10:41, Monday 24th February, 2014

THE ALGEBRA OF CALDERÓN-ZYGMUND KERNELS ON A HOMOGENEOUS GROUP IS INVERSE-CLOSED

PAWEŁ GŁOWACKI *(UNIVERSITY OF WROCŁAW)*

ABSTRACT. On a homogeneous group G we consider the algebra of convolution operators with Calderón-Zygmund kernels and show that this subalgebra is inverse-closed in the algebra of all bounded linear operators on the Hilbert space $L^2(G)$.

The main tool is a symbolic calculus where the convolution of distributions on the group is translated via the Abelian Fourier transform into a "twisted product" of symbols on the dual to the Lie algebra $\mathfrak{g}$ of G.

CONTENTS

1. Statement of the result 1
2. Notation and preliminaries 3
3. Symbolic calculus 7
4. Symbolic calculus (lemmas) 11
5. Kernels in $\mathcal{F}^m(\mathfrak{g})$ 14
6. Proof of the main theorem 20
References 22

1. STATEMENT OF THE RESULT

The term *Calderón-Zygmund kernel* on a homogeneous group G can be understood in many different ways depending on context and purpose (see, e.g. Stein [22] and Ricci [21]). In this paper the following definition has been adopted. A distribution $K \in \mathcal{S}'(G)$ is said to be a Calderón-Zygmund kernel if it is smooth away from the origin and satisfies the following conditions:

Size condition: For every multiindex α,

$$|D^\alpha K(x)| \leq C_\alpha |x|^{-Q-|\alpha|}, \qquad x \neq 0, \tag{1.1}$$

where Q stands for the homogeneous dimension of G.

Cancellation condition: There exists a continuous seminorm norm $\|\cdot\|$ in the Schwartz space $\mathcal{S}(G)$ such that for every $\varphi \in \mathcal{S}(G)$ and every $R > 0$

$$\left| \int \varphi(Rx) K(x)\, dx \right| \leq \|\varphi\|. \tag{1.2}$$

2000 *Mathematics Subject Classification.* 42B20, 42B15 (primary), 43A22, 43A15 (secondary).

Key words and phrases. singular integrals, Calderón-Zygmund kernels, homogeneous groups, inverse-closed algebras.

A characterization is given in Proposition 5.9 below.

It is well-known that such a Calderón-Zygmund kernel K gives rise to a bounded convolution operator

$$\mathrm{Op}(K)f(x) = f \star \widetilde{K}(x) = \int f(xy)K(y)\,dy, \qquad f \in \mathcal{S}(G),$$

on $L^p(G)$, $1 < p < \infty$ (see, e.g. Ricci [21]). To be more precise, it is the closure of $\mathrm{Op}(K)$ which does depend on p that is bounded on L^p, but we take the liberty here of disregarding this distinction.

The Calderón-Zygmund operators form a subalgebra of the algebra $\mathcal{B}(L^2(G))$ of all bounded operators on $L^2(G)$ (see, e.g. Coré-Geller [7] and also Theorem 5.16 below). In this paper the question is raised whether the subalgebra is inverse-closed. In other words, if K is such a kernel and $\mathrm{Op}(K)$ is invertible as a bounded operator on $L^2(G)$, is $\mathrm{Op}(K)^{-1}$ also an operator with a Calderón-Zygmund kernel?

The problem as to whether a given subalgebra $\mathcal{A} \subset \mathcal{B}(L^2(G))$ of singular integral operators is inverse-closed has been dealt with on several occasions by various authors starting with Calderón-Zygmund [1] and [2] where the Abelian algebra $\mathcal{A} = \mathcal{A}_q$ consists of homogeneous singular operators on the Euclidean space which are locally in L^q away from the origin, for a given $q > 1$. Christ and Geller [6] proved the inversion theorem for the algebra $\mathcal{A}$ of homogeneous singular integral operators with kernels smooth away from the origin on a graded homogeneous group. Subsequently, the result has been extended to arbitrary homogeneous groups in [13]. Another theorem of this kind is that of Christ [3] who took up the study of the Calderón-Zygmund algebras $\mathcal{A}_q$ in the non-Abelian context of a homogeneous group. For similar problems see also Christ [4]. From a more general point of view, the problem resembles that of regularity of solutions of PDE and in fact Christ's results have already found an application in the study of the $\bar{\partial}_b$ equation on CR manifolds (Christ [5]) as well as in that of Schrödinger operators (Dziubański-Głowacki [10]). Therefore, we believe that the following result may be of interest.

Theorem 1.3. *Let K be a Calderón-Zygmund kernel on a homogeneous group G. If the operator $\mathrm{Op}(K)$ has a bounded inverse on $L^2(G)$, then there exists a Calderón-Zygmund kernel L on G such that $\mathrm{Op}(K)^{-1} = \mathrm{Op}(L)$.*

The topology of the algebra of Calderón-Zygmund kernels is determined by a family of seminorms rather than a single norm, which seems to be a serious obstacle. The main tool employed is a symbolic calculus as created in Melin [20] and developed in Manchon [19] and Głowacki [12] where the convolution $\star$ is translated via the Abelian Fourier transform into a product # of symbols on the dual to the Lie algebra $\mathfrak{g}$ of G. Since the exponential map is a diffeomorphism of $\mathfrak{g}$ onto G, we can define

$$a\#b = \Big((a^\vee \circ \exp^{-1}) \star (b^\vee \circ \exp^{-1}) \circ \exp\Big)^\wedge, \qquad a, b \in \mathcal{S}(\mathfrak{g}^\star),$$

where $^\wedge$ and $^\vee$ denote the Fourier transforms on $\mathfrak{g}$ and $\mathfrak{g}^\star$, and study #, and therefore also $\star$, in terms of the properties of symbols. In the case of the Heisenberg group we obtain a calculus very closely related to the pseudodifferential one. In the simplest case of an Abelian group, the Fourier transform translates convolution into the ordinary product and no estimates on the derivatives are required. The basic class $S^0(G)$ of the Melin calculus consists of Calderón-Zygmund kernels which have no singularity at infinity and therefore their symbols are smooth everywhere. The symbols of general Calderón-Zygmund kernels are not differentiable at the origin so they stay outside the calculus. However, if K is such a kernel, then its partial Fourier transform K_λ, $\lambda \neq 0$, with respect to the central variable can be interpreted as an element of the class $S^0(G_0)$ on a quotient group G_0, which makes the necessary link. In principle, once we prove the inversion theorem for Calderón-Zygmund kernels with smooth symbols on the quotient group G_0, we can do the same for kernels on G.

