VERY SINGULAR INTEGRAL KERNELS ON EUCLIDEAN SPACES

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Abstract. We consider a generalisation of Calderón-Zygmund kernels on a Euclidean space with nonisotropic dilations. The resulting distributions, $\mathfrak{N}$-kernels, are highly singular. We look at the behaviour of $\mathfrak{N}$-kernels under the Fourier transform and also ask for conditions of convolvability of such kernels. Another interesting feature of $\mathfrak{N}$-kernels is their invariance for the change of variable $y = x+P(x)$, where P is a homogeneous triangular polynomial mapping. The tasks require the study of the multi-parameter homogeneity of the kernels.

1. Introduction

Suppose that K is a Calderón-Zygmund kernel on $\boldsymbol{R}^N$. This means that K is a tempered distribution which is C^∞ away from the origin and satisfies the *size* estimates

$$|D^\alpha K(x)| \lesssim |x|^{-N-|\alpha|}, \qquad x \neq 0,\ \alpha \in \boldsymbol{N}^N,$$

as well as the *cancellation* condition

$$\sup_{q(\varphi)\leq 1} \sup_{R>0} R^{-N} \left| \int_{\boldsymbol{R}^N} \varphi(Rx)K(x)\,dx \right| < \infty,$$

where q is a seminorm in the Schwartz space $\mathcal{S}(\boldsymbol{R}^N)$. An equivalent condition in terms of the Fourier transform is

$$|D^\alpha \widehat{K}(\xi)| \lesssim |\xi|^{-|\alpha|}, \qquad \xi \neq 0,\ \alpha \in \boldsymbol{N}^N,$$

where the cancellation property remains hidden. Operators of convolution with this kind of kernels arise in many problems of harmonic analysis and differential equations, see, e.g. Stein [11]. Important as they are they allow for interesting generalisations which go in various directions. In this paper we consider the generalisation which on the Fourier transform side takes the seemingly innocuous form

$$|D^\alpha \widehat{K}(\xi)| \lesssim \prod_{k=1}^{N} N_k^\star(\xi)^{-1-m_k-\alpha_k}, \qquad \xi_k \neq 0,\ \alpha \in \boldsymbol{N}^N,$$

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where $N_k^\star$ are suitable partial norms on subspaces of $\boldsymbol{R}^N$ and

$$|m_k| < \varepsilon, \qquad 1 \le k \le N,$$

for some $\varepsilon > 0$. This can be further extended to the context of a nilpotent Lie group with nonisotropic dilations. The resulting distributions, $\mathfrak{N}$-kernels, are highly singular and resist easy treatment.

In this paper, we look at the behaviour of $\mathfrak{N}$-kernels under the Fourier tranform and also ask for conditions of convolvability of such kernels. Another interesting feature of $\mathfrak{N}$-kernels is their invariance for the change of variable $y = x + P(x)$, where P is a homogeneous triangular polynomial mapping. The tasks require the study of the multi-parameter homogeneity of the kernels.

The most important is the case where $m_k = 0$, for $1 \le k \le N$, that is the case of *flag kernels*. The apt idea of a flag kernel is due to Müller-Ricci-Stein [7], [8], Nagel-Ricci-Stein [9], and Nagel-Ricci-Stein-Wainger [10] as a tool in the study of spectral multipliers on two-step nilpotent groups and $\Box_b$-complex operator on certain quadratic submanifolds. The key property under scrutiny is the L^p-boundedness of the operators of convolution with flag kernels. In [10] the problem of the boundedness is considered on arbitrary nilpotent groups with dilations (see also [6].) Closely related is the problem of convolution of flag kernels as tempered distributions.

An alternative approach to the same problems is presented in [6], where $\mathfrak{N}$-kernels of nonzero homogeneity prove instrumental in studying the flag kernels, which seems to sufficiently motivate and justify our interest in the extension of the concept.

2. Basic Setup

Let X be a real N-dimensional vector space with a fixed linear basis $\{e_k\}_{k=1}^N$. Accordingly, each element $x \in X$ has a representation as

$$x = \sum_{k \in \mathcal{N}} x_k e_k = (x_1, x_2, \ldots, x_N) = \big(x_k\big)_{k \in \mathcal{N}},$$

where $\mathcal{N} = \{1, 2, \ldots, N\}$. The space X is assumed to be homogeneous, that is, endowed with a family of dilations $\{\delta_t\}_{t>0}$. The vectors in the basis are supposed to be invariant under dilations:

$$\delta_t e_k = t^{p_k} e_k, \qquad t > 0,\ k \in \mathcal{N},$$

where $0 < p_1 \le p_2 \le \cdots \le p_N$. The number $Q = \sum_{k=1}^N p_k$ is called *the homogeneous dimension* of X. We have

$$d\delta_t x = t^Q dx. \qquad t > 0.$$

By $\alpha, \beta, \cdots$ we shall denote multiindices in $\boldsymbol{N}^N$, where $\boldsymbol{N}$ stands for the set of all nonnegative integers. Let

$$p(\alpha) = \sum_{k=1}^{N} p_k \alpha_k.$$

We shall adopt the following notation for partial derivatives:

$$D_k = \frac{\partial}{\partial x_k}, \qquad D^\alpha = \prod_k D_k^{\alpha_k}.$$

We also let $T_\alpha f(x) = x^\alpha f(x)$.

The Schwartz space of smooth functions which vanish rapidly at infinity along with all their derivatives will be denoted by $\mathcal{S}(X)$. The seminorms

$$q_m(f) = \max_{p(\alpha)+p(\beta)\le m} \sup_{x\in X} |x^\alpha D^\beta f(x)|, \qquad m \in \boldsymbol{N},$$

form a complete set of seminorms in $\mathcal{S}(X)$ giving it a structure of a locally convex Fréchet space. The subspace $C_c^\infty(X)$ of functions with compact support is dense in $\mathcal{S}(X)$. By $L^p(X)$ we denote the usual Lebesgue spaces. $\mathcal{S}(X)$ is a dense subspace of the Lebesgue spaces.

