, g$,

- $\Sigma_i$ is a connected sum of $\Sigma_{i-1}$ and a torus $T_{i-1}^2$ along the disc $D_{i-1}$ and some disc $D'_{i-1} \subset T_{i-1}^2$ along the disc $D_i$ is contained in the complement $T_{i-1}^2 \setminus D'_{i-1}$ and the map $f_i$ collapses $\Sigma_i$ to $\Sigma_{i-1}$ along a one-element family $\mathcal{D}_{i-1} = \{D_{i-1}\}$.

**Lemma 6.3.** Let $f : \Sigma \rightarrow \Sigma'$ collapse an orientable surface $\Sigma$ to an orientable surface $\Sigma'$ along a family $\mathcal{D} = \{D_1, \ldots, D_l\}$. Let $\Sigma_{g_1}, \Sigma_{g_2}, \ldots, \Sigma_{g_l}$ be orientable surfaces as in Definition 6.1. For $i = 1, 2, \ldots, l$ let $D'_i \subset \Sigma_{g_i}$ be discs as in Definition 6.1. Let the genus $g_j$ of the surface $\Sigma_{g_j}$ be greater than 1 for some $j \in \{1, 2, \ldots, l\}$. Then there exist:
such that

\[ f \] \text{ such that:}

and maps

\[ g \]

Note that Lemma 6.2 and Lemma 6.3 are also true for nonorientable surfaces.

**Remark 6.4.** Note that Lemma 6.2 and Lemma 6.3 are also true for nonorientable surfaces. The only difference is that collapsing maps are related to connected sums with projective planes rather than with tori.

As an immediate consequence we get the following:

**Corollary 6.5.** For \( n \geq 1 \) let the map \( \Pi'_{n+1} : S_{n+1} \to S_n \) be defined as before. For each vertex \( w \in (S_n)^{(0)} \) denote by \( g_w \) the genus of the link \( X_w \) (which is a closed orientable surface). Let \( g_n = \max\{g_w : w \in (S_n)^{(0)}\} \). Then there exist surfaces

\[ S_n = S_{n,0}, S_{n,1}, \ldots, S_{n,g_n} = S_{n+1} \]

and maps

\[
\begin{align*}
S_{n,0} & \xleftarrow{\Pi'_{n,1,0}} S_{n,1} \xleftarrow{\Pi'_{n,2,1}} S_{n,2} \xleftarrow{\Pi'_{n,3,2}} \cdots \xleftarrow{\Pi'_{n,g_n-1,g_n-2}} S_{n,g_n-1} \xleftarrow{\Pi'_{n,g_n-1,g_n-1}} S_{n,g_n}
\end{align*}
\]

such that:

- \( S_{n,k} \) is a connected sum of \( S_{n,k-1} \) and some tori \( T_{n,k,1}, T_{n,k,2}, \ldots, T_{n,k,m_n,k} \) (for some natural number \( m_n,k \geq 1 \)) along pairwise disjoint discs

\[ D_{n,k,1} \subset S_{n,k-1}, D_{n,k,2} \subset S_{n,k-1}, \ldots, D_{n,k,m_n,k} \subset S_{n,k-1} \]

and

\[ D'_{n,k,1} \subset T_{n,k,1}, D'_{n,k,2} \subset T_{n,k,2}, \ldots, D'_{n,k,m_n,k} \subset T_{n,k,m_n,k} \]

respectively,

- every disc \( D_{n,k+1,i} \) is contained in some torus \( T_{n,k,j} \) and is disjoint with the disc \( D'_{n,k,j} \),

- for each \( n > 0 \) and each \( k = 1, 2, \ldots, g_n \) the map \( \Pi'_{n+1,k,k-1} \) collapses \( S_{n,k} \) to \( S_{n,k-1} \) along the family \( \{D_{n,k,i} : i = 1, 2, \ldots, m_n,k\} \),

\[ \Pi'_{n+1} = \Pi'_{n,g_n-1} \] (where \( \Pi'_{n,g_n-0} \) denote the composition \( \Pi'_{n,1,0} \circ \Pi'_{n,2,1} \circ \cdots \circ \Pi'_{n,g_n-1,g_n-1} \)).
Corollary 6.5 gives the refined inverse system of orientable surfaces

\[(S_1 \leftarrow S_{1,1} \leftarrow S_{1,2-1} \leftarrow \ldots \leftarrow S_{2,1-1} \leftarrow S_{2,1-0} \ldots)\]

In this inverse system every surface is a connected sum of its predecessor and a finite number of tori (possibly only one). If the genus of the sphere \(S_1\) is greater than 1, we use Lemma 6.2 for the sphere \(S_1\) to get the inverse system

\[(S_0 \leftarrow S_{0,1} \leftarrow S_{0,2-1} \leftarrow \ldots \leftarrow S_{1,1} \leftarrow S_{1,1-0} \ldots)\]

with \(S_0\) a torus. We don’t do this if \(S_1\) is a torus.

The last condition of Corollary 6.5 implies that the refining of the inverse system does not change the inverse limit. Thus we get the following:

**Corollary 6.6.** Suppose that \(X\) is a 7-systolic normal orientable pseudomanifold of dimension 3. Then the Gromov boundary \(\partial_G X\) is homeomorphic to the inverse limit of the refined inverse system \(\lim(S_0 \leftarrow S_{0,1} \leftarrow S_{0,2-1} \leftarrow \ldots \leftarrow S_{1,1} \leftarrow S_{1,1-0} \ldots)\)

\(\lim(S_1 \leftarrow S_{1,1} \leftarrow S_{1,2-0} \ldots)\) if \(S_1\) is a torus).