Another feature of our approach is the use of "Calderón-Zygmund kernels" of order $m \neq 0$, which allows for greater flexibility. A distribution R on G is a kernel of class $\mathcal{F}^m(G)$ if it is smooth away from the origin and its Fourier transform satisfies the estimates

$$|D^\alpha \widehat{R}(\xi)| \leq C_\alpha |\xi|^{m-d(\alpha}$$

so that, by Proposition 5.9 below, the Calderón-Zygmund kernels are precisely the kernels of class $\mathcal{F}^0(G)$. By Coré-Geller [7],

$$\mathcal{F}^{m_1}(G) \star \mathcal{F}^{m_2}(G) \subset \mathcal{F}^{m_1+m_2}(G),$$

provided $m_1, m_2, m_1 + m_2 > -Q$. A model kernel of this type is a homogeneous distribution smooth away from the origin which is also a generalised laplacian. Such kernels are generating functionals of Poisson-like semigroups of measures and, as it seems, are natural replacements for the Laplace operator, or rather its fractional power. On homogeneous groups laplacians are not homogeneous and sublaplacians may not exist.

Theorem 1.3 belongs naturally in the context of our previous work (see [14]) and, ideally, should have made a part of it. Unfortunately, at the time of writing the paper technical difficulties prevented us from incorporating it and accommodating the relevant parts of the paper so as to put the whole thing nicely in one piece. The paper is heavily dependent on the Melin calculus for which we refer the reader to [12], Melin [20], and Manchon [19].

One more remark is in order. There is some overlap here with [14]. This is due to the fact that the proof a key lemma in [14], namely Lemma 3.6, is defective. To save the paper we give new proofs of Corollaries 3.7 and 3.8 that follow from the lemma. The claims of the corollaries are contained in our main theorem (Theorem 6.2) and its corollary (Corollary 6.11). Lemma 3.6 of [14] is replaced by Lemmas 4.1 and 4.2 below.

2. Notation and preliminaries

A homogeneous group G will be identified via the exponential map with its Lie algebra $\mathfrak{g}$. We change our notation from Section 1 and henceforth write $\mathfrak{g}$ rather than G for the nilpotent group in question. Of course, $\mathfrak{g}$ still has the Lie algebra structure, in particular it is a vector space. We shall denote by $\mathfrak{g}^\star$ its dual. Lebesgue measure on the vector space $\mathfrak{g}$ is a Haar measure on the group $\mathfrak{g}$. Whenever we refer to convolution of functions on $\mathfrak{g}$, we always think of

$$f \star g(x) = \int f(xy^{-1}) g(y)\, dy,$$

where $(x, y) \to xy$ is the Campell-Hausdorff multiplication

$$\begin{aligned} xy =& x + y + \frac{1}{2}[x, y] + \frac{1}{12}[x, [x, y]] - \frac{1}{12}[y, [x, y]] - \frac{1}{24}[y, [x, [x, y]]] \\ &+ \text{finite number of commutators in five or more terms} \\ &= x + y + r(x, y), \end{aligned}$$

where r is a polynomial mapping (see, e.g. Corwin-Greenleaf [8], section 1.2). Note that 0 is the identity and $x^{-1} = -x$, for $x \in \mathfrak{g}$. We also let

$$\widetilde{f}(x) = f(x^{-1}), \qquad f^\star(x) = \overline{f(x^{-1})}, \qquad f_t(x) = t^{-Q} f(\delta_{t^{-1}} x),$$

for $t > 0$. We shall employ the Abelian Fourier transform

$$\widehat{f}(\xi) = \int_{\mathfrak{g}} f(x) e^{-i\langle x,\xi\rangle}\, dx, \qquad f \in L^1(\mathfrak{g}),\ \xi \in \mathfrak{g}^\star,$$

where $(x,\xi) \to \langle x,\xi\rangle$ is a duality of vector spaces and $L^1(\mathfrak{g})$ denotes the usual Lebesgue space of integrable functions. We refer to it simply as the Fourier transform. The representation-theoretic group Fourier transform is never used.

Let $\{\delta_t\}_{t>0}$, be a family of group dilations on $\mathfrak{g}$ and let

$$\mathfrak{g}_j = \{x \in \mathfrak{g} : \delta_t x = t^{p_j} x\}, \qquad 1 \le j \le d,$$

where $1 = p_1 < p_2 < \cdots < p_d$. Then

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_d. \tag{2.1}$$

The number $Q = \sum_{k=1}^d Q_k$, where $Q_k = p_k \dim \mathfrak{g}_k$, is called *the homogeneous dimension* of $\mathfrak{g}$. We have $d\delta_t x = t^Q dx$.

We also pick an auxilliary Euclidean norm $\|\cdot\|$ such that the decomposition (2.1) is orthogonal and fix an orthonormal basis $\{e_{kj}\}_{j=1}^{n_k}$ in $\mathfrak{g}_k$, where $n_k = \dim \mathfrak{g}_k$. Thus the variable $x \in \mathfrak{g}$ splits into $x = (x_1, x_2, \ldots, x_d)$, where

$$x_k = (x_{k1}, x_{k2}, \ldots, x_{kn_k}) \in \mathfrak{g}_k.$$

A similar notation will be applied to the variable $\xi \in \mathfrak{g}^\star$ and to multiindices α. In particular,

$$d(\alpha) = \sum_{k=1}^d p_k |\alpha_k|, \qquad |\alpha| = \sum_{k=1}^d |\alpha_k|, \qquad |\alpha_k| = \sum_{j=1}^{n_k} |\alpha_{kj}|, \tag{2.2}$$

for $\alpha = (\alpha_k)_{k=1}^d = (\alpha_{kj}) \in \boldsymbol{N}^{\dim \mathfrak{g}}$, where $\boldsymbol{N}$ stands for the set of nonnegative integers. Let also

$$T_{kj} F(x) = i x_{kj} F(x), \qquad D_{kj} F(x) = F'(x) e_{kj},$$

and

$$T_\alpha F(x) = (ix)^\alpha F(x), \qquad D^\alpha F(x) = D_{11}^{\alpha_{11}} D_{12}^{\alpha_{12}} \ldots D_{dn_{d-1}}^{\alpha_{dn_{d-1}}} D_{dn_d}^{\alpha_{dn_d}} F(x).$$

Denote by Y_{kj} the right-invariant vector field such that

$$Y_{kj} f(0) = D_{kj} f(0), \qquad f \in C^\infty(\mathfrak{g}),$$

and let

$$Y^\alpha = Y_{11}^{\alpha_{11}} Y_{12}^{\alpha_{12}} \ldots Y_{dn_{d-1}}^{\alpha_{dn_{d-1}}} Y_{dn_d}^{\alpha_{dn_d}}.$$

A homogeneous norm on $\mathfrak{g}$ is a nonnegative function $x \mapsto |x|$ such that a) $|x| = 0$ implies $x = 0$, b) $|x^{-1}| = |x|$, c) $|\delta_t x| = t|x|$, for $t > 0$. There always exists a homogeneous norm on $\mathfrak{g}$ which is d) smooth away from the origin. In fact, we may take advantage of the implicit function theorem by letting

$$\|\delta_{|x|^{-1}} x\| = 1, \qquad x \in \mathfrak{g} \setminus \{0\},\ x \ne 0,$$

and $|x| = 0$. By Folland-Stein [11], page 8, $|\cdot|$ is a homogeneous norm. We define a homogeneous norm on $\mathfrak{g}^\star$ by duality.