By $\mathcal{S}'(X)$ we denote the space of tempered distributions, the dual to $\mathcal{S}(X)$. It will be convenient to denote the action of a distribution on a Schwartz function by

$$\langle K, \varphi\rangle = \int_X \varphi(x) K(x)\, dx$$

without thereby implying that K is a locally integrable function. If a distribution K coincides with a locally integrable function F on an open set $U \subset X$, we simply write $K(x) = F(x)$, for $x \in U$.

Analogous notation will be applied to the objects on the dual space $X^\star$ with the dual basis $\{e_k^\star\}_{k\in\mathcal{N}}$ and dual dilations still denoted by $\{\delta_t\}_{t>0}$.

We choose Lebesgue measures dx in X and $d\xi$ in $X^\star$ in such a way that Fourier transfoms take the form

$$f^\wedge(\xi) = \widehat{f}(\xi) = \int_X f(x) e^{-i\langle x,\xi\rangle} dx, \qquad g^\vee(\xi) = \int_{X^\star} g(\xi) e^{i\langle x,\xi\rangle} d\xi,$$

where $f \in \mathcal{S}(X),\ g \in \mathcal{S}(X^\star)$ and

$$\langle x, \xi\rangle = \sum_{k=1}^{N} x_k \xi_k$$

is the duality of vector spaces.

For a function F on X and $\varepsilon > 0$, let

$$F^\varepsilon(x) = F(\delta_\varepsilon x), \qquad F_\varepsilon(x) = \varepsilon^{-Q} F(\delta_{\varepsilon^{-1}} x), \qquad \widetilde{F}(x) = F(-x), \qquad x \in X.$$

Let A, B be positive quantities. We shall write $A \lesssim B$ to say that there exists a constant $c > 0$ whose precise value is irrelevant such that $A \leq cB$.

3. $\mathfrak{N}$-kernels

We assume that the set $\mathcal{N} = \{1, 2, \dots, N\}$ is endowed with a partial order $\prec$ such that $k \prec j$ implies $k < j$. A family $\mathfrak{N} = \{\mathcal{N}_k\}_{k\in\mathcal{N}}$ of subsets of $\mathcal{N}$ is called *a filtration* if, for every $k \in \mathcal{N}$ and every $j \in \mathcal{N}$,

a) $j \prec k$ implies $j \in \mathcal{N}_k$.

b) $j \in \mathcal{N}_k$ implies $\mathcal{N}_j \subset \mathcal{N}_k$.

A filtration $\mathfrak{N}$ is *closed* if, for every k,

c) $k \in \mathcal{N}_k$.

Any filtration $\mathfrak{N} = \{\mathcal{N}_k\}_{k\in\mathcal{N}}$ determines *partial norms*:

$$N_k(x) = \sum_{j\in\mathcal{N}_k} |x_j|^{1/p_j}, \qquad k \in \mathcal{N}.$$

If the sets $\mathcal{N}_k$ form a filtration in $\mathcal{N}$, then the sets

$$\mathcal{N}_k^\star = \{j \in \mathcal{N} : k \in \mathcal{N}_j\}$$

form *the dual filtration* in $\mathcal{N}$ which is a filtration in the sense of the above definition with respect to the *dual order*: $k \prec^\star j$ if and only if $j \prec k$. We also have

$$\mathcal{N}_k^{\star\star} = \mathcal{N}_k, \qquad k \in \mathcal{N}.$$

Let

$$N_k^\star(\xi) = \sum_{j\in\mathcal{N}_k^\star} |\xi_j|^{1/p_j}$$

be the *dual partial norms* on the dual vector space $X^\star$. Denote by $\mathcal{N}_{\min}$ (resp. $\mathcal{N}_{\max}$) the set of the minimal (resp. maximal) elements with respect to the order $\prec$.

For a given $I \subset \mathcal{N}$, denote by H_I *the singular subspace* of $X_I = \langle e_k\rangle_{k\in I}$ constisting of all $x_I = (x_k)_{k\in I}$ such that $x_k = 0$ for some $k \in I_{\min}$. Let $H = H_{\mathcal{N}}$. Similarly, denote by $H_I^\star$ *the singular subspace* of $X_I^\star = \langle e_k^\star\rangle_{k\in I}$ constisting of all $\xi_I = (\xi_k)_{k\in I}$ such that $\xi_k = 0$ for some $k \in I_{\max}$. Let $H^\star = H_{\mathcal{N}}^\star$.

For $\varphi \in \mathcal{S}(X)$ and $R = (r_k)_{k\in I} \in (0,\infty)^I$, let

$$\varphi^R(x_I) = \varphi(\Delta_R x_I),$$

where $\Delta_R x_I = \Big(\delta_{r_k} x_k\Big)_{k\in I}$. If $M = (m_1, m_2, \dots, m_N) \in \boldsymbol{R}^N$, then we set $M_I = (m_k)_{k\in I}$.

Given a distribution $K \in \mathcal{S}'(X)$ and a function $\varphi \in \mathcal{S}(X_I)$, we define a distribution K^φ on X_J, where $J = \mathcal{N} \setminus I$, by letting

$$\langle K^\varphi, f\rangle = \int_{X_I}\int_{X_J} \varphi(x_I) f(x_J) K(x_I, x_J)\, dx_I\, dx_J, \qquad f \in \mathcal{S}(X_J).$$

In other words,

$$\langle K^\varphi, f\rangle = \langle K, \varphi \otimes f\rangle, \qquad f \in \mathcal{S}(X_J).$$

Definition 3.1. Let $\mathfrak{N} = \{\mathcal{N}_k\}_{k\in\mathcal{N}}$ be a filtration, and $M = (m_k)_{k\in\mathcal{N}} \in \boldsymbol{R}^N$ a multiindex. We say that a tempered distribution K belongs to $\mathcal{F}^M(X, \mathfrak{N})$ if there exists a Schwartz seminorm q_m such that, for every $I \subset \mathcal{N}$, every Schwartz function $\varphi \in \mathcal{S}(X_I)$, and every $\alpha \in \boldsymbol{N}^N$, the distribution K^φ is smooth on X_J away from the singular space H_J and

$$C(K, \alpha, I) = \sup_{q_m(\varphi)\le 1}\ \sup_{R\in(0,\infty)^I} \prod_{k\in I} r_k^{-m_k}\, C(K, \alpha, I, \varphi^R) < \infty, \tag{3.2}$$

where

$$C(K, \alpha, I, \varphi) = \sup_{x_J\in X_J} \prod_{k\in\mathcal{N}} N_k(x_J)^{p_k+m_k+p_k\alpha_k} |D^\alpha K^\varphi(x_J)|.$$

(To avoid ambiguity we assume that $q_m = q$ is the minimal seminorm with the above property.)