### 7 Getting the structure of a Jakobsche inverse system

In this section we continue considerations of the previous one, under the same assumption that \(X\) is orientable. We define some finite families \(\mathcal{D}_{n,k}\) of pairwise disjoint discs in every surface \(S_{n,k}\). The inverse system \((S_{n,k}, \Pi'_{n,k})\) with families \(\mathcal{D}_{n,k}\) will satisfy all assumptions of the Jakobsche theorem, with \(L_0 = \mathbb{T}^2\).

To define these families we need some preparations. For \(n = 0, 1, \ldots\) let \(A_n \subset S_n\) be defined as \(A_n = \{\Pi'_{n,l}(w) : l \geq n, w \in S_i(0)\}\) and let \(A_{n,k}\) be the image \(\Pi'_{n,g_n-k}[A_{n+1}] \subset S_{n,k}\) of the set \(A_{n+1}\), where \(\Pi'_{n,l} : S_n \rightarrow S_l\) and \(\Pi'_{n,g_n-k} : S_{n+1} \rightarrow S_{n,k}\) are the compositions \(\Pi'_{l,k} \circ \ldots \circ \Pi'_{n}\) and \(\Pi'_{n,k+1-k} \circ \ldots \circ \Pi'_{n,g_n-k}\) respectively.

**Lemma 7.1.** \(A_n\) is a countable dense subset of \(S_n\) for all \(n \geq 1\), and \(A_{n,k}\) is a countable dense subset of \(S_{n,k}\) for all \(n \geq 0\) and all \(k = 0, 1, \ldots, g_n\).

**Proof:** Recall that every map \(\Pi_i\) is a \(C\)-contraction and the 0-skeleton \(S_i^{(0)}\) is a finite 1-net in the sphere \(S_i\) for all numbers \(i\). It follows that the set \(\{\Pi_{n,l}(v) : v \in S_i^{(0)}\}\) is a finite \(C^l-n\)-net in \(S_n\). Since the maps \(\Pi'_i\) satisfy the condition \(\Pi_{i}[S_i^{(0)}] \subset \Pi'_{i}[S_i^{(0)}]\) (see assertion 4 of Lemma 5.3) it follows that the assertion holds for the sets \(A_n\). Since the set \(A_{n,k}\) is the image of the countable dense set \(A_{n+1}\) by a surjection, it is itself countable and dense.

Now we define inductively families of discs \(\mathcal{D}_n\) and \(\mathcal{D}_{n,k}\) in the spheres \(S_n\) and \(S_{n,k}\) respectively to match all assumptions of Theorem 4.1.

Suppose the genus of the sphere \(S_1\) is equal to 1 (i.e. the sphere \(S_1\) is a torus). Let

\[\mathcal{D}_1 = \{D_w : w \in (S_1)^{(0)}\}\]

be a family of pairwise disjoint discs such that \(w \in \text{int}(D_w)\) with diameters \(\text{diam}(D_w) < \frac{1}{2}\) and such that the intersections \(\text{bd}(D_w) \cap A_1\) are empty. Note that since every disc is a union
of uncountable family of disjoint circles and a point (just take a homeomorphism to the unit plane disc and circles of radius $r \in (0, 1]$ centred at 0), it follows that such discs exist.

If the sphere $S_1$ has genus greater then 1, then as in Corollary 6.6 we start with the surface $S_0$. Let $\mathcal{D}_0$ be a family consisting of one small 2-disc $D$ in $S_0$ (contained in the disc $D_{0,0}$ as in Lemma 6.2 and satisfying the inequality $\text{diam}(D) < 1$), with $x \in \text{int}(D)$ (where a point $x \in \text{int}(D_{0,0})$ and a disc $D_{0,0}$ are given by the fact that the map $\Pi_{0,1-0} : S_{0,1} \to S_0$ is a collapsing map). Again we can assume that the intersection $\text{bd}(D) \cap A_0$ is empty.

Now suppose we have defined the families $\mathcal{D}_0, \ldots, \mathcal{D}_{n-1}$ and $\mathcal{D}_{i,j}$ for all $i \leq n - 1$ and $j = 0, 1, \ldots, g_i$. We define the family $\mathcal{D}_n$ as follows. For every vertex $u \in S_n^{(0)}$ we choose a small 2-disc $D_u$ containing $u$ in its interior such that:

- the discs $D_u$ are pairwise disjoint,
- the intersection $\text{bd}(D_u) \cap A_n$ is empty for all vertices $u \in S_n^{(0)}$,
- the intersection $\Pi'_{i,n}[D_u] \cap \text{bd}(D')$ is empty for all $i < n$ and for all discs $D' \in \mathcal{D}_i$,
- the intersection $\Pi'_{i,g_i-j} \circ \Pi'_{i+1,n}[D_u] \cap \text{bd}(D')$ is empty for all $i < n$, all $j = 0, 1, \ldots, g_i$ and for all discs $D' \in \mathcal{D}_{i,j}$,
- $\text{diam}_S(\Pi'_{i,n}[D_u]) < \frac{1}{2^n}$ for all $i < n$,
- $\text{diam}_{S,i,j}(\Pi'_{i,g_i-j} \circ \Pi'_{i+1,n}[D_u]) < \frac{1}{2^n}$ for all $i < n$ and all $j = 1, 2, \ldots, g_i$.

