VERY SINGULAR INTEGRAL KERNELS
ON EUCLIDEAN SPACES

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ABSTRACT. We consider a generalisation of Calderón-Zygmund kernels on a Euclidean space with nonisotropic dilations. The resulting distributions, N-kernels, are highly singular. We look at the behaviour of N-kernels under the Fourier transform and also ask for conditions of convolvability of such kernels. Another interesting feature of N-kernels is their invariance for the change of variable $y = x + P(x)$, where $P$ is a homogeneous triangular polynomial mapping. The tasks require the study of the multi-parameter homogeneity of the kernels.

1. INTRODUCTION

Suppose that $K$ is a Calderón-Zygmund kernel on $\mathbb{R}^N$. This means that $K$ is a tempered distribution which is $C^\infty$ away from the origin and satisfies the size estimates

$$|D^\alpha K(x)| \lesssim |x|^{-N-|\alpha|}, \quad x \neq 0, \, \alpha \in \mathbb{N}^N,$$

as well as the cancellation condition

$$\sup_{q(\phi) \leq 1} \sup_{R > 0} R^{-N} \left| \int_{\mathbb{R}^N} \phi(Rx)K(x) \, dx \right| < \infty,$$

where $q$ is a seminorm in the Schwartz space $\mathcal{S}(\mathbb{R}^N)$. An equivalent condition in terms of the Fourier transform is

$$|D^\alpha \hat{K}(\xi)| \lesssim |\xi|^{-|\alpha|}, \quad \xi \neq 0, \, \alpha \in \mathbb{N}^N,$$

where the cancellation property remains hidden. Operators of convolution with this kind of kernels arise in many problems of harmonic analysis and differential equations, see, e.g. Stein [11]. Important as they are they allow for interesting generalisations which go in various directions. In this paper we consider the generalisation which on the Fourier transform side takes the seemingly innocuous form

$$|D^\alpha \hat{K}(\xi)| \lesssim \prod_{k=1}^N N^*_k(\xi)^{-1-m_k-\alpha_k}, \quad \xi_k \neq 0, \, \alpha \in \mathbb{N}^N,$$

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where \( N_k^* \) are suitable partial norms on subspaces of \( \mathbb{R}^N \) and

\[ |m_k| < \varepsilon, \quad 1 \leq k \leq N, \]

for some \( \varepsilon > 0 \). This can be further extended to the context of a nilpotent Lie group with nonisotropic dilations. The resulting distributions, \( \mathfrak{N} \)-kernels, are highly singular and resist easy treatment.

In this paper, we look at the behaviour of \( \mathfrak{N} \)-kernels under the Fourier tranform and also ask for conditions of convolvability of such kernels. Another interesting feature of \( \mathfrak{N} \)-kernels is their invariance for the change of variable \( y = x + P(x) \), where \( P \) is a homogeneous triangular polynomial mapping. The tasks require the study of the multi-parameter homogeneity of the kernels.

The most important is the case where \( m_k = 0 \), for \( 1 \leq k \leq N \), that is the case of flag kernels. The apt idea of a flag kernel is due to Müller-Ricci-Stein [7], [8], Nagel-Ricci-Stein [9], and Nagel-Ricci-Stein-Wainger [10] as a tool in the study of spectral multipliers on two-step nilpotent groups and \( \Box_b \)-complex operator on certain quadratic submanifolds. The key property under scrutiny is the \( L^p \)-boundedness of the operators of convolution with flag kernels. In [10] the problem of the boundedness is considered on arbitrary nilpotent groups with dilations (see also [6].) Closely related is the problem of convolution of flag kernels as tempered distributions.

An alternative approach to the same problems is presented in [6], where \( \mathfrak{N} \)-kernels of nonzero homogeneity prove instrumental in studying the flag kernels, which seems to sufficiently motivate and justify our interest in the extension of the concept.

2. Basic Setup

Let \( X \) be a real \( N \)-dimensional vector space with a fixed linear basis \( \{e_k\}_{k=1}^N \). Accordingly, each element \( x \in X \) has a representation as

\[ x = \sum_{k \in \mathcal{N}} x_k e_k = (x_1, x_2, \ldots, x_N) = (x_k)_{k \in \mathcal{N}}, \]

where \( \mathcal{N} = \{1, 2, \ldots, N\} \). The space \( X \) is assumed to be homogeneous, that is, endowed with a family of dilations \( \{\delta_t\}_{t>0} \). The vectors in the basis are supposed to be invariant under dilations:

\[ \delta_t e_k = t^{p_k} e_k, \quad t > 0, \quad k \in \mathcal{N}, \]

where \( 0 < p_1 \leq p_2 \leq \cdots \leq p_N \). The number \( Q = \sum_{k=1}^N p_k \) is called the heterogeneous dimension of \( X \). We have

\[ d\delta_t x = t^Q dx, \quad t > 0. \]
By \( \alpha, \beta, \cdots \) we shall denote multiindices in \( \mathbb{N}^N \), where \( \mathbb{N} \) stands for the set of all nonnegative integers. Let
\[
p(\alpha) = \sum_{k=1}^{N} p_k \alpha_k.
\]
We shall adopt the following notation for partial derivatives:
\[
D_k = \frac{\partial}{\partial x_k}, \quad D^\alpha = \prod_k D_k^{\alpha_k}.
\]
We also let \( T_\alpha f(x) = x^\alpha f(x) \).

The Schwartz space of smooth functions which vanish rapidly at infinity along with all their derivatives will be denoted by \( S(X) \). The seminorms
\[
q_m(f) = \max_{p(\alpha)+p(\beta) \leq m} \sup_{x \in X} |x^\alpha D^\beta f(x)|, \quad m \in \mathbb{N},
\]
form a complete set of seminorms in \( S(X) \) giving it a structure of a locally convex Fréchet space. The subspace \( C_\infty^c(X) \) of functions with compact support is dense in \( S(X) \). By \( L^p(X) \) we denote the usual Lebesgue spaces. \( S(X) \) is a dense subspace of the Lebesgue spaces.