For $K \in \mathcal{F}^M(X, \mathfrak{N})$, we define seminorms

$$\|K\|_{\mathcal{F}^M, L} = \max_{p(\alpha)\le L}\ \max_{I\subset\mathcal{N}} C(K, \alpha, I), \qquad L \in \boldsymbol{N}.$$

A family $\mathcal{K}$ of elements of $\mathcal{F}^M(X, \mathfrak{N})$ is said to be *bounded* if, for every L,

$$\sup_{K\in\mathcal{K}} \|K\|_{\mathcal{F}^M, L} < \infty.$$

Remark 3.3. Let us look closer at the extreme cases of (3.2), that is when $I = \varnothing$ and $I = \mathcal{N}$. The first one implies that K itself is smooth away from H and

$$|D^\alpha K(x)| \le C_\alpha \prod_{k\in\mathcal{N}} N_k(x)^{-p_k-m_k-p_k\alpha_k}, \qquad x \notin H,\ \alpha \in \boldsymbol{N}^N. \tag{3.4}$$

The other one gives the *cancellation condition*:

$$|\langle K, \varphi^R\rangle| \le C q(\varphi) \prod_{k=1}^N r_k^{m_k}, \tag{3.5}$$

for every $\varphi \in \mathcal{S}(X)$ and every $R = (r_1, r_2, \dots, r_N) \in (0,\infty)^N$. Note that if the filtration is trivial with $\mathcal{N}_k = \mathcal{N}$ for all $k \in \mathcal{N}$, then (3.4) and (3.5) are the only conditions defining $\mathcal{F}^M(X, \mathfrak{N})$.

Proposition 3.6. *Let $\mathfrak{N}$ be a filtration, let $m_k < 0$, for $1 \le k \le N$, and let $F \in C^\infty(X \setminus H)$ satisfy* (3.4). *Then, F is locally integrable, and the distribution*

$$\langle K, f\rangle = \int_X f(x)F(x)dx$$

belongs to $\mathcal{F}^M(X, \mathfrak{N})$.

Proof. Note first that the condition $m_k < 0$ is equivalent to saying that F is locally integrable. Let $I \subset \mathcal{N}$ and $J = \mathcal{N} \setminus I$. Then, for $\varphi \in \mathcal{S}(X_I)$ and $\alpha \in \boldsymbol{N}^N$,

$$\begin{aligned} D^\alpha K^{\varphi^R}(x_J) &= \int_{X_I} \varphi^R(x_I) D^\alpha_{x_J} F(x_I, x_J)\, dx_I \\ &= \prod_{k\in I} R_k^{-p_k} \int_{X_I} \varphi(x_I) D^\alpha_{x_J} F(\Delta_{R^{-1}} x_I, x_J)\, dx_I, \end{aligned}$$

whence,

$$|D^\alpha K^{\varphi^R}(x_J)| \lesssim \prod_{k\in I} R_k^{-p_k} \int_{X_I} |\varphi(x_I)| \prod_{k\in\mathcal{N}} N_k\Big(\Delta_{R^{-1}} x_I, x_J\Big)^{-p_k - m_k - p_k\alpha_k}\, dx_I.$$

Now,

$$N_k(x_I, x_J)^{-1} \le N_k(x_I)^{-1}, \qquad N_k(x_I, x_J)^{-1} \le N_k(x_J)^{-1},$$

and $N_k(x_I)^{-1} \le |x_k|^{-1/p_k}$ if $k \in I$. Therefore,

$$|D^\alpha K^{\varphi_I^R}(x_J)| \lesssim q(\varphi) \prod_{k\in J} N_k(x_J)^{-p_k - m_k - p_k\alpha_k},$$

for $x_J \notin H_J$, where

$$q(\varphi) = \int_{X_I} |\varphi(x_I)| \prod_{k\in I} |x_k|^{-1-m_k/p_k}\, dx_I$$

is a seminorm on $\mathcal{S}(X_I)$ since $m_k < 0$. It is clear that all constants in the above estimates depend only on the constants C_α in (3.4). Our proof is complete. □

Proposition 3.7. *If $K \in \mathcal{F}^m(X, \mathfrak{N})$, then, for every α, $D^\alpha K \in \mathcal{F}^{m+p\alpha}(X, \mathfrak{N})$ and $T_\alpha K \in \mathcal{F}^{m-p\alpha}(X, \mathfrak{N})$, where $p\alpha = (p_1\alpha_1, p_2\alpha_2, \ldots, p_N\alpha_N)$.*

We leave it as an exercise for the reader.

Example 3.8. If $K = \varphi$ is a Schwartz function, then $K \in \mathcal{F}^M(X, \mathfrak{N})$ for every $\mathfrak{N}$ and every $M = (m_k)_{k\in\mathcal{N}}$ such that $-p_k < m_k < 0$.

Example 3.9. Let $0 < r \le N$ and

$$k_0 = 0 < k_1 < \cdots < k_{r-1} < k_r = N.$$

Let $\mathfrak{N} = \{\mathcal{N}_j\}_{j\in\mathcal{N}}$, where, for $k_{l-1} < j \le k_l$, $1 \le l \le r$,

$$\mathcal{N}_j = \{k \in \mathcal{N} : k \le k_l\}.$$

Then the class $\mathcal{F}^0(X,\mathfrak{N})$ is exactly the class of *flag kernels* of Nagel-Ricci-Stein-Wainger corresponding to the flag

$$\{0\} \subset \langle e_j\rangle_{j\in\mathcal{N}_1} \subset \cdots \subset \langle e_j\rangle_{j\in\mathcal{N}_l} \subset \cdots \subset \langle e_j\rangle_{j\in\mathcal{N}_{k_r}}.$$

This filtration is closed.

Example 3.10. On the other hand, the filtration

$$\mathcal{N}_j = \{k\in\mathcal{N} : k<j\}$$

is not closed.