We assume once for all that $d \geq 2$. Let $\mathfrak{z} = \mathfrak{g}_d$ be the central subalgebra corresponding to the largest eigenvalue of the dilations. Then,

$$\mathfrak{g} = \mathfrak{g}_0 \times \mathfrak{z}, \qquad \mathfrak{g}^\star = \mathfrak{g}_0^\star \times \mathfrak{z}^\star, \tag{2.3}$$

where

$$\mathfrak{g}_0 = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_{d-1}$$

may be identified with the quotient Lie algebra $\mathfrak{g}/\mathfrak{z}$. The homogeneous dimension of $\mathfrak{g}_0$ is $Q_0 = \sum_{k=1}^{d-1} Q_k$. Thus the variable x in $\mathfrak{g}$ splits as $x = (y, u)$ in accordance with the given decomposition. In a similar way we also split the variable $\xi = (\eta, \lambda)$ in $\mathfrak{g}^\star$. Then,

$$(x, u)(y, v) = (x \circ y, u + v + r_d(x, y)), \tag{2.4}$$

where $x \circ y$ denotes the multiplication in $\mathfrak{g}_0 = \mathfrak{g}/\mathfrak{g}_d$, and

$$r(x, y) = r_0(x, y) + r_d(x, y) \in \mathfrak{g}_0 \oplus \mathfrak{g}_d. \tag{2.5}$$

Note that $\mathfrak{g} \ni (x, u) \mapsto x \in \mathfrak{g}_0$ is the quotient homomorphism.

The Schwartz space of smooth functions which vanish rapidly at infinity along with all their derivatives will be denoted by $\mathcal{S}(\mathfrak{g})$. The seminorms

$$\|f\|_{(N)} = \max_{d(\alpha)+d(\beta)\leq N} \sup_{x\in\mathfrak{g}} |x^\alpha D^\beta f(x)|, \qquad N \in \boldsymbol{N},$$

form a complete set of seminorms in $\mathcal{S}(\mathfrak{g})$ giving it a structure of a locally convex Fréchet space. $\mathcal{S}(\mathfrak{g})$ is a dense subspace of both $L^1(\mathfrak{g})$ and $L^2(\mathfrak{g})$, the space of all square-integrable functions on $\mathfrak{g}$.

Let K be a tempered distribution, that is a continuous linear functional on $\mathcal{S}(\mathfrak{g})$. The action of K on a Schwartz function f will be denoted by

$$\langle K, f\rangle = \int_{\mathfrak{g}} f(x)K(x)\, dx$$

even when K is not locally integrable. We also let

$$\langle \widetilde{K}, f\rangle = \langle K, \widetilde{f}\rangle, \qquad \langle K^\star, f\rangle = \langle K, f^\star\rangle.$$

We define

$$f\#g = \left(f^\vee \star g^\vee\right)^\wedge \tag{2.6}$$

for $f, g \in \mathcal{S}(\mathfrak{g}^\star)$. By $f \mapsto f^\vee$ we denote the inverse Fourier transform.

By (1.22) of Folland-Stein [11], the group law is expressed by

$$(xy^{-1})_{kj} = x_{kj} - y_{kj} + P_{kj}(x, y),$$

where the polynomial P_{kj} is homogeneous of degree p_k and depends on the variables x_j, y_j, for $j < k$. Then, it is directly checked that

$$T_{kj}(f \star g) = T_{kj}f \star g + f \star T_{kj}g + \sum_{\substack{d(\alpha)+d(\beta)=p_k \\ 0<d(\alpha)<p_k}} c_{\alpha\beta} T_\alpha(f \star T_\beta g), \tag{2.7}$$

for some $c_{\alpha\beta} \in \boldsymbol{R}$. In particular,

$$T_{1j}(f \star g) = T_{1j}f \star g + f \star T_{1j}g. \tag{2.8}$$

Lemma 2.9. *For every $f, g \in \mathcal{S}(\mathfrak{g})$ and every $\gamma \neq 0$,*

$$T_\gamma(f \star g) = T_\gamma f \star g + f \star T_\gamma g + \sum_{\substack{d(\alpha)+d(\beta)=d(\gamma) \\ 0<d(\alpha)<d(\gamma)}} c_{\alpha\beta} T_\alpha f \star T_\beta g$$

$$= \sum_{\substack{d(\alpha)+d(\beta)=d(\gamma) \\ d(\alpha)\leq(\gamma)}} c_{\alpha\beta} T_\alpha f \star T_\beta g. \tag{2.10}$$

Equivalently, by applying the Fourier transform,

$$D^\gamma(f\#g) = D^\gamma f\#g + f\#D^\gamma g + \sum_{\substack{d(\alpha)+d(\beta)=d(\gamma) \\ 0<d(\alpha)<d(\gamma)}} c_{\alpha\beta}\, D^\alpha f\#D^\beta g$$

$$= \sum_{\substack{d(\alpha)+d(\beta)=d(\gamma) \\ d(\alpha)\leq d(\gamma)}} c_{\alpha\beta}\, D^\alpha f\#D^\beta g \tag{2.11}$$

for $f, g \in \mathcal{S}(\mathfrak{g}^\star)$.

Proof. The proof proceeds by induction on the lengh of γ. By (2.8), the claim is true for $d(\gamma) = 1$. We pick a $\gamma \neq 0$ and assume that (2.10) holds for all $d(\delta) < d(\gamma)$. We let $T_\gamma = T_{kj} T_\delta$ so that $d(\gamma) = d(\delta) + p_k$. By induction hypothesis and (2.7),

$$T_\gamma(f \star g) = T_{kj}\Big(T_\delta f \star g + f \star T_\delta g + \sum_{\substack{d(\alpha)+d(\beta)=d(\delta) \\ 0<d(\alpha)<d(\delta)}} c_{\alpha\beta} T_\alpha f \star T_\beta g\Big)$$

$$= T_\gamma f \star g + f \star T_\gamma g + R_\gamma(f, g),$$

where

$$R_\gamma(f, g) = T_\delta f \star T_{kj} g + T_{kj} f \star T_\delta g + \sum_{\substack{d(\alpha)+d(\beta)=d(\delta) \\ 0<d(\alpha)<d(\delta)}} c_{\alpha\beta} \sum_{\substack{d(\theta)+d(\zeta)=p_k \\ 0<d(\theta)<p_k}} d_{\theta\zeta} T_\theta(T_\alpha f \star T_\zeta T_\beta g).$$

To complete the proof one only needs to note that $d(\theta) < d(\gamma)$ and apply the induction hypothesis to the expressions $T_\theta(T_\alpha f \star T_\zeta T_\beta g)$. □

Denote by Δ the semigroup of nonnegative numbers generated by the exponents of homogeneity $\{p_k\}_{k=1}^d$. For $m \geq 0$, let

$$[m] = \max\{n \in \boldsymbol{N} : n \leq m\}, \qquad \widetilde{m} = \min\{p \in \Delta : p > m\}.$$

We shall make use of the following weak version of the Taylor inequality of Folland-Stein [11] (Theorem 1.37).