By \( S'(X) \) we denote the space of tempered distributions, the dual to \( S(X) \). It will be convenient to denote the action of a distribution on a Schwartz function by
\[
\langle K, \varphi \rangle = \int_X \varphi(x) K(x) \, dx
\]
without thereby implying that \( K \) is a locally integrable function. If a distribution \( K \) coincides with a locally integrable function \( F \) on an open set \( U \subset X \), we simply write \( K(x) = F(x) \), for \( x \in U \).

Analogous notation will be applied to the objects on the dual space \( X^* \) with the dual basis \( \{e_\xi^*\}_{\xi \in \mathbb{N}} \) and dual dilations still denoted by \( \{\delta_t\}_{t>0} \).

We choose Lebesgue measures \( dx \) in \( X \) and \( d\xi \) in \( X^* \) in such a way that Fourier transforms take the form
\[
f^\wedge(\xi) = \hat{f}(\xi) = \int_X f(x) e^{-i(x,\xi)} \, dx, \quad \quad g^\vee(\xi) = \int_X g(\xi) e^{i(x,\xi)} \, d\xi,
\]
where \( f \in S(X) \), \( g \in S(X^*) \) and
\[
\langle x, \xi \rangle = \sum_{k=1}^{N} x_k \xi_k
\]
is the duality of vector spaces.

For a function \( F \) on \( X \) and \( \varepsilon > 0 \), let
\[
F^\varepsilon(x) = F(\delta_\varepsilon x), \quad F_\varepsilon(x) = \varepsilon^{-Q} F(\delta_{\varepsilon^{-1}} x), \quad \bar{F}(x) = F(-x), \quad x \in X.
\]
Let $A, B$ be positive quantities. We shall write $A \lesssim B$ to say that there exists a constant $c > 0$ whose precise value is irrelevant such that $A \leq cB$.

3. $\mathcal{N}$-Kernels

We assume that the set $\mathcal{N} = \{1, 2, \ldots, N\}$ is endowed with a partial order $\prec$ such that $k \prec j$ implies $k < j$. A family $\mathcal{N} = \{\mathcal{N}_k\}_{k \in \mathcal{N}}$ of subsets of $\mathcal{N}$ is called a filtration if, for every $k \in \mathcal{N}$ and every $j \in \mathcal{N}$,

a) $j \prec k$ implies $j \in \mathcal{N}_k$.

b) $j \in \mathcal{N}_k$ implies $\mathcal{N}_j \subset \mathcal{N}_k$.

A filtration $\mathcal{N}$ is closed if, for every $k$,

c) $k \in \mathcal{N}_k$.

Any filtration $\mathcal{N} = \{\mathcal{N}_k\}_{k \in \mathcal{N}}$ determines partial norms:

$$N_k(x) = \sum_{j \in \mathcal{N}_k} |x_j|^{1/p_j}, \quad k \in \mathcal{N}. $$

If the sets $\mathcal{N}_k$ form a filtration in $\mathcal{N}$, then the sets

$$\mathcal{N}_k^* = \{j \in \mathcal{N} : k \in \mathcal{N}_j\}$$

form the dual filtration in $\mathcal{N}$ which is a filtration in the sense of the above definition with respect to the dual order: $k \prec^* j$ if and only if $j \prec k$. We also have

$$\mathcal{N}_k^{**} = \mathcal{N}_k, \quad k \in \mathcal{N}. $$

Let

$$N_k^*(\xi) = \sum_{j \in \mathcal{N}_k^*} |\xi_j|^{1/p_j}$$

be the dual partial norms on the dual vector space $X^*$. Denote by $\mathcal{N}_{\min}$ (resp. $\mathcal{N}_{\max}$) the set of the minimal (resp. maximal) elements with respect to the order $\prec$.

For a given $I \subset \mathcal{N}$, denote by $H_I$ the singular subspace of $X_I = \langle e_k \rangle_{k \in I}$ consisting of all $x_I = (x_k)_{k \in I}$ such that $x_k = 0$ for some $k \in I_{\min}$. Let $H = H_\mathcal{N}$. Similarly, denote by $H_I^*$ the singular subspace of $X^*_I = \langle e_k^* \rangle_{k \in I}$ consisting of all $\xi_I = (\xi_k)_{k \in I}$ such that $\xi_k = 0$ for some $k \in I_{\max}$. Let $H^* = H_\mathcal{N}^*$.

For $\varphi \in \mathcal{S}(X)$ and $R = (r_k)_{k \in I} \in (0, \infty)^I$, let

$$\varphi^R(x_I) = \varphi(\Delta_R x_I),$$

where $\Delta_R x_I = \left(\delta_{r_k} x_k\right)_{k \in I}$. If $M = (m_1, m_2, \ldots, m_N) \in \mathbb{R}^N$, then we set $M_I = (m_k)_{k \in I}$.
Given a distribution $K \in \mathcal{S}'(X)$ and a function $\varphi \in \mathcal{S}(X_I)$, we define a distribution $K^\varphi$ on $X_J$, where $J = \mathcal{N} \setminus I$, by letting
\[
\langle K^\varphi, f \rangle = \int_{X_I} \int_{X_J} \varphi(x_I)f(x_J)K(x_I, x_J)\,dx_I\,dx_J, \quad f \in \mathcal{S}(X_J).
\]
In other words,
\[
\langle K^\varphi, f \rangle = \langle K, \varphi \otimes f \rangle, \quad f \in \mathcal{S}(X_J).
\]

**Definition 3.1.** Let $\mathcal{M} = \{N_k\}_{k \in \mathbb{N}}$ be a filtration, and $M = (m_k)_{k \in \mathbb{N}} \in \mathbb{R}^N$ a multiindex. We say that a tempered distribution $K$ belongs to $\mathcal{F}_M(X, \mathcal{N})$ if there exists a Schwartz seminorm $q_m$ such that, for every $I \subset \mathcal{N}$, every Schwartz function $\varphi \in \mathcal{S}(X_I)$, and every $\alpha \in \mathbb{N}^N$, the distribution $K^\varphi$ is smooth on $X_J$ away from the singular space $H_J$ and
\[
C(K, \alpha, I) = \sup_{q_m(\varphi) \leq 1} \sup_{R \in (0, \infty)} \prod_{k \in I} r_k^{-m_k} C(K, \alpha, I, \varphi^R) < \infty,
\]
where
\[
C(K, \alpha, I, \varphi) = \sup_{x, \in X_J} \prod_{k \in I} N_k(x)_{pk + m_k + \alpha_k} |D^\alpha K^\varphi(x)|.
\]

(To avoid ambiguity we assume that $q_m = q$ is the minimal seminorm with the above property.)