It is possible to choose such a family $\mathcal{D}_n$. To see this consider a point $a = \Pi'_{i,n}(u) \in A_i$. This point is not contained in the boundary $\text{bd}(D')$ of any disc $D' \in \mathcal{D}_i$. Analogously, any point $a = \Pi'_{i,g_i-j} \circ \Pi'_{i+1,n}(y) \in A_{i,j}$ is not contained in the boundary $\text{bd}(D')$ of any disc $D' \in \mathcal{D}_{i,j}$. Thus for small enough $\epsilon > 0$ the intersection $\Pi'_{i,n}[B_{S_n}(u, \epsilon)] \cap \text{bd}(D')$ is empty for all discs $D' \in \mathcal{D}_i$ and the intersection $\Pi'_{i,g_i-j} \circ \Pi'_{i+1,n}[B_{S_n}(u, \epsilon)] \cap \text{bd}(D')$ is empty for all discs $D' \in \mathcal{D}_{i,j}$ (there are only finitely many such discs $D'$). Since $S_n$ is a surface, the metric ball $B_{S_n}(u, \epsilon)$ contains a 2-disc $D_u$ containing $u$ in its interior. Again we can assume that the intersection $\text{bd}(D_u) \cap A_n$ is empty.

Now suppose we have defined the families $\mathcal{D}_0, \mathcal{D}_1, \ldots, \mathcal{D}_n$, the families $\mathcal{D}_{i,j}$ for all $i < n$ and $j = 0, 1, \ldots, g_i$ and the families $\mathcal{D}_{n,j}$ for all $j < k$. We define the family $\mathcal{D}_{n,k}$ as follows. For all points $x_{n,k,l}$ given by the fact that the map $\Pi'_{n,k+1-k} : S_{n,k+1} \to S_{n,k}$ is a collapsing map let $D_{n,k,l}$ be a small disc in $S_{n,k}$ containing $x_{n,k,l}$ in its interior such that:

- the discs $D_{n,k,l}$ are pairwise disjoint,
- the intersection $\text{bd}(D_{n,k,l}) \cap A_{n,k}$ is empty for all $l$,
- the intersection $\Pi'_{i,n} \circ \Pi'_{n,k-0}[D_{n,k,l}] \cap \text{bd}(D')$ is empty for all $i < n$ and for all discs $D' \in \mathcal{D}_i$,
- the intersection $\Pi'_{i,g_i-j} \circ \Pi'_{i+1,n} \circ \Pi'_{n,k-0}[D_{n,k,l}] \cap \text{bd}(D')$ is empty for all $i < n$, all $j = 0, 1, \ldots, g_i$ and all discs $D' \in \mathcal{D}_{i,j}$,
- $\text{diam}_{S,i,j}(\Pi'_{i,n} \circ \Pi'_{n,k-0}[D_{n,k,l}]) < \frac{1}{2^n}$ for all $i < n$,
- $\text{diam}_{S,i,j}(\Pi'_{i,g_i-j} \circ \Pi'_{i+1,n} \circ \Pi'_{n,k-0}[D_{n,k,l}]) < \frac{1}{2^n}$ for all $i < n$ and all $j = 0, 1, \ldots, g_i$. 

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Choosing such discs is possible and we do it analogously as we have defined the elements of the families $D_n$.

Now the families of discs $D_n$ and $D_{n,k}$ together with the maps $\Pi'_n$ and $\Pi'_{n,k-1}$ satisfy assumptions of Theorem 4.1 (note that $A_n$ is dense in $S_n$ and $A_{n,k}$ is dense in $S_{n,k}$). Since the maps $\Pi'_n$ were chosen to be close enough to the maps $\Pi_n$ to preserve the inverse limit, as a corollary we get:

**Theorem 7.2.** The Gromov boundary of a 7-systolic normal orientable pseudomanifold of dimension 3 is a Jakobsche tree of tori, i.e. the Pontriagin sphere.

8 Nonorientable trees of surfaces

In this section we examine properties of Jakobsche inverse systems of non-orientable surfaces. An extension of the Jacobsche’s construction for nonorientable case was considered in [S]. In dimension 2, i.e. for non-orientable surfaces, it is possible and more convenient to follow rather Jakobsche’s approach then that of Stallings. We sketch here some details of this.

We call a family $D$ of pairwise disjoint closed discs contained in the interior of a manifold $M$ a good family, if it is null family and the family $\{\text{int}(D) : D \in D\}$ is a dense family in $M$.

The following lemma is a simple extension of Toruńczyk’s lemma (see [J, Lemma 3.1]).

**Lemma 8.1.** Let $\Sigma$ and $\Sigma'$ be nonorientable surfaces (with or without boundaries) and let $f : \Sigma \to \Sigma'$ be a homeomorphism. Let $Z$ and $Z'$ be two good families of closed 2-discs in $\Sigma$ and $\Sigma'$ respectively. Then there exists a bijective function $p : Z \to Z'$ and a homeomorphism $f' : \Sigma \setminus \bigcup_{D \in Z} \text{int}(D) \to \Sigma' \setminus \bigcup_{D' \in Z'} \text{int}(D')$ such that

$$f'|_{\text{bd}(\Sigma)} = f|_{\text{bd}(\Sigma)} \quad \text{and} \quad f'([\text{bd}(D)]) = \text{bd}(p(D)) \quad \text{for each} \quad D \in Z.$$

The proof of this lemma is the same as in [J], thus we omit it.