Proposition 2.12. *Let $m \geq 0$. For every $f \in C^\infty(\mathfrak{g})$ and every x in a fixed bounded set,*

$$\Big| f(x) - \sum_{d(\alpha)\leq m} \frac{D^\alpha f(0)}{\alpha!} x^\alpha \Big| \leq \Big(C \sum_{\substack{|\alpha|\leq[m]+1 \\ d(\alpha)>m}} \|D^\alpha f\|_\infty \Big) |x|^{\widetilde{m}},$$

where $\|f\|_\infty = \sup_{x\in\mathfrak{g}} |f(x)|$.

3. Symbolic calculus

Let T be a tempered distribution. By $\mathrm{Op}(T)$ we shall denote the linear convolution operator

$$\mathcal{S}(\mathfrak{g}) \ni f \mapsto f \star T \in C^\infty(\mathfrak{g}).$$

T is called an L^2-*convolver* if $\mathrm{Op}(T)$ extends to a bounded endomorphism of $L^2(\mathfrak{g})$. The norm of $\mathrm{Op}(T)$ acting on $L^2(\mathfrak{g})$ will be denoted by $\|\mathrm{Op}(T)\|$ and referred to simply as the operator norm of $\mathrm{Op}(T)$. If T, S are convolvers, then there exists a convolver R such that

$$\mathrm{Op}(T)\mathrm{Op}(S) = \mathrm{Op}(R).$$

We write $R = T \star S$. We say that a convolver T is invertible, if there exists another convolver S such that

$$\mathrm{Op}(T)\mathrm{Op}(S) = \mathrm{Op}(S)\mathrm{Op}(T) = I,$$

where I stands for the identity operator on $L^2(\mathfrak{g})$, which is of course equivalent to saying that the operator $\mathrm{Op}(T)$ is invertible on $L^2(\mathfrak{g})$. If T is a convolver, then $\mathrm{Op}(T)^\star = \mathrm{Op}(T^\star)$.

Let $m \in \boldsymbol{R}$. By $S^m(\mathfrak{g})$ we denote the class of $A \in \mathcal{S}'(\mathfrak{g})$ whose Fourier transforms $\widehat{A}$ are smooth functions on $\mathfrak{g}^\star$ such that

$$|D^\alpha \widehat{A}(\xi)| \le C_\alpha (1 + |\xi|)^{m-|\alpha|}, \qquad \xi \in \mathfrak{g}^\star, \quad \text{all } \alpha.$$

$S^m(\mathfrak{g})$ is a Fréchet space with the family of seminorms

$$\|A\|_N = \max_{d(\alpha) \le N} \sup_{\xi \in \mathfrak{g}^\star} |(1 + |\xi|)^{-m+|\alpha|} D^\alpha \widehat{A}(\xi)|. \tag{3.1}$$

It is not hard to see that an $A \in S^m(\mathfrak{g})$ is smooth away from the origin and satisfies

$$|D^\alpha A(x)| \le C_{\alpha,N} |x|^{-N}, \qquad |x| \ge 1, \tag{3.2}$$

for every α and every $N > 0$. Thus, A can be represented as a sum of a compactly supported distribution and a Schwartz function.

Let $U \subset \mathfrak{z}^\star$ be open. Let $\mathcal{S}_U(\mathfrak{g})$ denote the space of all $f \in \mathcal{S}(\mathfrak{g})$ such that the $\mathfrak{z}^\star$-support of $\widehat{f}$ is contained in U. In other words, $f \in \mathcal{S}_U(\mathfrak{g})$, if there exists a closed set $E \subset U$ such that $\widehat{f}(\eta, \lambda) = 0$, for $(\eta, \lambda) \notin \mathfrak{g}_0^\star \times E$.

Lemma 3.3. *Let U be open. The class $\mathcal{S}_U(\mathfrak{g})$ is invariant under left and right group translations.*

Proof. Note first that $f \in \mathcal{S}_U(\mathfrak{g})$ if and only if, for every $\lambda_0 \notin U$, there exists a neighbourhood V of λ_0 such that $\widehat{f}\varphi = 0$, for all $\varphi \in C_c^\infty(V)$. Thus, our claim follows from the following identities

$$(\widehat{\mu \star f})\varphi = (\widehat{\mu} \# \widehat{f})\varphi = \widehat{\mu} \# \widehat{f}\varphi,$$

and

$$(\widehat{f \star \mu})\varphi = (\widehat{f} \# \widehat{\mu})\varphi = \widehat{f}\varphi \# \widehat{\mu},$$

for every bounded measure μ on $\mathfrak{g}$. The identities are due to the fact that φ considered as a function on $\mathfrak{g}^\star$ independent of the variable η is the Fourier transform of a central measure. □

The convergence in the Fréchet topology will be referred to as the *strong convergence* in $S^m(\mathfrak{g})$. Apart from that we shall also consider a *weak convergence*. We say that a bounded sequence $\{A_n\}$ of elements of $S^m(\mathfrak{g})$ is *weakly convergent* if, for every α, the sequence $\{D^\alpha \widehat{A_n}\}$ is uniformly convergent on compact subsets of $\mathfrak{g}^\star$.

Proposition 3.4. *The convolution mapping*

$$\mathcal{S}(\mathfrak{g}) \times \mathcal{S}(\mathfrak{g}) \ni (f,g) \mapsto f \star g \in \mathcal{S}(\mathfrak{g})$$

extends uniquely to a mapping

$$S^{m_1}(\mathfrak{g}) \times S^{m_2}(\mathfrak{g}) \ni (A,B) \mapsto A \star B \in S^{m_1+m_2}(\mathfrak{g})$$

which is continuous when all three spaces are endowed simultaneously with either strong or weak topology.