4. Fourier transform

Proposition 4.1. *Let $\mathfrak{N}$ be a closed filtration in $\mathcal{N}$. Let*

$$M = (m_1, m_2, \ldots, m_N) \in \boldsymbol{R}^N$$

and let $K \in \mathcal{F}^M(X,\mathfrak{N})$. Let $\varphi_k \in C_c^\infty(X_k)$ be equal to 1 in a neighbourhood of zero and

$$\Phi(x) = \prod_{k\in\mathcal{N}} \varphi_k(x_k), \qquad x\in X. \tag{4.2}$$

Then, for every $\varepsilon>0$,

$$\left|\int_X e^{-ix\xi}\Phi(\delta_\varepsilon x)K(x)dx\right| \lesssim \prod_{k\in\mathcal{N}}\big(\varepsilon + N_k^\star(\xi)\big)^{m_k}, \qquad \xi\in X^\star.$$

Proof. Let us fix $\xi \in X^\star\setminus H^\star$. Let $R_k = N_k^\star(\xi)$. Let $\eta_k \in C_c^\infty(X_k)$ be equal to 1, for $|x_k|\le R_k^{-p_k}$. Let

$$\eta_I(x) = \prod_{k\in I}\eta_k(x_k), \qquad \eta_I^c(x) = \prod_{k\in I^c}\Big(1-\eta_k(x_k)\Big), \qquad x\in X,$$

where $I^c = \mathcal{N}\setminus I$. Note that

$$1 = \sum_{I\subset\mathcal{N}} \eta_I(x)\eta_I^c(x), \qquad x\in\mathfrak{g}.$$

Under the notation introduced above

$$\int_X e^{-ix\xi}\Phi(\delta_\varepsilon x)K(x)\,dx = \sum_{I\subset\mathcal{N}}\int_X e^{-ix\xi}\eta_I(\Delta_R x)\eta_I^c(\Delta_R x)\Phi(\delta_\varepsilon x)K(x)\,dx.$$

We are going to estimate each term

$$S_I(\xi) = \int_X e^{-ix\xi}\eta_I(\Delta_R x)\eta_I^c(\Delta_R x)\Phi(\delta_\varepsilon x)K(x)\,dx, \qquad I\subset\mathcal{N},$$

separately. We begin with

$$\begin{aligned} S_{\mathcal{N}}(\xi) &= \int_X e^{-ix\xi}\eta_{\mathcal{N}}(\Delta_R x)\Phi(\delta_\varepsilon x)K(x)\,dx \\ &= \int_X F_{R,\varepsilon}(\Delta_R x)K(x)\,dx, \end{aligned} \tag{4.3}$$

where the two-parameter family of $C_c^\infty(X)$-functions

$$F_{R,\varepsilon}(x) = e^{-ix\Delta_{R^{-1}}\xi}\eta_{\mathcal{N}}(x)\Phi(\Delta_{\varepsilon R^{-1}}x), \qquad 0<\varepsilon<\min_{k\in\mathcal{N}} R_k,$$

is bounded in $\mathcal{S}(X)$. Therefore, by definition,

$$|S_{\mathcal{N}}(\xi)| \lesssim \prod_{k\in\mathcal{N}} R_k^{m_k}.$$

As for $S_\varnothing$, we need an extra twist. Recall that $\xi\in X^\star\setminus H^\star$ is fixed. For every $k\in\mathcal{N}$, let $\lambda(k)$ be the biggest index $l\in\mathcal{N}_k^\star$ such that

$$|\xi_l|^{1/p_l} = \max_{j\in\mathcal{N}_k^\star}|\xi_j|^{1/p_j}.$$

Then $|\xi_l|^{1/p_l}\approx N_k^\star(\xi)$. Of course, the mapping $\lambda:\mathcal{N}\to\mathcal{N}$ and the set $\Lambda=\lambda(\mathcal{N})$ depend on ξ. Let $a=(a_1,a_2,\dots,a_N)$ be a multiindex such that $m_k+p_{\lambda(k)}a_k>0$, for every k, and let

$$D^A = D_{\lambda(1)}^{a_1}D_{\lambda(2)}^{a_2}\dots D_{\lambda(N)}^{a_N} = D_1^{A_1}D_2^{A_2}\dots D_N^{A_N},$$

so that $A_l=\sum_{\lambda(k)=l}a_k$ if $l\in\Lambda$ and $A_l=0$ if $l\notin\Lambda$. If

$$W_{R,\varepsilon}(x) = \eta_\varnothing^c(\Delta_R x)\Phi(\delta_\varepsilon x) = \prod_{k\in\mathcal{N}}\Big(1-\eta_k(R_k^{p_k}x_k)\Big)\varphi_k(\varepsilon^{p_k}x_k)$$

then

$$D^A\big(W_{R,\varepsilon}K\big)(x)=0,$$

if $|x_k|\le R_k^{-p_k}$, for some $k\in\mathcal{N}$, and

$$|D^A\big(W_{R,\varepsilon}K\big)(x)| \lesssim \prod_k |x_k|^{-p_k-m_k-p_kA_k}. \tag{4.4}$$

This follows from the fact that $R_k\approx|x_k|^{-1}$ on the support of the derivative of η_k and $\varepsilon\approx|x_k|^{-1}$ on the support of the derivatives of φ_k. The filtration is closed, so we also have $N_k(x)\ge|x_k|^{1/p_k}$.