Using Lemma 8.1 and the fact that every homeomorphism of the boundary of a closed nonorientable surface with the interior of a disc removed can be extended to a homeomorphism of this surface, by the same argument as in the proof of Theorem 4.6 in [J], we get the following:

**Theorem 8.2.** Let $(L_0 \xleftarrow{\alpha_1} L_1 \xleftarrow{\alpha_2} L_2 \leftarrow \ldots)$ be an inverse system of connected closed nonorientable surfaces and for each $k \geq 0$ let $D_k$ be a finite collection of pairwise disjoint discs in $L_k$ such that:

1. each $L_k$ is a connected sum of finitely many copies of $L_0$,
2. every map $\alpha_{k+1}$ restricted to the preimage

$$\alpha_{k+1}^{-1}\left[ L_k \setminus \bigcup\{\text{int}(D) : D \in D_k\} \right]$$

is a homeomorphism onto the set

$$L_k \setminus \bigcup\{\text{int}(D) : D \in D_k\}$$
3. every preimage $\alpha^{-1}_{k+1}[D]$ (for $D \in \mathcal{D}_k$) is homeomorphic to a copy of $L_0$ with the interior of a disc removed,

4. the family $\{\alpha_j[D] : i \geq j, D \in \mathcal{D}_i\}$ is null and dense in $L_j$ for all $j$,

5. the intersection $\alpha_j[D] \cap \partial(D')$ is empty for for all $i > j$ and for all discs $D \in \mathcal{D}_i$ and $D' \in \mathcal{D}_j$.

Then the inverse limit $\lim_{\leftarrow} (L_0 \xrightarrow{\alpha_1} L_1 \xrightarrow{\alpha_2} L_2 \leftarrow \ldots)$ depends only on $L_0$.

As in the orientable case we denote this space by $X(L_0)$ and call it a Jacobsche tree of nonorientable surfaces $L_0$. Similarly as in the orientable case, we call the system $(L_k, \alpha_k, \mathcal{D}_k)_{k \geq 0}$ satisfying assumptions 1-5 of Theorem 8.2 a Jacobsche inverse system for $L_0$. If the system $(L_k, \alpha_k, \mathcal{D}_k)_{k \geq 0}$ satisfies assumptions 2, 4, 5 and the condition:

3a. every preimage $\alpha^{-1}_{k+1}[D]$ (for $D \in \mathcal{D}_k$) is homeomorphic to a connected closed nonorientable surfaces with the interior of a disc removed,

then we call it a Jacobsche inverse system of nonorientable surfaces.

**Remark 8.3.** 1. For $L_0 = \mathbb{RP}^2$, the projective plane, the space $X(\mathbb{RP}^2)$ is known as the nonorientable Pontriagin surface and denoted by $\Sigma_P$.

2. For $L_0 = \Sigma_g$, the nonorientable surface of genus $g > 1$, the space $X(\Sigma_g)$ is homeomorphic to the nonorientable Pontriagin surface. Actually the tree of nonorientable surfaces is homeomorphic to the nonorientable Pontriagin surface (see Remark 8.6)

The next two lemmas show that if nonorientable surfaces occur densely enough in a tree of surfaces, then this tree is homeomorphic to the nonorientable Pontriagin surface.

**Lemma 8.4.** Let $(X_0 \xleftarrow{s_1} X_1 \xleftarrow{s_2} X_2 \ldots)$ and $(Y_0 \xleftarrow{t_1} Y_1 \xleftarrow{t_2} Y_2 \ldots)$ be two inverse systems of topological spaces such that the maps $s_i$ and $t_i$ are continuous and onto for all natural numbers $i$ and such that there exist:

- increasing sequences $\{n_k\}$, $\{m_k\}$, $\{n'_k\}$ and $\{m'_k\}$ of natural numbers satisfying $n_{k-1} \leq n'_k \leq n_k$ and $m_{k-1} \leq m'_k \leq m_k$,

- continuous maps $f_k : X_{n_k} \to Y_{m_k}$ and $g_k : Y_{m'_k} \to X_{n'_k}$ being onto for all $k$,

such that the following diagrams are commutative:

\[
\begin{array}{ccc}
X_{n'_k} & \xleftarrow{s_{k,n'_k}} & X_{n_k} \\
\uparrow g_k & & \downarrow f_k \\
Y_{m'_k} & \xleftarrow{t_{m'_k,m_k}} & Y_{m_k}
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
X_{n_{k-1}} & \xleftarrow{s_{k-1,n'_k}} & X_{n'_k} \\
\downarrow f_{k-1} & & \uparrow g_k \\
Y_{m_{k-1}} & \xleftarrow{t_{m_{k-1},m'_k}} & Y_{m'_k}
\end{array}
\]

i.e. it holds $g_k \circ t_{m'_k,m_k} \circ f_k = s_{n'_k,n_k}$ and $f_{k-1} \circ s_{n_{k-1},n'_k} \circ g_k = t_{m_{k-1},m'_k}$. Then the inverse limits $\lim_{\leftarrow} (X_k, s_k)$ and $\lim_{\leftarrow} (Y_k, t_k)$ are homeomorphic.
Proof: Define maps

\[ F : \lim\limits_{\rightarrow}(X_{n_0} \xleftarrow{s_{n_0, n_1'}} X_{n_1'} \xleftarrow{s_{n_1', n_1}} \ldots) \rightarrow \lim\limits_{\rightarrow}(Y_{m_0} \xleftarrow{t_{m_0, m_1'}} Y_{m_1'} \xleftarrow{t_{m_1', m_1}} \ldots) \]