Proof. This is Corollary 5.2 of [12] specialized to the metric

$$g_\xi(\zeta)^2 = (1+|\xi|)^{-2} \sum_{k=1}^{d} \|\zeta_k\|^2, \qquad \xi,\zeta \in \mathfrak{g}^\star.$$

□

Let $m \in \boldsymbol{R}$. By $S_0^m(\mathfrak{g})$ we denote the class of $A \in \mathcal{S}'(\mathfrak{g})$ whose Fourier transforms $\widehat{A}$ are smooth functions on $\mathfrak{g}^\star$ such that

$$|D_\eta^\alpha D_\lambda^\beta \widehat{A}(\xi)| \le C_{\alpha\beta}(1+|\eta|+|\lambda|)^{m-|\alpha|}, \qquad \xi \in \mathfrak{g}^\star, \quad \text{all } \alpha.$$

$S_0^m(\mathfrak{g})$ is a Fréchet space with the family of seminorms

$$\|A\|_N = \max_{d(\alpha)+d(\beta)\le N} \sup_{(\eta,\lambda)\in\mathfrak{g}^\star} |(1+|\eta|+|\lambda|)^{-m+|\alpha|} D_\eta^\alpha D_\lambda^\beta \widehat{A}(\xi)|. \tag{3.5}$$

The notions of weak and strong convergence in $S_0^m(\mathfrak{g})$ are analogous to those in $S^m(\mathfrak{g})$.

Proposition 3.6. *The convolution mapping*

$$\mathcal{S}(\mathfrak{g}) \times \mathcal{S}(\mathfrak{g}) \ni (f,g) \mapsto f \star g \in \mathcal{S}(\mathfrak{g})$$

extends uniquely to a mapping

$$S_0^{m_1}(\mathfrak{g}) \times S_0^{m_2}(\mathfrak{g}) \ni (A,B) \mapsto A \star B \in S_0^{m_1+m_2}(\mathfrak{g})$$

which is continuous when all three spaces are endowed simultaneously with either strong or weak topology.

Proof. This is Corollary 5.2 of [12] specialized to the metric

$$g_{(\eta,\lambda)}(\zeta,\mu)^2 = (1+|\eta|+|\lambda|)^{-2} \sum_{k=1}^{d-1} \|\zeta_k\|^2 + \|\mu\|^2, \qquad (\eta,\lambda),(\zeta,\mu) \in \mathfrak{g}^\star.$$

□

The Fourier transform of a distribution $A \in \mathcal{S}'(\mathfrak{g})$ will be called the *symbol* of A. The twisted product

$$a\#b = \left(a^\vee \star b^\vee\right)^\wedge$$

as defined in (2.6) makes sense whenever the convolution on the right-hand side makes sense. This happens when, e.g. a, b are Fourier transforms of convolvers or when they are symbols of elements of some classes $S^m(\mathfrak{g})$. Whenever convenient we will work with the spaces of symbols $\widehat{S}^m(\mathfrak{g}^\star)$ and $\widehat{S}_0^m(\mathfrak{g}^\star)$ which are natural equivalents of the corresponding spaces $S^m(\mathfrak{g})$ and $S_0^m(\mathfrak{g})$.

Proposition 3.7. *There exists an integer N such that, for every $A \in S^0(\mathfrak{g})$ and every $f \in \mathcal{S}(\mathfrak{g})$,*

$$\|\mathrm{Op}(A)f\| \le \|A\|_N \|f\|,$$

where $\|f\|^2 = \int_{\mathfrak{g}} |f(x)|^2\,dx$. Thus, every element of $S^0(\mathfrak{g})$ is a convolver.

Proof. This is a consequence of Theorem 7.4 of [12]. Alternatively it can be seen as a corollary to the Ricci theorem invoked below in (6.1), see Ricci [21]. □

We shall need a slight generalization of the calculus. First, let us recall that, for $f, g \in \mathcal{S}(\mathfrak{g}^\star)$,

$$\begin{aligned} f\#g(\eta,\lambda) &= (f^\vee \star g^\vee)^\wedge(\eta,\lambda) \\ &= \iint_{\mathfrak{g}_0\times\mathfrak{g}_0} f(\cdot,\lambda)^\vee(x) g(\cdot,\lambda)^\vee(y) H(x,y,\eta,\lambda) e^{-i\langle x+y,\eta\rangle}\,dx\,dy, \end{aligned}$$

where

$$H(x,y,\eta,\lambda) = e^{-i\langle r_0(x,y),\eta\rangle} e^{-i\langle r_d(x,y),\lambda\rangle}.$$

(Here r_0 and r_d are as in (2.5).) For each $\theta \in (0,1)$, we define a new bilinear mapping

$$f\#^\theta g(\eta,\lambda) = \iint_{\mathfrak{g}_0\times\mathfrak{g}_0} f(\cdot,\lambda)^\vee(x) g(\cdot,\lambda)^\vee(y) H^\theta(x,y,\eta,\lambda) e^{-i\langle x+y,\eta\rangle}\,dx\,dy,$$

where

$$H^\theta(x,y,\eta,\lambda) = e^{-i\langle r_0(x,y),\eta\rangle} e^{-i\theta\langle r_d(x,y),\lambda\rangle}.$$

Let

$$f \star^\theta g = \left(\widehat{f}\#^\theta \widehat{g}\right)^\vee, \qquad f, g \in \mathcal{S}(\mathfrak{g}), \quad \theta \in (0,1).$$

Proposition 3.8. *Let $m_1, m_2 \in \boldsymbol{R}$. The mappings*

$$\mathcal{S}(\mathfrak{g}) \times \mathcal{S}(\mathfrak{g} \ni (f,g) \mapsto f \star^\theta g \in \mathcal{S}(\mathfrak{g}), \qquad 0 < \theta < 1,$$

extend uniquely to mappings

$$S^{m_1}(\mathfrak{g}) \times S^{m_2}(\mathfrak{g}) \ni (A,B) \mapsto A \star^\theta B \in S^{m_1+m_2}(\mathfrak{g})$$

which are equicontinuous when all three spaces are endowed simultaneously with either strong or weak topology. The same holds true if the spaces $S^m(\mathfrak{g})$ are replaced with the spaces $S^m_0(\mathfrak{g})$.

Proof. Observe that H^θ corresponds to another group multiplication on $\mathfrak{g}$ generated by the commutator

$$[x,y]_\theta = [x,y]' + \theta[x,y]'', \qquad x, y \in \mathfrak{g},$$

where z' denotes the orthogonal projection of $z \in \mathfrak{g}$ onto $\mathfrak{g}_0$, and z'' the orthogonal projection onto $\mathfrak{z}$. Thus, Theorem 5.1 of [12], where all the estimates stay trivially unchanged independently of $0 < \theta \le 1$, applies. □

For a smooth function a on $\mathfrak{g}^\star$ and $\lambda \in \mathfrak{z}^\star$, let

$$a^\lambda(\eta) = a(\eta,\lambda), \qquad \eta \in \mathfrak{g}_0^\star.$$

The following proposition shows that the twisted product on $\mathfrak{g}^\star$ can be viewed as a perturbation of the twisted product on $\mathfrak{g}_0^\star$. This is our version of Proposition II.2.3 (c) of Manchon [19]. Recall that $n_d = \dim\mathfrak{z} = \dim\mathfrak{z}^\star$.