Now, by integration by parts and application of (4.4),

$$\begin{aligned}
|S_\varnothing(\xi)| &= \left|\int_X e^{-ix\xi}W_{R,\varepsilon}(x)K(x)dx\right| \\
&\approx \prod_{k=1}^N |\xi_{\lambda(k)}|^{-p_kA_k}\left|\int_X e^{-ix\xi}D^A\big(W_{R,\varepsilon}K\big)(x)\,dx\right| \\
&\lesssim \prod_{k=1}^N R_k^{-p_kA_k}\int_{U_1\times U_2\times\dots\times U_N}\prod_k |x_k|^{-p_k-m_k-p_kA_k}dx,
\end{aligned}$$

where $U_k = [R_k^{-p_k}, \infty)$, for $k \in \mathcal{N}$. Consequently,

$$\begin{aligned}\prod_{k\in\mathcal{N}} R_k^{p_k A_k} |S_\varnothing(\xi)| &\lesssim \int_{U_1\times U_2\times\ldots U_N} \prod_{k\in\mathcal{N}} |x_k|^{-p_k-m_k-p_{\lambda(k)}a_k}\, dx,\\ &\le \prod_{k\in\mathcal{N}} \int_{U_k} \Big(|x_k|^{1/p_k}\Big)^{-p_k-m_k-p_{\lambda(k)}a_k}\, dx,\\ &\lesssim \prod_{k\in\mathcal{N}} R_k^{m_k+p_{\lambda(k)}a_k} = \prod_{k\in\mathcal{N}} R_k^{m_k+p_kA_k},\end{aligned}$$

and, finally,

$$|S_\varnothing(\xi)| \lesssim \prod_{k\in\mathcal{N}} R_k^{m_k}.$$

It remains to consider those cases where I is a proper subset of $\mathcal{N}$. We have

$$\begin{aligned}S_I(\xi) &= \int_X \Big(e^{-ix_I\xi_I}\eta_I(\Delta_{R_I}x_I)\Phi_I(\delta_\varepsilon x_I)\Big)\Big(e^{-ix_J\xi_J}\eta_I^c(\Delta_{R_J}x_J)\Phi_J(\delta_\varepsilon x_J)\Big)K(x)\,dx\\ &= \int_X F_{R_I,\varepsilon}(\Delta_{R_I}x_I)G_{R_J,\varepsilon}(\Delta_{R_J}x_J)K(x)\,dx\\ &= \int_{X_I} F_{R_I,\varepsilon}(\Delta_{R_I}x_I)K^{G^{R_I}_{R_J,\varepsilon}}(x_I)\,dx_I,\end{aligned}$$

where $J = I^c = \mathcal{N}\setminus I$,

$$F_{R_I,\varepsilon}(x_I) = e^{-ix_I\Delta_{R_I^{-1}}\xi_I}\eta_I(x_I)\Phi_I(\Delta_{R_I^{-1}\varepsilon}x_I), \qquad 0<\varepsilon<\min_{k\in I} R_k,$$

and

$$G_{R_J,\varepsilon}(x_J) = e^{-ix_J\Delta_{R_J^{-1}}\xi_J}\eta_I^c(x_J)\Phi_J(\Delta_{R_J^{-1}\varepsilon}x_J), \qquad 0<\varepsilon<\min_{k\in J} R_k.$$

The two-parameter familiy $G_{R_J,\varepsilon}$ is bounded in $\mathcal{S}(X_J)$, so

$$K_{R_I,\varepsilon} = \Big(\prod_{k\in I} R_k^{-m_k}\Big)K^{G^{R_J}_{R_J,\varepsilon}}$$

is a family of kernels bounded in $\mathcal{F}^M(X_J)$. The integral

$$\int_{X_I} F_{R_I,\varepsilon}(\Delta_{R_I}x_I)K_{R_J,\varepsilon}(x_I)\,dx_I = \Big(\prod_{k\in I} R_k^{-m_k}\Big)S_I(\xi)$$

is a lower-dimensional counterpart of the integral (4.3), so that

$$\Big(\prod_{k\in I} R_k^{-m_k}\Big)|S_I(\xi)| \lesssim \prod_{k\in J} R_k^{m_k},$$

that is

$$|S_I(\xi)| \lesssim \prod_{k\in\mathcal{N}} R_k^{m_k}.$$

This completes the proof. $\square$

Corollary 4.5. *Under the hypotheses of Proposition 4.1,*

$$|D^\alpha\big(\widehat{K}\star\widehat{\Phi}_\varepsilon\big)(\xi)| \lesssim \prod_{k\in\mathcal{N}}\big(\varepsilon+N_k^\star(\xi)\big)^{m_k-p_k\alpha_k}, \qquad \xi\in X^\star,$$

uniformly for $\varepsilon>0$.

Proof. The function $\widehat{\Phi}_\varepsilon\star\widehat{K}$ is smooth and

$$|D^\alpha\big(\widehat{\Phi}_\varepsilon\star\widehat{K}\big)(\xi)| = |\widehat{\Phi}_\varepsilon\star\ D^\alpha\widehat{K}(\xi)|, \qquad \xi\in X^\star,\ \alpha\in\boldsymbol{N}^N.$$

Since $|D^\alpha\widehat{K}| = \big|\big(T_\alpha K\big)^\wedge\big|$ and $T_\alpha K\in\mathcal{F}^{M-p\alpha}(X,\mathfrak{N})$, our claim follows by Proposition 4.1.

□

For $M=(m_1,m_2,\dots,m_N)\in\boldsymbol{R}^N$, let

$$M^\star=(-p_1-m_1,-p_2-m_2,\dots,-p_N-m_N).$$

Theorem 4.6. *Let $\mathfrak{N}$ be a closed filtration in $\mathcal{N}$. Let $M\in\boldsymbol{R}^N$. If $K\in\mathcal{F}^M(X,\mathfrak{N})$, then $\widehat{K}\in\mathcal{F}^{M^\star}(X^\star,\mathfrak{N}^\star)$.*

Proof. Let us keep the notation of Proposition 4.1. We have $\widehat{\Phi}_\varepsilon\star\widehat{K}\to\widehat{K}$ in the sense of distributions. The estimates of Corollary 4.5 show that all derivatives $D^\alpha\big(\widehat{\Phi}_\varepsilon\star\widehat{K}\big)$ are locally bounded in $X^\star\setminus H^\star$, so by the usual argument based on the Ascoli theorem we conclude that the convergence is almost uniform in $X^\star\setminus H^\star$, which implies that $\widehat{K}$ is smooth away from $H^\star$ and satisfies

$$|D^\alpha\widehat{K}(\xi)| \lesssim \prod_{k\in\mathcal{N}} N_k^\star(\xi)^{m_k-p_k\alpha_k}, \qquad \xi\in X^\star\setminus H^\star.$$