and

\[ G : \lim\limits_{\rightarrow}(Y_{m_0} \xleftarrow{t_{m_0, m_1'}} Y_{m_1'} \xleftarrow{t_{m_1', m_1}} \ldots) \rightarrow \lim\limits_{\rightarrow}(X_{n_0} \xleftarrow{s_{n_0, n_1'}} X_{n_1'} \xleftarrow{s_{n_1', n_1}} \ldots) \]

by formulas

\[ F((x_0, x_1', x_1, x_2', \ldots)) = (f_0(x_0), t_{m_0, m_1}(f_1(x_1)), f_1(x_1), t_{m_1', m_2}(f_2(x_2)), \ldots) \]

and

\[ G((y_0, y_1', y_1, y_2', \ldots)) = (s_{n_0, n_1'}(g_1(y_1')), g_1(y_1'), s_{n_1', n_2} g_2(y_2'), g_2(y_2'), \ldots) \]

respectively.

These maps are well defined, continuous and inverse one to the other. Thus they are both homeomorphisms. Moreover, the inverse limits

\[ \lim\limits_{\rightarrow}(X_k, s_k) \text{ and } \lim\limits_{\rightarrow}(X_{n_0} \xleftarrow{s_{n_0, n_1'}} X_{n_1'} \xleftarrow{s_{n_1', n_1}} X_{n_1} \xleftarrow{s_{n_1, n_2}} \ldots) \]

and similarly

\[ \lim\limits_{\rightarrow}(Y_k, t_k) \text{ and } \lim\limits_{\rightarrow}(Y_{m_0} \xleftarrow{t_{m_0, m_1'}} Y_{m_1'} \xleftarrow{t_{m_1', m_1}} Y_{m_1} \xleftarrow{t_{m_1, m_2'}} \ldots) \]

are naturally homeomorphic. Thus the assertion holds.

Lemma 8.5. Let \((L_0 \xrightarrow{\alpha_1} L_1 \xrightarrow{\alpha_2} L_2 \ldots)\) be an inverse system of connected closed nonorientable surfaces and for each \(k \geq 0\) let \(D_k\) be a finite collection of pairwise disjoint discs in \(L_k\) such that:

1. \((L_k, \alpha_k, D_k)\) is a Jakobsche inverse system of surfaces, \(^2\)

2. for every natural number \(k\) and for every disc \(D \in D_k\) there is a natural number \(l_D > k\) such that the preimage \((\alpha_k, l_D)^{-1}[D]\) is a nonorientable surface with the interior of a disc removed,

3. every map \(\alpha_{k+1}\) collapses \(L_{k+1}\) to \(L_k\) along \(D_k\).

Then the inverse limit \(\lim\limits_{\rightarrow}(L_0 \xrightarrow{\alpha_1} L_1 \xrightarrow{\alpha_2} L_2 \ldots)\) is homeomorphic to the nonorientable Pontriagin surface.

Proof: We shall define the following collection of data:

- an infinite increasing sequence \(\{n_k\}\) of natural numbers,
- a sequence \(\{L'_k\}\) of nonorientable closed surfaces,
- a sequence \(\{D'_k\}\) of finite families of pairwise disjoint discs in every surface \(L'_k\).

\(^2\)In particular we require that the preimage \(\alpha_{k+1}^{-1}[D]\) (for \(D \in D_k\)) is a closed surface (orientable or not) with the interior of a disc removed.
• sequences of maps \( \{f_k : L_{nk} \to L'_{k-1}\}, \{g_k : L'_k \to L_{nk}\} \) and \( \{\alpha'_k : L'_k \to L'_{k-1}\} \) satisfying the following:

a) the diagrams:

\[
\begin{array}{ccc}
L_{nk} & \xrightarrow{\alpha_{nk} : L_{nk} \to L_{nk+1}} & L_{nk+1} \\
\downarrow{g_k} & & \downarrow{f_{k+1}} \\
L'_k & & L'_k
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
L_{nk} & \xleftarrow{\alpha'_k : L'_k \to L'_k} & L_{nk} \\
\downarrow{g_k} & & \downarrow{f_k} \\
L'_k & & L'_k
\end{array}
\]

are commutative, i.e. \( g_k \circ f_{k+1} = \alpha_{nk,nk+1} \) and \( f_k \circ g_k = \alpha'_k \),

b) \( g_k \) maps

\[
L'_k \setminus \bigcup_{D \in \mathcal{D}'_{k-1}} (\alpha'_k)^{-1}[\text{int}(D)]
\]

homeomorphically onto

\[
L_{nk} \setminus \bigcup_{D \in \mathcal{D}'_{k-1}} f_k^{-1}[\text{int}(D)]
\]

and maps \((\alpha'_k)^{-1}[D]\) onto \(f_k^{-1}[D]\) for all discs \(D \in \mathcal{D}'_{k-1}\),

c) \( f_{k+1} \) maps

\[
L_{nk+1} \setminus \bigcup_{D \in \mathcal{D}'_k} f_{k+1}^{-1}[\text{int}(D)]
\]

homeomorphically onto

\[
L'_k \setminus \bigcup_{D \in \mathcal{D}'_k} \text{int}(D)
\]

d) \( \alpha'_k \) collapses \( L'_k \) to \( L'_{k-1} \) along \( \mathcal{D}'_{k-1} \),

e) \((L'_k, \alpha'_k, \mathcal{D}'_k)\) is a Jakobsche inverse system of nonorientable surfaces.