Proposition 3.9. *Let $a \in \widehat{S}_0^{m_1}(\mathfrak{g}^\star)$ and $b \in \widehat{S}_0^{m_2}(\mathfrak{g}^\star)$. Then, for every $\lambda \in \mathfrak{z}^\star$,*

$$(a\#b)(\eta,\lambda) = a^\lambda \#_0 b^\lambda(\eta) + \sum_{j=1}^{n_d} \lambda_j h_j(\eta,\lambda), \qquad \eta \in \mathfrak{g}_0^\star,$$

where $h_j \in \widehat{S}_0^{m_1+m_2-p_d}(\mathfrak{g}^\star)$, and the mappings

$$\widehat{S}_0^{m_1}(\mathfrak{g}^\star) \times \widehat{S}_0^{m_2}(\mathfrak{g}^\star) \ni (a,b) \mapsto h_j \in \widehat{S}_0^{m_1+m_2-p_d}(\mathfrak{g}^\star)$$

are continuous if all the spaces are endowed simultaneously with either weak or strong topology. The same holds true if the spaces $\widehat{S}_0^m(\mathfrak{g})$ are replaced with the spaces $\widehat{S}^m(\mathfrak{g})$.

Proof. By the Taylor formula,

$$e^{-i\langle r_d(x,y),\lambda\rangle} = 1 - \sum_{j=1}^{\dim \mathfrak{z}^\star} i\lambda_j r_{dj}(x,y) \int_0^1 e^{-i\theta\langle r_d(x,y),\lambda\rangle}\, d\theta,$$

where $r_{dj}(x,y) = \langle r(x,y), e_{dj}\rangle$, whence, for $f, g \in \mathcal{S}(\mathfrak{g})$,

$$\begin{aligned} f\#g(\eta,\lambda) &= \iint_{\mathfrak{g}\times\mathfrak{g}} f(\cdot,\lambda)^\vee(x) g(\cdot,\lambda)^\vee(y) e^{-i\langle r_0(x,y),\eta\rangle} e^{-i\langle x+y,\eta\rangle} dxdy \\ &\quad - \sum_{j=1}^{n_d} \lambda_j \int_0^1 \Phi_j^\theta(\eta,\lambda)\, d\theta = f_\lambda \#_0 g_\lambda(\eta) - \sum_{j=1}^{n_d} \lambda_j \int_0^1 \Phi_j^\theta(\eta,\lambda)\, d\theta \\ &= f_\lambda \#_0 g_\lambda(\eta) - \sum_{j=1}^{\dim \mathfrak{z}^\star} \lambda_j h_j(\eta,\lambda), \end{aligned}$$

where $\Phi_j^\theta(\eta,\lambda)$ is equal to

$$\iint_{\mathfrak{g}_0\times\mathfrak{g}_0} \{r_{dj}(x,y) f(\cdot,\lambda)^\vee(x) g(\cdot,\lambda)^\vee(y)\}\, H^\theta(x,y,\eta,\lambda) e^{-i\langle x+y,\eta\rangle} dxdy.$$

Now, r_{dj} is a homogeneous polynomial of degree p_d so that

$$\begin{aligned} \Phi_j^\theta(\eta,\lambda) &= \sum_k c_k \iint_{\mathfrak{g}_0\times\mathfrak{g}_0} (f_{j,k})_\lambda^\vee(x)(g_{j,k})_\lambda^\vee(y) H_\theta(x,y,\eta,\lambda) e^{-i\langle x+y,\eta\rangle} dxdy \\ &= \sum_k c_k f_{j,k} \#^\theta\, g_{j,k}(\eta,\lambda), \end{aligned}$$

where

$$f_{j,k} \in \widehat{S}_0^{m_1-s_1}(\mathfrak{g}^\star), \qquad g_{j,k} \in \widehat{S}_0^{m_2-s_2}(\mathfrak{g}^\star), \qquad s_1+s_2 = p_d,$$

and the constants c_k are dependent only on the group multiplication. Thus, by Proposition 3.8,

$$h_j(\eta,\lambda) = \sum_k c_k \int_0^1 f_{j,k} \#^\theta\, g_{j,k}(\eta,\lambda)\, d\theta, \qquad (\eta,\lambda) \in \mathfrak{g}_0^\star \times \mathfrak{z}^\star,$$

is an element of $\widehat{S}_0^{m_1+m_2-p_d}(\mathfrak{g}^\star)$. The continuous dependence of h_j on f, g follows from Proposition 3.8. The proof is completed by routine approximations. □

4. Symbolic calculus (lemmas)

Let us denote the twisted product on $\mathfrak{g}_0^\star$ by $\#_0$.

Lemma 4.1. *Let $a \in \widehat{S}^0(\mathfrak{g}^\star)$. Suppose that a^0 is invertible in $S^0(\mathfrak{g}_0^\star)$. Let $\varphi \in C_c^\infty(\mathfrak{z}^\star)$ and $\varphi(\lambda) = 1$, for $|\lambda| < 1$. Then, there exists $p \in S_0^0(\mathfrak{g}^\star)$ and $q \in S_0^{-p_d}(\mathfrak{g}^\star)$ such that*

$$p\#a = \varphi^2 - q.$$

Proof. Let $b_0 \in \widehat{S}^0(\mathfrak{g}_0^\star)$ be such that $a^0\#_0 b_0 = 1$. Denote by ρ a smooth function on $\mathfrak{g}^\star$ such that $\rho(\eta) \geq 1$, for every $\eta \in \mathfrak{g}_0^\star$, and

$$\rho(\eta) = 1 + |\eta|, \qquad |\eta| \geq 2.$$

Let

$$s(\eta, \lambda) = \varphi\left(\frac{\lambda}{\rho(\eta)}\right).$$

Then $s \in \widehat{S}^0(\mathfrak{g}^\star)$, and

$$p(\eta, \lambda) = \varphi^2(\eta) s(\eta, \lambda) b_0(\eta)$$

is an element of $\widehat{S}_0^0(\mathfrak{g}^\star)$. By Proposition 3.9,

$$p\#a(\eta, \lambda) = \varphi^2 s b_0 \# a(\eta, \lambda) = \varphi(\lambda)^2 s^\lambda b_0 \#_0 a^\lambda(\eta) + h(\eta, \lambda),$$