It remains to take care of the lower-dimensional conditions. Let I,J and φ be as in Definition 3.1. Let

$$\mathfrak{N}_J=\{E\cap J : E\in\mathfrak{N}\}.$$

This is a closed filtration in J. The distribution K^{φ^R} is a kernel in $\mathcal{F}^{M_J}(X_J,\mathfrak{N}_J)$, so

$$\widehat{K}^{\varphi^R}=\Big(\prod_{k\in I}R^{-p_k}\Big)\Big(K^{(\varphi^\vee)^{R^{-1}}}\Big)^\wedge$$

is in $C^\infty(X_J^\star\setminus H_J^\star)$ and satisfies the necessary condition for the decay of derivatives. The estimates are uniform in $R\in(0,\infty)^I$ and $\varphi\in\mathcal{S}(X_J)$. □

5. APPROXIMATION

We shall consider the following classes of distributions on X associated with a fixed closed filtration $\mathfrak{N}$. For $M=\big(m_k\big)_{k\in\mathcal{N}}\in\boldsymbol{R}^N$ and

$\varepsilon > 0$, let $S_\varepsilon(M)$ be the totality of distributions A on X whose Fourier transform $\widehat{A}$ is smooth and satisfies

$$(5.1) \qquad |D^\alpha \widehat{A}(\xi)| \lesssim \prod_{k\in\mathcal{N}} \big(\varepsilon + N_k^\star(\xi)\big)^{m_k - p_k\alpha_k}, \qquad \xi \in X^\star,$$

cf. Corollary 4.5. The class is a Fréchet space if equipped with the family of natural seminorms

$$\|A\|_{S_\varepsilon(M),l} = \max_{p(\alpha)\le l} \sup_{\xi\in\mathfrak{g}^\star} \prod_{k\in\mathcal{N}} \big(\varepsilon + N_k^\star(\xi)\big)^{-m_k+p_k\alpha_k} |D^\alpha \widehat{A}(\xi)|, \qquad l \in \boldsymbol{N}.$$

A sequence $K_\nu \in S_{1/\nu}(M)$ is said to be an M-*approximating sequence* if K_ν is convergent in $\mathcal{S}'(\mathfrak{g})$ and *bounded*, that is

$$\sup_\nu \|K_\nu\|_{S_{1/\nu},l} < \infty,$$

for every $l \in \boldsymbol{N}$.

Proposition 5.2. *Let $\mathfrak{N}$ be a closed filtration and $M \in \boldsymbol{R}^N$ an multi-index with $m_k > -p_k$. Then, $K \in \mathcal{F}^M(X,\mathfrak{N})$ if and only if there exists an M-aproximating sequence convergent in $\mathcal{S}'(X)$ to K.*

Proof. Let $K \in \mathcal{F}^M(X,\mathfrak{N})$. For $\nu \in \boldsymbol{N}$, let

$$K_\nu(x) = \Phi(\delta_{1/\nu}x)K(x), \qquad x \in X,$$

where Φ is as in Proposition 4.1. Then, by Corollary 4.5, the sequence $K_\nu \in S(1/\nu)$ is bounded and convergent to K in $\mathcal{S}'(X)$.

Now let K be a limit of an M-approximate sequence K_ν. We have

$$(1/\nu + N_k(\xi))^{m_k-p_k\alpha_k} \le (1 + N_k(\xi))^{m_k-p_k\alpha_k},$$

if $m_k - p_k\alpha_k \ge 0$, and

$$(1/\nu + N_k(\xi))^{m_k-p_k\alpha_k} \le N_k(\xi)^{m_k-p_k\alpha_k},$$

if $m_k - p_k\alpha_k \le 0$. Therefore, by the Ascoli theorem, $D^\alpha \widehat{K}_\nu$ converges uniformly, for every α, on every compact subset of $X^\star \setminus H^\star$. Thus,

$$\widehat{K}(\xi) = \lim_{\nu\to\infty} \widehat{K}_\nu(\xi)$$

satisfies the necessary estimates for the derivatives on $X^\star \setminus H^\star$. In particular,

$$|\widehat{K}(\xi)| \lesssim \prod_{k\in\mathcal{N}} N_k^\star(\xi)^{m_k},$$

which shows that $\widehat{K}$ is locally integrable. Therefore, by Proposition 3.6, $\widehat{K} \in \mathcal{F}^{M^\star}(X^\star,\mathfrak{N}^\star)$, so, by Theorem 4.6, $K \in \mathcal{F}^M(X,\mathfrak{N})$. $\square$

6. Change of variable

A mapping $P : X \to X$ is said to be a *homogeneous triangular polynomial mapping*, if

$$P(x) = \big(P_1(x), P_2(x), \dots P_N(x)\big),$$

where P_k are polynomials of degree at least 2 such that

$$P_k(x) = P_k(x_1, x_2, \dots, x_{k-1}) \tag{6.1}$$

and

$$P(\delta_t x) = \delta_t P(x), \qquad x \in X,\ t > 0. \tag{6.2}$$

Condition (6.2) means that, for every k, P_k is a homogeneous polynomial of degree p_k.

The mapping P determines an order in $\mathcal{N}$. We write $k \prec j$ if $D_k P_j \neq 0$ and extend this relation to the smallest order containg it. By (6.1), $k \prec j$ implies $k < j$.

Let the operator $\mathbf{S}_P$ be defined as

$$\mathbf{S}_P f(x) = f(x + P(x)), \qquad x \in X,$$

for $f \in \mathcal{S}(X)$ and

$$\langle \mathbf{S}_P A, f\rangle = \langle A, \mathbf{S}_{\widetilde{P}} f\rangle, \qquad f \in \mathcal{S}(X),$$

for $A \in \mathcal{S}'(X)$, where $\widetilde{P}(x + P(x)) = -P(x)$. In other words, $y = x + P(x)$ is equivalent to $x = y + \widetilde{P}(y)$.