Note that by Lemma 8.4 the inverse limits \(\lim(L_k, \alpha_k)\) and \(\lim(L'_k, \alpha'_k)\) are homeomorphic.

By the nonorientable analogues of Lemma 6.2 and Lemma 6.3 the inverse limit \(\lim(L'_k, \alpha'_k)\) is homeomorphic to the Jakobsche tree of projective planes. Thus, by Theorem 8.2, both of these inverse limits are homeomorphic to the nonorientable Pontriagin surface.

It remains to construct the desired data. We proceed to do this inductively. Let \(n_0 = 0\), \(L'_0 = L_0\), \(g_0 = Id_{L_0}\), \(\mathcal{D}'_0 = \mathcal{D}_0\). Let \(n_1 = 1\), \(f_1 = \alpha_1\).

Suppose that we have defined the following:

• natural numbers \(n_j\) for \(j = 0, 1, \ldots, k\) satisfying \(n_j < n_{j+1}\) for \(j = 0, 1, \ldots, k - 1\),

• nonorientable closed surfaces \(L'_j\) for \(j = 0, 1, \ldots, k - 1\),

• finite families \(\mathcal{D}'_j\) (for \(j = 0, 1, \ldots, k - 1\)) of pairwise disjoint discs in every surface \(L'_j\) respectively,

• maps \(f_j : L_{nj} \to L'_{j-1}\) for \(j = 0, 1, \ldots, k\), \(g_j : L'_j \to L_{nj}\) for \(j = 0, 1, \ldots, k - 1\) and \(\alpha'_j : L'_j \to L'_{j-1}\) for \(j = 0, 1, \ldots, k - 1\) (if \(k > 0\))

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satisfying (a), (b), (c), (d) and an additional condition: 
f) every preimage \((\alpha'_j)^{-1}[D]\) (for \(D \in \mathcal{D}'_{j-1}\)) is homeomorphic to a nonorientable closed surface with the interior of a disc removed.

Let \(n_{k+1} > n_k\) be the smallest integer such that the preimage \((f_k \circ \alpha_{n_k,n_{k+1}})^{-1}[D]\) is a nonorientable surface with the interior of a disc removed for all discs \(D \in \mathcal{D}'_{k-1}\) (such a number exists due to assumption 2).

Let 

\[ L'_k = \left( L_{k-1}^c \setminus \bigcup_{D \in \mathcal{D}'_{k-1}} \text{int}(D) \right) \cup \bigcup_{D \in \mathcal{D}'_{k-1}} (f_k \circ \alpha_{n_k,n_{k+1}})^{-1}[D] \]

where points \(x \in \text{bd}(D)\) are identified with their preimages \((f_k \circ \alpha_{n_k,n_{k+1}})^{-1}[x]\) due to (c) and assumption 1.

Define the maps \(g_k : L'_k \to L_{n_k}\), \(f_{k+1} : L_{n_{k+1}} \to L'_k\) and \(\alpha'_k : L'_k \to L'_{k-1}\) as follows:

\[
g_k(x) = \begin{cases} 
  f_k^{-1}(x) & \text{if } x \in L_{k-1}^c \setminus \bigcup_{D \in \mathcal{D}'_{k-1}} \text{int}(D) \text{ (by (c) for } f_k) \\
  \alpha_{n_k,n_{k+1}}(x) & \text{if } x \in \bigcup_{D \in \mathcal{D}'_{k-1}} (f_k \circ \alpha_{n_k,n_{k+1}})^{-1}[D] 
\end{cases}
\]

\[
f_{k+1}(x) = \begin{cases} 
  x & \text{if } x \in \bigcup_{D \in \mathcal{D}'_{k-1}} (f_k \circ \alpha_{n_k,n_{k+1}})^{-1}[D] \\
  f_k \circ \alpha_{n_k,n_{k+1}}(x) & \text{otherwise} 
\end{cases}
\]

and

\[
\alpha'_k(x) = \begin{cases} 
  x & \text{if } L_{k-1}^c \setminus \bigcup_{D \in \mathcal{D}'_{k-1}} \text{int}(D) \text{ (by (c) for } \alpha_k) \\
  f_k \circ \alpha_{n_k,n_{k+1}}(x) & \text{otherwise} 
\end{cases}
\]

These maps are of course well defined and continuous. They satisfy (a), (b) and (d) in an obvious way.

To define the family \(\mathcal{D}'_j\) we need some technical definition. For \(n_k \leq j < n_{k+1}\) let

\[ \mathcal{D}'_j^+ = \{ D \in \mathcal{D}_j : D \cap (f_k \circ \alpha_{n_k,j})^{-1}[D'] = \emptyset \text{ for } D' \in \mathcal{D}'_{k-1} \text{ and } \}

\[ D \cap \alpha_{s,j}^{-1}[D''] = \emptyset \text{ for } n_k \leq s < j \text{ and } D'' \in \mathcal{D}_s \} \]

Define the family \(\mathcal{D}'_k\) by

\[ \mathcal{D}'_k = \{ f_{k+1} \left[ \alpha_{j,n_{k+1}}^{-1}[D] \right] : n_k \leq j < n_{k+1}, D \in \mathcal{D}'_j^+ \} \]

We skip the straightforward checking of conditions (c) and (e).

\[ \square \]

**Remark 8.6.** 1. Note that the assumption 3 in Lemma 8.5 is not necessary. Indeed, as in the proof of Theorem 4.6 in [J] it can be shown that it is possible to change the maps \(\alpha_k : L_k \to L_{k-1}\) on the preimages \(\alpha_k^{-1}[D]\) (for \(D \in \mathcal{D}_{k-1}\)), while keeping the inverse limit unchanged, to get the collapsing maps.