For $K \in \mathcal{F}_M(X, \mathcal{N})$, we define seminorms
\[
\|K\|_{\mathcal{F}_M, L} = \max_{p(\alpha) \leq L} \max_{I \subset \mathcal{N}} C(K, \alpha, I), \quad L \in \mathbb{N}.
\]
A family $\mathcal{K}$ of elements of $\mathcal{F}_M(X, \mathcal{N})$ is said to be bounded if, for every $L$,
\[
\sup_{K \in \mathcal{K}} \|K\|_{\mathcal{F}_M, L} < \infty.
\]

**Remark 3.3.** Let us look closer at the extreme cases of (3.2), that is when $I = \emptyset$ and $I = \mathcal{N}$. The first one implies that $K$ itself is smooth away from $H$ and
\[
|D^\alpha K(x)| \leq C_\alpha \prod_{k \in \mathcal{N}} N_k(x)^{-p_k m_k - \alpha_k}, \quad x \notin H, \quad \alpha \in \mathbb{N}^N.
\]
The other one gives the cancellation condition:
\[
|\langle K, \varphi^R \rangle| \leq C q(\varphi) \prod_{k=1}^N r_k^m,
\]
for every $\varphi \in \mathcal{S}(X)$ and every $R = (r_1, r_2, \ldots, r_N) \in (0, \infty)^N$. Note that if the filtration is trivial with $N_k = \mathcal{N}$ for all $k \in \mathcal{N}$, then (3.4) and (3.5) are the only conditions defining $\mathcal{F}_M(X, \mathcal{N})$. 
Proposition 3.6. Let $\mathcal{N}$ be a filtration, let $m_k < 0$, for $1 \leq k \leq N$, and let $F \in C^\infty(X \setminus H)$ satisfy (3.4). Then, $F$ is locally integrable, and the distribution
\[ \langle K, f \rangle = \int_X f(x)F(x)dx \]
belongs to $\mathcal{F}^M(X, \mathcal{N})$.

Proof. Note first that the condition $m_k < 0$ is equivalent to saying that $F$ is locally integrable. Let $I \subset N$ and $J = N \setminus I$. Then, for $\varphi \in S(X_I)$ and $\alpha \in N^N$,
\[ D^\alpha K^\varphi_R(x_J) = \int_{X_I} \chi^\varphi(x_I)D^\alpha x_J F(x_I, x_J) dx_I = \prod_{k \in I} R_k^{-p_k} \int_{X_I} \varphi(x_I)D^\alpha x_J F(\Delta R^{-1} x_I, x_J) dx_I, \]
whence,
\[ |D^\alpha K^\varphi_R(x_J)| \lesssim \prod_{k \in I} R_k^{-p_k} \int_{X_I} |\varphi(x_I)| \prod_{k \in N} N_k \left( \Delta R^{-1} x_I, x_J \right)^{-p_k - m_k - p_k \alpha_k} dx_I. \]
Now,
\[ N_k(x_I, x_J)^{-1} \leq N_k(x_I)^{-1}, \quad N_k(x_I, x_J)^{-1} \leq N_k(x_J)^{-1}, \]
and $N_k(x_I)^{-1} \leq |x_k|^{-1/p_k}$ if $k \in I$. Therefore,
\[ |D^\alpha K^\varphi_R(x_J)| \lesssim q(\varphi) \prod_{k \in J} N_k(x_J)^{-p_k - m_k - p_k \alpha_k}, \]
for $x_J \notin H_J$, where
\[ q(\varphi) = \int_{X_I} |\varphi(x_I)| \prod_{k \in I} |x_k|^{-1-m_k/p_k} dx_I \]
is a seminorm on $S(X_I)$ since $m_k < 0$. It is clear that all constants in the above estimates depend only on the constants $C_\alpha$ in (3.4). Our proof is complete. \qed

Proposition 3.7. If $K \in \mathcal{F}^m(X, \mathcal{N})$, then, for every $\alpha$, $D^\alpha K \in \mathcal{F}^{m+\alpha}(X, \mathcal{N})$ and $T_\alpha K \in \mathcal{F}^{m-\alpha}(X, \mathcal{N})$, where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$.

We leave it as an exercise for the reader.

Example 3.8. If $K = \varphi$ is a Schwartz function, then $K \in \mathcal{F}^M(X, \mathcal{N})$ for every $\mathcal{N}$ and every $M = (m_k)_{k \in N}$ such that $-p_k < m_k < 0$.

Example 3.9. Let $0 < r \leq N$ and
\[ k_0 = 0 < k_1 < \cdots < k_{r-1} < k_r = N. \]
Let $\mathcal{N} = \{ \mathcal{N}_j \}_{j \in N}$, where, for $k_{l-1} < j \leq k_l$, $1 \leq l \leq r$,
\[ \mathcal{N}_j = \{ k \in \mathcal{N} : k \leq k_l \}. \]
Then the class $\mathcal{F}^0(X, \mathfrak{N})$ is exactly the class of flag kernels of Nagel-Ricci-Stein-Wainger corresponding to the flag
\[ \{0\} \subset \langle e_j \rangle_{j \in \mathcal{N}_1} \subset \cdots \subset \langle e_j \rangle_{j \in \mathcal{N}_l} \subset \cdots \subset \langle e_j \rangle_{j \in \mathcal{N}_{kr}}. \]
This filtration is closed.