where

$$h(\eta, \lambda) = \sum_j \lambda_j h_j(\eta, \lambda)$$

is an element of $\widehat{S}_0^{-p_d}(\mathfrak{g}^\star)$. Let us take care of the first term of the sum on the right-hand side. We have

$$\begin{aligned}
\varphi^2 s^\lambda b_0 \#_0 a^\lambda &= \varphi^2 b_0 \#_0 a^\lambda + \varphi^2(1-s) b_0 \#_0 a^\lambda \\
&= \varphi^2 + \varphi b_0 \#_0 \varphi(a^\lambda - a^0) + \varphi^2(1-s) b_0 \#_0 a^\lambda \\
&= \varphi^2 + \varphi b_0 \#_0 c_\lambda + d_\lambda \# a^\lambda,
\end{aligned}$$

where

$$c_\lambda = \varphi(a^\lambda - a^0), \qquad d_\lambda = \varphi^2(1-s) b_0.$$

We are going to show that c_λ and d_λ are elements of $\widehat{S}_0^{-p_d}(\mathfrak{g}_0^\star)$. In fact, by the meanvalue theorem,

$$c_\lambda(\eta) = \varphi(\lambda)(a(\eta, \lambda) - a(\eta, 0)) = \varphi(\lambda) \sum_j \lambda_j \int_0^1 D_{\lambda_j} a(\eta, t\lambda)\, dt,$$

so that

$$|D_\eta^\alpha c_\lambda(\eta)| \leq C_\alpha |\lambda|^{p_d} \int_0^1 (1 + |\eta| + t^{\frac{1}{p_d}} |\lambda|)^{-p_d - d(\alpha)} dt \leq C'_\alpha |\lambda|^{p_d} (1 + |\eta|)^{-p_d - d(\alpha)},$$

where λ stays in a bounded set. Similarly,

$$d_\lambda(\eta) = \varphi^2(\lambda)(1 - s(\eta, \lambda)) = -\varphi^2(\lambda) \sum_j \lambda_j \int_0^1 D_\lambda s(\eta, t\lambda)\, dt,$$

and the same argument applies.

Consequently, $\varphi b_0 \#_0 c_\lambda \in \widehat{S}_0^{-p_d}(\mathfrak{g}_0^\star)$ and $d_\lambda \#_0 a^\lambda \in \widehat{S}_0^{-p_d}(\mathfrak{g}_0^\star)$. Since both functions $(\eta,\lambda) \mapsto c_\lambda(\eta)$ and $(\eta,\lambda) \mapsto d_\lambda(\eta)$ are smooth and λ stays in a bounded set,

$$q_1(\eta,\lambda) = \varphi b_0 \#_0 c_\lambda + d_\lambda \#_0 a^\lambda$$

is an element of $\widehat{S}_0^{-p_d}(\mathfrak{g}^\star)$, so that, finally,

$$p\#a = \varphi^2 - q,$$

where $q = -q_1 - h \in \widehat{S}_0^{-p_d}(\mathfrak{g}^\star)$.

□

Our next lemma goes one step further.

Lemma 4.2. *Let $p, a \in \widehat{S}_0^0(\mathfrak{g}^\star)$, $q \in \widehat{S}_0^{-m}(\mathfrak{g}^\star)$. Let $\psi \in C^\infty(\mathfrak{z}^\star)$ have bounded derivatives. If*

$$p\#a = \psi - q,$$

then, for every positive integer N, there exists $p_N \in \widehat{S}_0^0(\mathfrak{g}^\star)$ such that

$$p_N\#a = \psi^{2^N} - q_N.$$

where $q_N \in \widehat{S}_0^{-2^N m}(\mathfrak{g}^\star)$.

Proof. We let $p_0 = p$ and

$$p_{N+1} = (\psi + q^{2^N})\#p_N, \qquad q_N = q^{2^N}, \qquad N \geq 0,$$

where the power is understood in the sense of the twisted product. The proof follows by an easy induction. □

Lemma 4.3. *Let $a, b \in \widehat{S}_0^0(\mathfrak{g}^\star)$. Assume that*

$$a\#b(\eta,\lambda) = 1, \qquad \eta \in \mathfrak{g}_0^\star,\ \lambda \in U,$$

where $U \subset \mathfrak{z}^\star$ is open. Let $V \subset \overline{V} \subset U$ be another open set. If a satisfies

$$|D^\alpha a(\xi)| \leq C_\alpha (1 + |\xi|)^{-d(\alpha)} \tag{4.4}$$

on $\mathfrak{g}_0^\star \times U$, then so does b on $\mathfrak{g}_0^\star \times V$. Each of the constants C_α in the case of b depends on finitely many of those in the case of a.

Proof. By Lemma 2.9,

$$D^\gamma b = b\#D^\gamma(a\#b) - \sum_{\substack{d(\alpha)+d(\beta)=d(\gamma)\\ d(\beta)<d(\gamma)}} c_{\alpha\beta}\, b\#D^\alpha a\#D^\beta b, \qquad \text{all } \gamma,$$

where $b\#D^\gamma(a\#b) = 0$ on $\mathfrak{g}_0^\star \times U$. Let $\varphi, \chi, \psi \in C_c^\infty(U)$ be such that $\varphi\psi = \chi\psi = \psi$ and ψ is equal to 1 on a neighbourhood of $\overline{V}$. We have

$$\psi D^\gamma b = - \sum_{\substack{d(\alpha)+d(\beta)=d(\gamma)\\ d(\beta)<d(\gamma)}} c_{\alpha\beta}\, \psi b\#\chi D^\alpha a\#\varphi D^\beta b,$$

which, by symbolic calculus, shows that if $\varphi D^\beta b \in \widehat{S}_0^{-d(\beta)}(\mathfrak{g}^\star)$, for all $d(\beta) < d(\gamma)$, then $\psi D^\gamma b \in \widehat{S}_0^{-d(\gamma)}$. By induction, we see that b satisfies (4.4) on $\mathfrak{g}^\star \times V$. The required dependence of constants follows from the proof. □

Let $U \subset \boldsymbol{R}^k$ be open. A family $A_u \in S^m(\mathfrak{g})$, where $u \in U$, is said to depend smoothly on the parameter u, if the the function

$$\mathfrak{g}^\star \times U \ni (\xi, u) \mapsto \widehat{A_u}(\xi) \in \boldsymbol{C}$$

is smooth.

Lemma 4.5. *Let $\{A_u\}_{u\in U}$ be a family elements of $S^m(\mathfrak{g})$ depending smoothly on $u \in U$. If A_u are invertible and the family $\{A_u^{-1}\}_{u\in U}$ is bounded in $S^{-m}(\mathfrak{g})$, then A_u^{-1} also depends smoothly on u.*

Proof. Let $u_n \to u$. The sequence $A_n = A_{u_n}$ is weakly convergent to $A = A_u$, and the sequence A_n^{-1} is bounded in $S^{-m}(\mathfrak{g})$. As such A_n^{-1} has weakly convergent subsequences. To prove that the family A_u^{-1} depends continuously on u, it is enough to show that every such subsequence is convergent to A^{-1}.