2. The same argument shows that the tree of orientable surfaces is homeomorphic to the Pontriagin sphere.
9 Proof of part b) of Main Theorem

In this section we extend Theorem 7.2 to the nonorientable case. We need some preparations. We start with the following property of group actions on metric spaces, the proof of which we skip.

Lemma 9.1. Let $X$ be a proper metric space and let a group $G$ act on $X$ cocompactly by isometries. Then there is a positive constant $R > 0$ such that for all points $x \in X$ translates of the metric ball $B_X(x, R)$ under elements of $G$ cover $X$, i.e. it holds $G \cdot B_X(x, R) = X$.

Consider now a 3-dimensional 7-systolic normal pseudomanifold $X$ with a cocompact action of a group $G$ by simplicial automorphisms. For a vertex $w \in X$ and a simplex $\sigma \subset X_w$ consider a subcomplex

$$X_{w,\sigma} = (X_w \setminus B_2(\sigma, X_w)) \cup S_2(\sigma, X_w)$$

Let $K_{w,\sigma}$ denote the diameter $\text{diam}(X_{w,\sigma}^{(1)})$ (in the intrinsic metric $d_{X_{w,\sigma}}^{(1)}$). Note that the number $K = \max\{K_{w,\sigma} : w \in X^{(0)}, \sigma \subset X_w\}$ is finite.

The next lemma describes the relationship between distances in successive spheres in a 7-systolic normal pseudomanifold of dimension 3.

Lemma 9.2. Let $X$ be a 7-systolic 3-dimensional normal pseudomanifold with a cocompact action of a group $G$ by simplicial automorphisms. Let $K$ be as above. Let $p$ and $q$ be two vertices in the sphere $S_k$ and let $p'$ and $q'$ be two vertices in the sphere $S_{k+1}$ connected by an edge with $p$ and $q$ respectively. Then

$$d_{S_{k+1}}^{(1)}(p', q') \leq K \cdot (d_{S_k}^{(1)}(p, q) + 1)$$

Proof: Let $p = p_0, p_1, \ldots, p_n = q$ be a geodesic in the 1-skeleton $S_k^{(1)}$. For $i = 1, 2, \ldots, n$ let $p'_i$ be a vertex in the intersection $X_{p_{i-1}p_i} \cap S_{k+1}$. Note that $\text{diam}((X_{p_i} \cap S_{k+1})^{(1)}) \leq K$, since $X_{p_i} \cap S_{k+1} = X_{p_i, \rho}$, where $\rho = \Pi_k(p_i)$. Thus

$$d_{S_{k+1}}^{(1)}(p', q') \leq d_{S_k}^{(1)}(p'_1) + \sum_{j=1}^{n-1} d_{S_k}^{(1)}(p'_j, p'_{j+1}) + d_{S_{k+1}}^{(1)}(p'_n, q') \leq K \cdot (n + 1)$$

The next lemma shows that if a nonorientable complex $X$ is as in Lemma 9.2 then there are enough many vertices with nonorientable links in $X$, in certain precise sense.

Lemma 9.3. Let $X$ be a 7-systolic nonorientable pseudomanifold of dimension 3 with a cocompact action of a group $G$ by simplicial isometries. Let $v \in X$ be a vertex. Let $w \in S_k = S_k(v, X)$ be a vertex. Then for every positive number $\epsilon > 0$ there are a number $k' > k$ and a vertex $u \in S_{k'}$ such that the link $X_u$ is a nonorientable surface and $\Pi_{k,k'}(u) \in B_{S_k}(w, \epsilon)$.

Proof: Let $\epsilon$ be a positive number. By Lemma 9.1 there is a positive number $R$ such that for all points $x \in X$ translates of a metric ball $B_X(x, R)$ under elements of $G$ cover $X$, i.e. it holds $G \cdot B_X(x, R) = X$. Thus there is a positive integer $N$ such that $G \cdot B_N(w, X) = X$ for all vertices $w \in X$ (where $B_N(w, X)$ denote the combinatorial ball).

For a natural number $l > 0$ and for $i = 0, 1, \ldots, 2N$ consider the combinatorial spheres $S_{k+l+i} = S_{k+i}(v, X)$. Let $u_i \in S_{k+i}$ be such points that $\Pi_{k+i+1}(u_{i+1}) = u_i$ for $i =$
0, 1, . . . , 2N − 1 and \( \Pi_{k,k+l}(u_0) = w \). There is a vertex \( u \in B_N(u_N, X) \) such that the link \( X_u \) is a nonorientable surface. Since \( B_N(u_N, X) \subseteq B_{k+l+2N}(v, X) \setminus B_{k+l+1}(v, X) \), it is enough to show that for \( l \) large enough for all vertices \( z \in B_N(u_N, X) \) we have \( d_{S_k}(\Pi_{k,k+l+1}(z), w) < \epsilon \), where \( k + l + i = d_{X(t)}(v, z) \) (here we use the convention that \( \Pi_{k,k} = \text{Id}_{S_k} \)).