**Example 3.10.** On the other hand, the filtration
\[ N_j = \{ k \in \mathcal{N} : k < j \} \]
is not closed.

## 4. Fourier transform

**Proposition 4.1.** Let $\mathfrak{N}$ be a closed filtration in $\mathcal{N}$. Let
\[ M = (m_1, m_2, \ldots, m_N) \in \mathbb{R}^N \]
and let $K \in \mathcal{F}^M(X, \mathfrak{N})$. Let $\varphi_k \in C_c^\infty(X_k)$ be equal to 1 in a neighbourhood of zero and
\[ \Phi(x) = \prod_{k \in \mathcal{N}} \varphi_k(x_k), \quad x \in X. \]
Then, for every $\varepsilon > 0$,
\[ \left| \int_X e^{-ix\xi} \Phi(\delta_c x) K(x) dx \right| \lesssim \prod_{k \in \mathcal{N}} (\varepsilon + p_k^*(\xi))^{m_k}, \quad \xi \in X^*. \]

**Proof.** Let us fix $\xi \in X^* \setminus H^*$. Let $R_k = N_k^*(\xi)$. Let $\eta_k \in C_c^\infty(X_k)$ be equal to 1, for $|x_k| \leq R_k^{-p_k}$. Let
\[ \eta_I(x) = \prod_{k \in I} \eta_k(x_k), \quad \eta^c_I(x) = \prod_{k \in I^c} (1 - \eta_k(x_k)), \quad x \in X, \]
where $I^c = \mathcal{N} \setminus I$. Note that
\[ 1 = \sum_{I \subset \mathcal{N}} \eta_I(x) \eta^c_I(x), \quad x \in g. \]
Under the notation introduced above
\[ \int_X e^{-ix\xi} \Phi(\delta_c x) K(x) dx = \sum_{I \subset \mathcal{N}} \int_X e^{-ix\xi} \eta_I(\Delta_R x) \eta^c_I(\Delta_R x) \Phi(\delta_c x) K(x) dx. \]
We are going to estimate each term
\[ S_I(\xi) = \int_X e^{-ix\xi} \eta_I(\Delta_R x) \eta^c_I(\Delta_R x) \Phi(\delta_c x) K(x) dx, \quad I \subset \mathcal{N}, \]
separately. We begin with
\[ S_{\mathcal{N}}(\xi) = \int_X e^{-ix\xi} \eta_N(\Delta_R x) \Phi(\delta_c x) K(x) dx \]
(4.3)\[ = \int_X F_{R,\xi}(\Delta_R x) K(x) dx, \]
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where the two-parameter family of $C^\infty_c(X)$-functions

$$F_{R,\varepsilon}(x) = e^{-ix\Delta R - 1 - \xi} \eta_N(x) \Phi(\Delta_{\varepsilon R - 1} x), \quad 0 < \varepsilon < \min_{k \in \mathbb{N}} R_k,$$

is bounded in $S(X)$. Therefore, by definition,

$$|S_N(\xi)| \lesssim \prod_{k \in \mathbb{N}} R_k^{m_k}.$$

As for $S_\varnothing$, we need an extra twist. Recall that $\xi \in X^* \setminus H^*$ is fixed. For every $k \in \mathcal{N}$, let $\lambda(k)$ be the biggest index $l \in \mathbb{N}_k^*$ such that

$$|\xi_l|^{1/p_l} = \max_{j \in \mathbb{N}_k^*} |\xi_j|^{1/p_j}.$$

Then $|\xi_l|^{1/p_l} \approx N_k^*(\xi)$. Of course, the mapping $\lambda: \mathcal{N} \to \mathcal{N}$ and the set $\Lambda = \lambda(\mathcal{N})$ depend on $\xi$. Let $a = (a_1, a_2, \ldots, a_N)$ be a multiindex such that $m_k + p_{\lambda(k)} a_k > 0$, for every $k$, and let

$$D^A = D_{\lambda(1)}^{a_1} D_{\lambda(2)}^{a_2} \ldots D_{\lambda(N)}^{a_N} = D_1^{A_1} D_2^{A_2} \ldots D_N^{A_N},$$

so that $A_l = \sum_{\lambda(k)=l} a_k$ if $l \in \Lambda$ and $A_l = 0$ if $l \notin \Lambda$. If

$$W_{R,\varepsilon}(x) = \eta_\varnothing^c(\Delta R x) \Phi(\delta R x) = \prod_{k \in \mathcal{N}} \left(1 - \eta_k(R_k^{p_k} x_k)\right) \varphi_k(\varepsilon^{p_k} x_k)$$

then

$$D^A(W_{R,\varepsilon}K)(x) = 0,$$

if $|x_k| \leq R_k^{-p_k}$, for some $k \in \mathcal{N}$, and

(4.4) $$|D^A(W_{R,\varepsilon}K)(x)| \lesssim \prod_k |x_k|^{-p_k - m_k - p_k A_k}.$$

This follows from the fact that $R_k \approx |x_k|^{-1}$ on the support of the derivative of $\eta_k$ and $\varepsilon \approx |x_k|^{-1}$ on the support of the derivatives of $\varphi_k$. The filtration is closed, so we also have $N_k(x) \geq |x_k|^{1/p_k}$.