Recall from [6] the operator

$$\mathbf{U}_P F(\xi) = \iint_{X\times X} e^{-i(\langle x,\xi\rangle)} e^{-i\langle P(x,y),\xi\rangle} F^\vee(x,y))\, dx\, dy, \qquad \xi \in X^\star,$$

which we here consider as acting on $\mathcal{S}(X) \times \mathcal{S}(X)$:

$$\mathbf{U}_P(f,g)(\xi) = \iint_{X\times X} e^{-i(\langle x,\xi\rangle)} e^{-i\langle P(x,y),\xi\rangle} f^\vee(x) g^\vee(,y))\, dx\, dy.$$

By Theorem 7.1 of [6], $\mathbf{U}_P$ has an extension, still denoted by $\mathbf{U}_P$, to a Fréchet continuous mapping $S_1(M) \times S_1(M') \to S_1(M + M')$, where $M, M' \in \boldsymbol{R}^N$. (This extension is unique in the sense that is irrelevant here.) As a matter of fact, the Fréchet continuity has not been made explicit in the statement of the theorem, but the final part of the proof (see the last line of the page 1657) shows that the Fréchet continuity claim holds true.

Let

$$\mathcal{P}(x,y) = \big(P(x), P(y)\big), \qquad x, y \in X.$$

If we apply the above to the homogeneous triangular polynomial mapping $\mathcal{P}$, we get the operator $\mathbf{U}_{\mathcal{P}}$ acting on distributions on $X^\star \times X^\star$

whose restriction to $\mathcal{S}(X^\star) \times \mathcal{S}(X^\star)$ is

$$\mathbf{U}_{\mathcal{P}}(f,g)(\xi,\eta) = \iint_{X\times X} e^{-i(\langle x,\xi\rangle)+\langle y,\eta\rangle)} e^{-i\langle P(x),\xi\rangle+\langle P(u),\eta\rangle)} f^\vee(x,y) g^\vee(u,v))\, dxdydudv.$$

Let

$$\mathbf{V}_P(A)(\xi) = \mathbf{U}_{\mathcal{P}}(A,\mathbf{1})(\xi,0), \qquad A \in \mathcal{S}'(X^\star),\ \xi \in X^\star.$$

Fix $M \in \boldsymbol{R}^N$. It follows that $\mathbf{V}_P : S_1(M) \to S_1(M)$ is Fréchet continuous. For $A \in \mathcal{S}'(\mathfrak{g})$ and $\varepsilon > 0$, let

$$A_\varepsilon(x) = \varepsilon^{-Q-\sum_{k=1}^N m_k} A(\varepsilon^{-1}x).$$

Clearly,

$$\mathbf{V}_P(A_\varepsilon) = \mathbf{V}_P(A)_\varepsilon, \qquad A \in \mathcal{S}'(X^\star), \quad \varepsilon > 0,$$

so $\mathbf{V}_P : S_\varepsilon(M) \to S_\varepsilon(M)$ is Fréchet continuous uniformly in ε. If $A = f \in \mathcal{S}(X^\star)$, then

$$\mathbf{V}_P f(\xi) = \int_X e^{-i\langle x,\xi\rangle} e^{-i\langle P(x),\xi\rangle} f^\vee(x)\, dx,$$

so by the change of variable $z = x + P(x)$,

$$\mathbf{S}_P = \mathcal{F} \circ \mathbf{V}_P \circ \mathcal{F}^{-1},$$

where $\mathcal{F}$ denotes the Fourier transform.

Theorem 6.3. *Let $\mathfrak{N} = \{\mathcal{N}_k\}_{k\in\mathcal{N}}$ be a filtration, and $M = (m_k)$ a multiindex. If $K \in \mathcal{F}^M(X,\mathfrak{N})$, the so does $\mathbf{S}_P K$.*

Proof. Let $K \in \mathcal{F}^M(X,\mathfrak{N})$. Let K_ν be an M-approximating sequence for K. Then,

$$L_\nu = \mathbf{S}_P K_\nu = \Big(\mathbf{V}_P(K_\nu^\wedge)\Big)^\vee$$

is another M-approximating sequence convergent to $\mathbf{S}_P K$. Hence, by Proposition 5.2, $\mathbf{S}_P K \in \mathcal{F}^M(X,\mathfrak{N})$. □

7. CONVOLUTION

We keep the setup of Section 2. Furthermore, we endow X (since now denoted by $\mathfrak{g}$) with a Lie algebra commutator $(x,y) \mapsto [x,y]$ such that the dilations δ_t become automorphisms. The commutator determines a Campbell-Hausdorff multiplication (see Corwin-Greenleaf [2])

$$xy = x + y + P(x,y), \tag{7.1}$$

where

$$P(x,y) = \frac{1}{2}[x,y] + \frac{1}{12}[x,[x,y]] + \frac{1}{12}[y,[y,x]] + \dots$$

is a homogeneous triangular polynomial mapping. This makes $\mathfrak{g}$ into a nilpotent Lie group with the same underlying manifold X.

If $A \in S_\varepsilon(M)$ for some $M \in \boldsymbol{R}^N$ and $\varepsilon > 0$, then for every compact neighbourhood U of 0 in $\mathfrak{g}$, $A = A_0 + F$, where A_0 is supported in U and $F \in \mathcal{S}(\mathfrak{g})$. Thus, for every two members of the classes $S_\varepsilon(M)$, convolution makes sense. Let $A \in S_1(M)$, $B \in S_1(M')$. Then, by Proposition 8.1 of [6],

$$(A \star B)^\wedge = \mathbf{U}_P(A^\wedge, B^\wedge).$$

Consequently, the mappings

$$S_\varepsilon(M) \times S_\varepsilon(M') \ni (A, B) \mapsto A \star B \in S_\varepsilon(M + M'), \qquad \varepsilon > 0, \tag{7.2}$$

are continuous.