For this let \( z \) be a vertex in the intersection \( B_N(u_N, X) \cap S_{k+l+1} \), for some \( i = 0, 1, \ldots, 2N \). Let \( u_0 = z_0, 0, z_0, 1, \ldots, z_{0,j_0}, z_{1,1}, \ldots, z_i, j_i, \ldots, z_{i-1,j_{i-1}}, z_i, j_i = z \) (for some natural numbers \( j_0 \geq 1, j_i \geq 1, \ldots, j_i \geq 1 \)) be a geodesic in the 1-skeleton \( X^{(1)} \) satisfying \( z_{m,n} \in B_N(u_N, X) \cap S_{k+l+m}(v, X) \) for \( m = 0, 1, \ldots, i \) and \( n = 1, 2, \ldots, j_m \) (actually all geodesics between \( z \) and \( u_0 \) have this form, since combinatorial balls are convex (see [JS, Corollary 7.5]) and thus geodesically convex (see [HS, Proposition 4.9])).

Let \( K \) be a constant as in Lemma 9.2 and let \( L = \max\{K, 2N + 2\} \). We will show that

\[
d_{S_{k+l+1}}(z, u_i) < L^{2N+3}.
\]

Using this inequality we get that

\[
d_{S_k}(\Pi_{k,k+l+i}(z), w) < C^{i+1}L^{2N+3} < C L^{2N+3}
\]

where \( C \) is a constant given by Fact 2.7. Thus for \( l \) large enough the assertion holds.

To prove the inequality, we inductively show that for all \( t = 0, 1, \ldots, i \) it holds

\[
d_{S_{k+l+t}}^{(1)}(z_{t,j_t}, u_t) < tL^{t+1} + 2R \quad \text{and} \quad d_{S_{k+l+t}}^{(1)}(z_{t,0}, u_t) < tL^{t+1}
\]

Since \( z_{0,j_0} \) and \( u_0 \) are vertices in the intersection \( B_N(u_N, X) \cap B_{k+l}(v, X) \), it follows that

\[
d_{S_{k+l+t}}^{(1)}(z_{0,j_0}, u_0) = d_{X(t)}(z_{0,j_0}, u_0) \leq 2N
\]

Suppose that

\[
d_{S_{k+l+t}}^{(1)}(z_{t,j_t}, u_t) < tL^{t+1} + 2N
\]

By Lemma 9.2 it holds

\[
d_{S_{k+l+t+1}}^{(1)}(z_{t+1,0}, u_{t+1}) < K(tL^{t+1} + 2N + 1) < L(t+1)L^{t+1} = (t+1)L^{t+2}
\]

And thus

\[
d_{S_{k+l+t+1}}^{(1)}(z_{t+1,j_{t+1}}, u_{t+1}) < (t+1)L^{t+2} + 2N
\]

It follows that

\[
d_{S_{k+l+t}}^{(1)}(z, u_t) < tL^{t+1} + 2N < L^{2N+3}
\]

and the lemma follows. \( \square \)

**Lemma 9.4.** Let \( X \) and \( G \) be as in Lemma 9.3. Let \( v \in X \) be a vertex. Let \( w \in S_k = S_k(v, X) \) be a vertex. Then there are a number \( k' > k \) and a vertex \( u \in S_{k'} \) such that the link \( X_u \) is a nonorientable surface and \( \Pi'_{k,k'}(u) = w \).
Proof: Let \( w' \in S_{k+1} \) be a vertex such that \( \Pi_{k+1}[Res(w', S_{k+1})] = w \). Let \( \epsilon > 0 \) be such a number that \( \epsilon + \epsilon_l \frac{1}{1-C} < 1 \) for \( l = 2, 3, \ldots \) (where \( \epsilon_l \) are numbers given by the proof of Lemma 5.3). Due to Lemma 9.3 there are a number \( k' > k + 1 \) and a vertex \( u \in S_{k'} \) such that the link \( X_u \) is a nonorientable surface and \( \Pi_{k+1,k'}(u) \in B_{S_{k+1}}(w', \epsilon) \).

Note, that if \( d_{S_l}(x, y) \leq \delta \), then

\[
d_{S_{l-1}}(\Pi'_l(x), \Pi_l(y)) \leq d_{S_{l-1}}(\Pi'_l(x), \Pi_l(x)) + d_{S_{l-1}}(\Pi_l(x), \Pi_l(y)) \leq C \delta + \epsilon_l
\]

It follows that

\[
d_{S_{k+1}}(\Pi'_{k+1,k'}(u), \Pi_{k+1,k'}(u)) \leq \epsilon_{k+1} + C \epsilon_{k+2} + \ldots + C^{k'-k-1} \epsilon_{k'} < \epsilon_{k+1} \frac{1}{1-C} < 1 - \epsilon
\]

Thus \( \Pi'_{k+1,k'}(u) \in B_{S_{k+1}}(w', 1) \subset Res(w', S_{k+1}) \), so \( \Pi_{k,k'}(u) = w \). \qed

Now we can prove part b) of Main Theorem.

Theorem 9.5. Let \( X \) be a 7-systolic nonorientable pseudomanifold of dimension 3. Let a group \( G \) act cocompactly on \( X \) by simplicial automorphisms. Then the Gromov boundary \( \partial G X \) is homeomorphic to the nonorientable Pontriagin surface.

Proof: By Sections 3, 5, 6 and 7 we can assume that the Gromov boundary \( \partial G X \) is homeomorphic to the inverse limit of a system of nonorientable surfaces satisfying assumptions 1 and 3 of Lemma 8.5. By Lemma 9.4 we can assume that assumption 2 is also satisfied. Thus the assertion holds by Lemma 8.5. \qed

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