Now, by integration by parts and application of (4.4),

$$|S_\varnothing(\xi)| = \left| \int_X e^{-ix\xi} W_{R,\varepsilon}(x) K(x) dx \right|$$

$$\approx \prod_{k=1}^N |\xi_{\lambda(k)}|^{-p_k A_k} \left| \int_X e^{-ix\xi} D^A(W_{R,\varepsilon}K)(x) dx \right|$$

$$\lesssim \prod_{k=1}^N R_k^{-p_k A_k} \int_{U_1 \times U_2 \times \cdots \times U_N} \prod_k |x_k|^{-p_k - m_k - p_k A_k} dx,$$
where \( U_k = [R_k^{-p_k}, \infty) \), for \( k \in \mathcal{N} \). Consequently,
\[
\prod_{k \in \mathcal{N}} R_k^{-p_k} |S_\varnothing(\xi)| \lesssim \int_{U_1 \times U_2 \times \ldots \times U_N} \prod_{k \in \mathcal{N}} |x_k|^{-p_k-m_k-p_{\lambda(k)}a_k} \, dx,
\[
\leq \prod_{k \in \mathcal{N}} \int_{U_k} |x_k|^{1/p_k} \, dx,
\[
\lesssim \prod_{k \in \mathcal{N}} R_k^{m_k+p_{\lambda(k)}a_k} = \prod_{k \in \mathcal{N}} R_k^{m_k+p_kA_k},
\]

and, finally,
\[
|S_\varnothing(\xi)| \lesssim \prod_{k \in \mathcal{N}} R_k^{m_k}.
\]

It remains to consider those cases where \( I \) is a proper subset of \( \mathcal{N} \). We have
\[
S_I(\xi) = \int_X \left( e^{-ix_I \xi_I} \eta_I(\Delta R_I x_I) \Phi_I(\delta_x x_I) \right) \left( e^{-ix_J \xi_J} \eta_J^c(\Delta R_J x_J) \Phi_J(\delta_x x_J) \right) K(x) \, dx
\]
\[
= \int_X F_{R_I,\epsilon}(\Delta R_I x_I) G_{R_J,\epsilon}(\Delta R_J x_J) K(x) \, dx
\]
\[
= \int_{X_I} F_{R_I,\epsilon}(\Delta R_I x_I) K^{G_{R_J,\epsilon}}(x) \, dx_I,
\]
where \( J = I^c = \mathcal{N} \setminus I \),
\[
F_{R_I,\epsilon}(x_I) = e^{-ix_I \Delta R_I^{-1} \xi_I} \eta_I(x_I) \Phi_I(\Delta R_I^{-1} x_I), \quad 0 < \epsilon < \min_{k \in I} R_k,
\]
and
\[
G_{R_J,\epsilon}(x_J) = e^{-ix_J \Delta R_J^{-1} \xi_J} \eta_J^c(x_J) \Phi_J(\Delta R_J^{-1} x_J), \quad 0 < \epsilon < \min_{k \in J} R_k.
\]
The two-parameter family \( G_{R_J,\epsilon} \) is bounded in \( \mathcal{S}(X_J) \), so
\[
K_{R_I,\epsilon} = \left( \prod_{k \in I} R_k^{-m_k} \right) K_{G_{R_J,\epsilon}}^{R_I}
\]
is a family of kernels bounded in \( \mathcal{F}^M(X_J) \). The integral
\[
\int_{X_I} F_{R_I,\epsilon}(\Delta R_I x_I) K_{R_J,\epsilon}(x_I) \, dx_I = \left( \prod_{k \in I} R_k^{-m_k} \right) S_I(\xi)
\]
is a lower-dimensional counterpart of the integral \([4.3]\), so that
\[
\left( \prod_{k \in I} R_k^{-m_k} \right) |S_I(\xi)| \lesssim \prod_{k \in I} R_k^{m_k},
\]
that is
\[
|S_I(\xi)| \lesssim \prod_{k \in \mathcal{N}} R_k^{m_k}.
\]
This completes the proof. \( \square \)
Corollary 4.5. Under the hypotheses of Proposition 4.1,

\[ |D^\alpha(\hat{K} \ast \hat{\Phi}_\varepsilon)(\xi)| \lesssim \prod_{k \in \mathcal{N}} (\varepsilon + N_k^*(\xi))^{m_k - p_k \alpha_k}, \quad \xi \in X^*, \]

uniformly for \( \varepsilon > 0 \).

Proof. The function \( \hat{\Phi}_\varepsilon \ast \hat{K} \) is smooth and

\[ |D^\alpha(\hat{\Phi}_\varepsilon \ast \hat{K})(\xi)| = |\hat{\Phi}_\varepsilon \ast D^\alpha \hat{K}(\xi)|, \quad \xi \in X^*, \alpha \in \mathbb{N}^\mathcal{N}. \]

Since \( |D^\alpha \hat{K}| = |(T_\alpha K)^\wedge| \) and \( T_\alpha K \in \mathcal{F}^{M-p\alpha}(X, \mathfrak{N}) \), our claim follows by Proposition 4.1. \( \Box \)

For \( M = (m_1, m_2, \ldots, m_N) \in \mathbb{R}^N \), let

\[ M^* = (-p_1 - m_1, -p_2 - m_2, \ldots, -p_N - m_N). \]

Theorem 4.6. Let \( \mathfrak{N} \) be a closed filtration in \( \mathcal{N} \). Let \( M \in \mathbb{R}^N \). If \( K \in \mathcal{F}^M(X, \mathfrak{N}) \), then \( \hat{K} \in \mathcal{F}^{M^*}(X^* \setminus H^*) \).

Proof. Let us keep the notation of Proposition 4.1. We have \( \hat{\Phi}_\varepsilon \ast \hat{K} \rightarrow \hat{K} \) in the sense of distributions. The estimates of Corollary 4.5 show that all derivatives \( D^\alpha(\hat{\Phi}_\varepsilon \ast \hat{K}) \) are locally bounded in \( X^* \setminus H^* \), so by the usual argument based on the Ascoli theorem we conclude that the convergence is almost uniform in \( X^* \setminus H^* \), which implies that \( \hat{K} \) is smooth away from \( H^* \) and satisfies

\[ |D^\alpha \hat{K}(\xi)| \lesssim \prod_{k \in \mathcal{N}} N_k^*(\xi)^{m_k - p_k \alpha_k}, \quad \xi \in X^* \setminus H^*. \]

It remains to take care of the lower-dimensional conditions. Let \( I, J \) and \( \varphi \) be as in Definition 3.1. Let

\[ \mathfrak{N}_J = \{ E \cap J : E \in \mathfrak{N} \}. \]