Let $p > 1$. We say that a multiindex $M = (m_k)_{k \in \mathcal{N}}$ is *p-admissible* if, for every $k \in \mathcal{N}$,

$$p_k(1/p - 1) < m_k < p_k/p.$$

Proposition 7.3. *Let $\mathfrak{N}$ be a closed filtration. Let $M = \big(m_k\big)_{k \in \mathcal{N}}$ be p-admissible for some $p > 1$. For $\nu \in \boldsymbol{N}$, let K_ν be an M-approximating sequence convergent to K in the sense of distributions. Then, for every $\varphi \in \mathcal{S}(\mathfrak{g})$ and every ν, $\varphi \star K_\nu \in L^p(\mathfrak{g})$ and $\varphi \star K_\nu \to \varphi \star K$ in $L^p(\mathfrak{g})$. Thus $\varphi \star K \in L^p(\mathfrak{g})$.*

Proof. Let $\varphi \in \mathcal{S}(\mathfrak{g})$. We know that $\varphi \in \mathcal{F}^T(\mathfrak{g}, \mathfrak{N})$ for every T with $-p_k < t_k < 0$ (see Example 3.8). Also $\varphi \in S_{1/\nu}(T)$ uniformly in ν. By symbolic calculus (7.2), $F_\nu = K_\nu \star \varphi \in S_{1/\nu}(M + T)$. It is not hard to see that F_ν is an $(M + T)$-approximating sequence for $F = K \star \varphi$. Hence, by Proposition 5.2, $F \in \mathcal{F}^{M+T}(\mathfrak{g})$, which implies

$$|F(x)| \lesssim \prod_{k \in \mathcal{N}} |x_k|^{-p_k - m_k - t_k}, \qquad x \in X \setminus H,$$

for any $-p_k < t_k < 0$. If T is chosen so that $m_k + t_k < p_k/p$ if $|x_k| \leq 1$ and $m_k + t_k > p_k(1/p - 1)$ if $|x_k| > 1$, the right-hand side is in $L^p(\mathfrak{g})$, so $K \star \varphi \in L^p(\mathfrak{g})$. We can do so since M is p-admissible.

The same argument implies uniformly to each F_ν. The sequence is convergent to F almost everywhere, hence, by the Lebesgue dominated convergence theorem, it is convergent in $L^p(\mathfrak{g})$. □

The following definition is due to Chevalley. Tempered distributions K, L are *convolvable* if

$$(\widetilde{K} \star \varphi)(L \star \widetilde{\psi}) \in L^1(\mathfrak{g}),$$

for all $\varphi, \psi \in \mathcal{S}(\mathfrak{g})$. Then, by Chevalley [1], Section 8, there exists a unique tempered distribution $K \star L$ such that

$$\langle K \star L, \varphi \star \psi \rangle = \int_{\mathfrak{g}} \widetilde{K} \star \varphi(x)\, L \star \widetilde{\psi}(x)\, dx,$$

for all $\varphi, \psi \in \mathcal{S}(\mathfrak{g})$. Recall that functions of the form $\varphi \star \psi$ span the whole of $\mathcal{S}(\mathfrak{g})$. (In fact, by Dixmier-Malliavin [3], every function in $\mathcal{S}(\mathfrak{g})$ is of this form, but we need not this difficult result.)

Remark 7.4. If $K_j \in S_1(M_j)$, $j = 1, 2$, then it is not hard to see that they are convolvable in the sense of Chevalley and the Chevalley convolution coincides with the usual convolution $K_1 \star K_2$ of distributions.

Theorem 7.5. *Let $\mathfrak{N}$ be a closed filtration. Let $1/p + 1/q \leq 1$, $1 < p, q < \infty$. Let $K \in \mathcal{F}^{M_1}(\mathfrak{g}, \mathfrak{N})$, $L \in \mathcal{F}^{M_2}(\mathfrak{g}, \mathfrak{N})$, where M_1 is p-admissible and M_2 is q-admissible. Then the distributions K and L are convolvable and $K \star L \in \mathcal{F}^{M_1+M_2}(\mathfrak{g}, \mathfrak{N})$.*

Proof. We may assume that $1/p+1/q = 1$. By Proposition 7.3, K and L are convolvable. Let K_ν and L_ν be the approximate sequences converging to K and L, respectively (Proposition 5.2). Then, by symbolic calculus, the sequence

$$U_\nu = K_\nu \star L_\nu \in S(1/\nu, M_1 + M_2) \tag{7.6}$$

is bounded. By (3.5), the distributions U_ν are equicontinuous in $\mathcal{S}'(\mathfrak{g})$. Once we show that U_ν is convergent to $K \star L$ in $\mathcal{S}'(\mathfrak{g})$, we shall be able to conclude that U_ν is an $(M_1 + M_2)$-approximate sequence and $K \star L \in \mathcal{F}^{M_1+M_2}(\mathfrak{g}, \mathfrak{N})$.

Let $f, g \in \mathcal{S}(\mathfrak{g})$. By (7.6),

$$\langle U_\nu, f \star g\rangle = \int_{\mathfrak{g}} \widetilde{K_\nu} \star f(x)\, L_\nu \star \widetilde{g}(x)\, dx,$$

where, by Proposition 7.3, $\widetilde{K_\nu} \star f \to \widetilde{K} \star f$ in $L^p(\mathfrak{g})$ and $L_\nu \star \widetilde{g} \to L_\nu \star \widetilde{g}$ in $L^q(\mathfrak{g})$. Therefore,

$$\langle K \star L, f \star g\rangle = \lim_{\nu\to\infty} \int_{\mathfrak{g}} \widetilde{K_\nu} \star f(x)\, L_\nu \star \widetilde{g}(x)\, dx = \lim_{\nu} \langle U_\nu, f \star g\rangle,$$

which shows that, in fact, $U_\nu \to K \star L$ in $\mathcal{S}'(\mathfrak{g})$. □

Corollary 7.7. *Let $\mathfrak{N}$ be a closed filtration. Let $K_j \in \mathcal{F}^{M_j}(\mathfrak{g}, \mathfrak{N})$, $1 \leq j \leq 3$, where M_j is r_j-admisible and $r_1^{-1} + r_2^{-1} + r_3^{-1} \leq 2$. Then,*

$$K_1 \star (K_2 \star K_3) = (K_1 \star K_2) \star K_3.$$

Proof. By Theorem 7.5, all convolutions are legitimate. Let $K_{j,\nu} \in S_{1/\nu}(M_j)$ be a M_j-approximating sequence for K_j. Then,

$$K_{1,\nu} \star (K_{2,\nu} \star K_{3,\nu}) = (K_{1,\nu} \star K_{2,\nu}) \star K_{3,\nu},$$

for every ν. Now, the sequence $K_{2,\nu} \star K_{3,\nu}$ is an (M_2+M_3)-approximating sequence for $K_2 \star K_3$ and the sequence $K_{1,\nu} \star K_{2,\nu}$ an $(M_1 + M_2)$-approximating sequence for $K_1 \star K_2$ By passing to the limit with $\nu \to \infty$, we get our assertion. □

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