This is a closed filtration in \( J \). The distribution \( K^{\varphi^R} \) is a kernel in \( \mathcal{F}^{M_J}(X_J, \mathfrak{N}_J) \), so

\[ \hat{K}^{\varphi^R} = \left( \prod_{k \in I} R^{-p_k} \right) \left( K^{(\varphi^R)^R} \right)^\wedge \]

is in \( C^\infty(X_J^* \setminus H_J^*) \) and satisfies the necessary condition for the decay of derivatives. The estimates are uniform in \( R \in (0, \infty)^J \) and \( \varphi \in \mathcal{S}(X_J) \). \( \Box \)

5. APPROXIMATION

We shall consider the following classes of distributions on \( X \) associated with a fixed closed filtration \( \mathfrak{N} \). For \( M = (m_k)_{k \in \mathcal{N}} \in \mathbb{R}^N \) and
$\varepsilon > 0$, let $S_\varepsilon(M)$ be the totality of distributions $A$ on $X$ whose Fourier transform $\hat{A}$ is smooth and satisfies

\begin{equation}
|D^\alpha \hat{A}(\xi)| \lesssim \prod_{k \in \mathbb{N}} (\varepsilon + N_k^*(\xi))^{m_k - p_k \alpha_k}, \quad \xi \in X^*,
\end{equation}

cf. Corollary \[4.5\] The class is a Fréchet space if equipped with the family of natural seminorms

$$
\|A\|_{S_\varepsilon(M),l} = \max_{p(\alpha) \leq l} \sup_{\xi \in \theta^*} \prod_{k \in \mathbb{N}} (\varepsilon + N_k^*(\xi))^{-m_k + p_k \alpha_k} |D^\alpha \hat{A}(\xi)|, \quad l \in \mathbb{N}.
$$

A sequence $K_\nu \in S_{1/\nu}(M)$ is said to be an $M$-approximating sequence if $K_\nu$ is convergent in $S'(\mathfrak{g})$ and bounded, that is

$$
\sup_{\nu} \|K_\nu\|_{S_{1/\nu},l} < \infty,
$$

for every $l \in \mathbb{N}$.

**Proposition 5.2.** Let $\mathfrak{N}$ be a closed filtration and $M \in \mathbb{R}^N$ an multi-index with $m_k > -p_k$. Then, $K \in \mathcal{F}^M(X, \mathfrak{N})$ if and only if there exists an $M$-approximating sequence convergent in $S'(X)$ to $K$.

**Proof.** Let $K \in \mathcal{F}^M(X, \mathfrak{N})$. For $\nu \in \mathbb{N}$, let

$$
K_\nu(x) = \Phi(\delta_{1/\nu}x)K(x), \quad x \in X,
$$

where $\Phi$ is as in Proposition \[4.1\] Then, by Corollary \[4.5\] the sequence $K_\nu \in S(1/\nu)$ is bounded and convergent to $K$ in $S'(X)$.

Now let $\hat{K}$ be a limit of an $M$-approximating sequence $\hat{K}_\nu$. We have

$$
(1/\nu + N_k^*(\xi))^{m_k - p_k \alpha_k} \leq (1 + N_k^*(\xi))^{m_k - p_k \alpha_k},
$$

if $m_k - p_k \alpha_k \geq 0$, and

$$
(1/\nu + N_k^*(\xi))^{m_k - p_k \alpha_k} \leq N_k^*(\xi)^{m_k - p_k \alpha_k},
$$

if $m_k - p_k \alpha_k \leq 0$. Therefore, by the Ascoli theorem, $D^\alpha \hat{K}_\nu$ converges uniformly, for every $\alpha$, on every compact subset of $X^* \backslash H^*$. Thus,

$$
\hat{K}(\xi) = \lim_{\nu \to \infty} \hat{K}_\nu(\xi)
$$

satisfies the necessary estimates for the derivatives on $X^* \backslash H^*$. In particular,

$$
|\hat{K}(\xi)| \lesssim \prod_{k \in \mathbb{N}} N_k^*(\xi)^{m_k},
$$

which shows that $\hat{K}$ is locally integrable. Therefore, by Proposition \[3.6\] $\hat{K} \in \mathcal{F}^M(X^*, \mathfrak{N}^*)$, so, by Theorem \[4.6\] $K \in \mathcal{F}^M(X, \mathfrak{N})$. \[\square\]
6. Change of variable

A mapping $P : X \to X$ is said to be a \textit{homogeneous triangular polynomial mapping}, if

$$P(x) = (P_1(x), P_2(x), \ldots, P_N(x)),$$

where $P_k$ are polynomials of degree at least 2 such that

$$(6.1) \quad P_k(x) = P_k(x_1, x_2, \ldots, x_{k-1})$$

and

$$(6.2) \quad P(\delta_t x) = \delta_t P(x), \quad x \in X, \quad t > 0.$$ 

Condition (6.2) means that, for every $k$, $P_k$ is a homogeneous polynomial of degree $p_k$.

The mapping $P$ determines an order in $\mathbb{N}$. We write $k \prec j$ if $D_k P_j \neq 0$ and extend this relation to the smallest order containing it. By (6.1), $k \prec j$ implies $k < j$.

Let the operator $S_P$ be defined as

$$S_P f(x) = f(x + P(x)), \quad x \in X,$$

for $f \in \mathcal{S}(X)$ and

$$\langle S_P A, f \rangle = \langle A, S_P f \rangle, \quad f \in \mathcal{S}(X),$$

for $A \in \mathcal{S}'(X)$, where $\widetilde{P}(x + P(x)) = -P(x)$. In other words, $y = x + P(x)$ is equivalent to $x = y + \widetilde{P}(y)$.

Recall from [6] the operator

$$U_P F(\xi) = \int\int_{X \times X} e^{-i(x, \xi)} e^{-i(P(x,y), \xi)} F^\vee(x,y) \, dx \, dy, \quad \xi \in X^*,$$

which we here consider as acting on $\mathcal{S}(X) \times \mathcal{S}(X)$:

$$U_P (f,g)(\xi) = \int\int_{X \times X} e^{-i(x, \xi)} e^{-i(P(x,y), \xi)} f^\vee(x) g^\vee(y) \, dx \, dy.$$

By Theorem 7.1 of [6], $U_P$ has an extension, still denoted by $U_P$, to a Fréchet continuous mapping $S_1(M) \times S_1(M') \to S_1(M + M')$, where $M, M' \in \mathbb{R}^N$. (This extension is unique in the sense that is irrelevant here.) As a matter of fact, the Fréchet continuity has not been made explicit in the statement of the theorem, but the final part of the proof (see the last line of the page 1657) shows that the Fréchet continuity claim holds true.

Let

$$P(x,y) = (P(x), P(y)), \quad x, y \in X.$$ 

If we apply the above to the homogeneous triangular polynomial mapping $P$, we get the operator $U_P$ acting on distributions on $X^* \times X^*$.
whose restriction to $\mathcal{S}(X^*) \times \mathcal{S}(X^*)$ is
\[
U_P(f, g)(\xi, \eta) = \int_{X \times X} e^{-i((x, \xi) + (y, \eta))} e^{-i(P(x, \xi) + (P(u, \eta))} f^\vee(x, y) g^\vee(u, v) \, dx dy du dv.
\]
Let
\[
V_P(A)(\xi) = U_P(A, 1)(\xi, 0), \quad A \in \mathcal{S}'(X^*), \ \xi \in X^*.
\]
Fix $M \in \mathbb{R}^N$. It follows that $V_P : S_1(M) \rightarrow S_1(M)$ is Fréchet continuous. For $A \in S'(g)$ and $\varepsilon > 0$, let
\[
A_{\varepsilon}(x) = \varepsilon^{-Q-\sum_{k=1}^N m_k} A(\varepsilon^{-1} x).
\]
Clearly,
\[
V_P(A_{\varepsilon}) = V_P(A), \quad A \in \mathcal{S}'(X^*), \ \varepsilon > 0,
\]
so $V_P : S_\varepsilon(M) \rightarrow S_\varepsilon(M)$ is Fréchet continuous uniformly in $\varepsilon$. If $A = f \in \mathcal{S}(X^*)$, then
\[
V_P f(\xi) = \int_X e^{-i(x, \xi)} e^{-i(P(x, \xi))} f^\vee(x) \, dx,
\]
so by the change of variable $z = x + P(x)$,
\[
S_P = \mathcal{F} \circ V_P \circ \mathcal{F}^{-1},
\]
where $\mathcal{F}$ denotes the Fourier transform.

**Theorem 6.3.** Let $\mathcal{R} = \{N_k\}_{k \in \mathbb{N}}$ be a filtration, and $M = (m_k)$ a multiindex. If $K \in \mathcal{F}^M(X, \mathcal{R})$, then so does $S_P K$.

**Proof.** Let $K \in \mathcal{F}^M(X, \mathcal{R})$. Let $K_\nu$ be an $M$-approximating sequence for $K$. Then,
\[
L_\nu = S_P K_\nu = \left( V_P(K_\nu^\vee) \right)^\vee
\]
is another $M$-approximating sequence convergent to $S_P K$. Hence, by Proposition 5.2 $S_P K \in \mathcal{F}^M(X, \mathcal{R})$. \hfill $\square$

7. CONVOLUTION

We keep the setup of Section 2. Furthermore, we endow $X$ (since now denoted by $g$) with a Lie algebra commutator $(x, y) \mapsto [x, y]$ such that the dilations $\delta_t$ become automorphisms. The commutator determines a Campbell-Hausdorff multiplication (see Corwin-Greenleaf [2])

\[
(7.1) \quad xy = x + y + P(x, y),
\]
where
\[
P(x, y) = \frac{1}{2} [x, y] + \frac{1}{12} [x, [x, y]] + \frac{1}{12} [y, [y, x]] + \ldots
\]
is a homogeneous triangular polynomial mapping. This makes $g$ into a nilpotent Lie group with the same underlying manifold $X$. 

If $A \in S_{\varepsilon}(M)$ for some $M \in \mathbb{R}^N$ and $\varepsilon > 0$, then for every compact neighbourhood $U$ of 0 in $\mathfrak{g}$, $A = A_0 + F$, where $A_0$ is supported in $U$ and $F \in S(\mathfrak{g})$. Thus, for every two members of the classes $S_{\varepsilon}(M)$, convolution makes sense. Let $A \in S_1(M)$, $B \in S_1(M')$. Then, by Proposition 8.1 of [6],

$$(A \ast B)^{\wedge} = U_p(A^{\wedge},B^{\wedge}).$$

Consequently, the mappings

$$(7.2) \quad S_{\varepsilon}(M) \times S_{\varepsilon}(M') \ni (A,B) \mapsto A \ast B \in S_{\varepsilon}(M + M'), \quad \varepsilon > 0,$$

are continuous.

Let $p > 1$. We say that a multiindex $M = (m_k)_{k \in \mathbb{N}}$ is $p$-admissible if, for every $k \in \mathbb{N}$,

$$p_k(1/p - 1) < m_k < p_k/p.$$  

**Proposition 7.3.** Let $\mathcal{N}$ be a closed filtration. Let $M = (m_k)_{k \in \mathbb{N}}$ be $p$-admissible for some $p > 1$. For $\nu \in \mathbb{N}$, let $K_{\nu}$ be an $M$-approximating sequence convergent to $K$ in the sense of distributions. Then, for every $\varphi \in S(\mathfrak{g})$ and every $\nu$, $\varphi \ast K_{\nu} \in L^p(\mathfrak{g})$ and $\varphi \ast K_{\nu} \rightarrow \varphi \ast K$ in $L^p(\mathfrak{g})$.

Thus $\varphi \ast K \in L^p(\mathfrak{g})$.

**Proof.** Let $\varphi \in S(\mathfrak{g})$. We know that $\varphi \in \mathcal{F}^T(\mathfrak{g},\mathcal{N})$ for every $T$ with $-p_k < t_k < 0$ (see Example 3.8). Also $\varphi \in S_{1/\nu}(T)$ uniformly in $\nu$. By symbolic calculus (7.2), $F_{\nu} = K_{\nu} \ast \varphi \in S_{1/\nu}(M + T)$. It is not hard to see that $F_{\nu}$ is an $(M + T)$-approximating sequence for $F = K \ast \varphi$.

Hence, by Proposition 5.2, $F \in \mathcal{F}^{M+T}(\mathfrak{g})$, which implies

$$|F(x)| \lesssim \prod_{k \in \mathbb{N}} |x_k|^{-p_k-m_k-t_k}, \quad x \in X \setminus H,$$

for any $-p_k < t_k < 0$. If $T$ is chosen so that $m_k + t_k < p_k/p$ if $|x_k| \leq 1$ and $m_k + t_k > p_k(1/p - 1)$ if $|x_k| > 1$, the right-hand side is in $L^p(\mathfrak{g})$, so $K \ast \varphi \in L^p(\mathfrak{g})$. We can do so since $M$ is $p$-admissible.

The same argument implies uniformly to each $F_{\nu}$. The sequence is convergent to $F$ almost everywhere, hence, by the Lebesgue dominated convergence theorem, it is convergent in $L^p(\mathfrak{g})$. \qed

The following definition is due to Chevalley. Tempered distributions $K,L$ are **convolvable** if

$$(\tilde{K} \ast \varphi)(L \ast \tilde{\psi}) \in L^1(\mathfrak{g}),$$

for all $\varphi, \psi \in S(\mathfrak{g})$. Then, by Chevalley [1], Section 8, there exists a unique tempered distribution $K \ast L$ such that

$$(K \ast L, \varphi \ast \psi) = \int_{\mathfrak{g}} \tilde{K} \ast \varphi(x) L \ast \tilde{\psi}(x) \, dx,$$

for all $\varphi, \psi \in S(\mathfrak{g})$. Recall that functions of the form $\varphi \ast \psi$ span the whole of $S(\mathfrak{g})$. (In fact, by Dixmier-Malliavin [3], every function in $S(\mathfrak{g})$ is of this form, but we need not this difficult result.)
Remark 7.4. If \( K_j \in S_1(M_j) \), \( j = 1, 2 \), then it is not hard to see that they are convolvable in the sense of Chevalley and the Chevalley convolution coincides with the usual convolution \( K_1 \ast K_2 \) of distributions.

Theorem 7.5. Let \( \mathfrak{N} \) be a closed filtration. Let \( 1/p + 1/q \leq 1 \), \( 1 < p, q < \infty \). Let \( K \in \mathcal{F}^{M_1}(g, \mathfrak{N}) \), \( L \in \mathcal{F}^{M_2}(g, \mathfrak{N}) \), where \( M_1 \) is \( p \)-admissible and \( M_2 \) is \( q \)-admissible. Then the distributions \( K \) and \( L \) are convolvable and \( K \ast L \in \mathcal{F}^{M_1 + M_2}(g, \mathfrak{N}) \).

Proof. We may assume that \( 1/p + 1/q = 1 \). By Proposition 7.3, \( K \) and \( L \) are convolvable. Let \( K_\nu \) and \( L_\nu \) be the approximate sequences converging to \( K \) and \( L \), respectively (Proposition 5.2). Then, by symbolic calculus, the sequence

\[
U_\nu = K_\nu \ast L_\nu \in S(1/\nu, M_1 + M_2)
\]

is bounded. By (3.5), the distributions \( U_\nu \) are equicontinuous in \( S'(g) \). Once we show that \( U_\nu \) is convergent to \( K \ast L \) in \( S'(g) \), we shall be able to conclude that \( U_\nu \) is an \((M_1 + M_2)\)-approximate sequence and \( K \ast L \in \mathcal{F}^{M_1 + M_2}(g, \mathfrak{N}) \).

Let \( f, g \in S(g) \). By (7.6),

\[
\langle U_\nu, f \ast g \rangle = \int g \widehat{K_\nu} \ast f(x) L_\nu \ast \widehat{g}(x) \, dx,
\]

where, by Proposition 7.3, \( \widehat{K_\nu} \ast f \to \widehat{K} \ast f \) in \( L^p(g) \) and \( L_\nu \ast \widehat{g} \to L_\nu \ast \widehat{g} \) in \( L^q(g) \). Therefore,

\[
\langle K \ast L, f \ast g \rangle = \lim_{\nu \to \infty} \int g \widehat{K_\nu} \ast f(x) L_\nu \ast \widehat{g}(x) \, dx = \lim_{\nu \to \infty} \langle U_\nu, f \ast g \rangle,
\]

which shows that, in fact, \( U_\nu \to K \ast L \) in \( S'(g) \). \( \square \)

Corollary 7.7. Let \( \mathfrak{N} \) be a closed filtration. Let \( K_j \in \mathcal{F}^{M_j}(g, \mathfrak{N}) \), \( 1 \leq j \leq 3 \), where \( M_j \) is \( r_j \)-admissible and \( r_1^{-1} + r_2^{-1} + r_3^{-1} \leq 2 \). Then,

\[
K_1 \ast (K_2 \ast K_3) = (K_1 \ast K_2) \ast K_3.
\]

Proof. By Theorem 7.5, all convolutions are legitimate. Let \( K_{j, \nu} \in S_{1/\nu}(M_j) \) be a \( M_j \)-approximating sequence for \( K_j \). Then,

\[
K_{1, \nu} \ast (K_{2, \nu} \ast K_{3, \nu}) = (K_{1, \nu} \ast K_{2, \nu}) \ast K_{3, \nu},
\]

for every \( \nu \). Now, the sequence \( K_{2, \nu} \ast K_{3, \nu} \) is an \((M_2 + M_3)\)-approximating sequence for \( K_2 \ast K_3 \) and the sequence \( K_{1, \nu} \ast K_{2, \nu} \) an \((M_1 + M_2)\)-approximating sequence for \( K_1 \ast K_2 \). By passing to the limit with \( \nu \to \infty \), we get our assertion. \( \square \)

References


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