Suppose then that $A_{n_k}^{-1} \to B$ weakly in $S^m(\mathfrak{g})$. Then, by Proposition 3.4,

$$I = A_{n_k}^{-1} A_{n_k} = A_{n_k} A_{n_k}^{-1} \to BA = AB,$$

which implies $B = A^{-1}$.

Let $a(\cdot, u) = \widehat{A_u}$. Let $b(\cdot, u) = a(\cdot, u)^{-1}$. We are going to show that, for every α, the mapping

$$u \mapsto D_u^\alpha b(\cdot, u) \in \widehat{S}^{-m}(\mathfrak{g}^\star)$$

is weakly continuous, which implies our assertion.

If $\alpha = 0$, then the assertion follows by the first part of the proof. Assume that $\alpha \neq 0$ and the assertion holds for all α' such that $d(\alpha') < d(\alpha)$. Let $v \in \boldsymbol{R}^k$. Then,

$$\lim_{t\to 0} \frac{b(\cdot, u+tv) - b(\cdot, u)}{t} = \lim_{t\to 0} b(\cdot, u) \# \frac{a(\cdot, u) - a(\cdot, u+tv)}{t} \# b(\cdot, u+tv),$$

where

$$\frac{a(\cdot, u) - a(\cdot, u+tv)}{t} \to -\nabla_v a(\cdot, u)$$

weakly in $\widehat{S}^0(\mathfrak{g}^\star)$, so

$$D_{u_j} b(\cdot, u) = b(\cdot, u) \# D_{u_j} a(\cdot, u) \# b(\cdot, u), \qquad 1 \le j \le p.$$

By induction, it follows that

$$D_u^\alpha b(\cdot, u) = \sum_{\beta+\gamma+\delta=\alpha,\, d(\gamma)>0} D_u^\beta b(\cdot, u) \# D_u^\gamma a(\cdot, u) \# D_u^\delta b(\cdot, u), \qquad \text{all } \alpha,$$

which, by hypothesis and Proposition 3.4, implies that $D_u^\alpha b(\cdot, u) \in S^{-m}(\mathfrak{g}^\star)$ with a weakly continuous dependence on u. □

Recall that the class $\mathcal{S}_U(\mathfrak{g})$, where $U \subset \mathfrak{z}^\star$ is open, has been defined in Section 3.

Lemma 4.6. *Let $U \subset \mathfrak{z}^\star$ be open. Let $A \in S_0^0(\mathfrak{g})$. Then* $\mathrm{Op}(A)$ *maps continuously $\mathcal{S}_U(\mathfrak{g})$ into $\mathcal{S}_U(\mathfrak{g})$. If $B \in \mathcal{S}'(\mathfrak{g})$ is a convolver such that*

$$\mathrm{Op}(A)\mathrm{Op}(B)f = \mathrm{Op}(B)\mathrm{Op}(A)f = f, \qquad f \in \mathcal{S}_U(\mathfrak{g}),$$

then also $\mathrm{Op}(B)$ *maps continuously $\mathcal{S}_U(\mathfrak{g})$ into $\mathcal{S}_U(\mathfrak{g})$. To be more precise, for every N, there exists a constant C_N and an integer M_N such that*

$$\|\mathrm{Op}(B)f\|_{(N)} \le C_N \|f\|_{(M_N)}, \qquad f \in \mathcal{S}_U(\mathfrak{g}), \tag{4.7}$$

where each of the constants C_N depends only on a seminorm of A in $S^0_0(\mathfrak{g})$ and the operator norm of $\mathrm{Op}(B)$.

Proof. That $\mathrm{Op}(A)$ maps $\mathcal{S}_U(\mathfrak{g})$ continuously into $\mathcal{S}_U(\mathfrak{g})$ follows from (3.2) and the fact that $\mathfrak{z} = \mathfrak{g}_d$ is central. Thus, we turn to $\mathrm{Op}(B)$. Being a convolution operator bounded on $L^2(\mathfrak{g})$, it commutes with right-invariant derivatives Y^γ. Therefore, by the Sobolev inequality, it is sufficient to show that for every γ, there exists a constant C_γ depending on a finite number of seminorms $\|A\|_N$ and such that

$$\|T_\gamma \mathrm{Op}(B) f\|_{L^2(\mathfrak{g})} \le C_\gamma \max_{d(\alpha)\le d(\gamma)} \|T_\alpha f\|_{L^2(\mathfrak{g})}, \qquad f \in \mathcal{S}_U(\mathfrak{g}).$$

Let

$$\langle A_\alpha, f\rangle = \langle A, x^\alpha f\rangle.$$

Then $A_\alpha \in S^{-d(\alpha)}(\mathfrak{g})$ and, by Lemma 2.9,

$$[T_\gamma, \mathrm{Op}(A)] = \mathrm{Op}(A_\gamma) + \sum_{\substack{d(\alpha)+d(\beta)=d(\gamma)\\ 0<d(\alpha)<d(\gamma)}} c_{\alpha\beta} \mathrm{Op}(A_\alpha) T_\beta$$

so

$$\begin{aligned} T_\gamma \mathrm{Op}(B) &= \mathrm{Op}(B)T_\gamma - \mathrm{Op}(B)[T_\gamma, \mathrm{Op}(A)]\mathrm{Op}(B) \\ &= \mathrm{Op}(B)T_\gamma - \mathrm{Op}(B)\mathrm{Op}(A_\gamma)\mathrm{Op}(B) \\ &\quad - \sum_{\substack{d(\alpha)+d(\beta)=d(\gamma)\\ 0<d(\alpha)<d(\gamma)}} c_{\alpha\beta}\, \mathrm{Op}(B)\mathrm{Op}(A_\alpha)T_\beta \mathrm{Op}(B). \end{aligned} \tag{4.8}$$

Since $A_\alpha \in S^{-d(\alpha)}(\mathfrak{g}) \subset S^0(\mathfrak{g})$, by Proposition 3.7, the operators $\mathrm{Op}(A_\alpha)$ are bounded. The proof is completed by induction. The required dependence of seminorms and the constants C_N follows from the proof. □

5. Kernels in $\mathcal{F}^m(\mathfrak{g})$

Let $m \in \boldsymbol{R}$. A tempered distribution K belongs to $\mathcal{F}^m(\mathfrak{g})$, if it is smooth away from the origin, satisfies the size condition

$$|D^\alpha K(x)| \le C_\alpha |x|^{-Q-m-|\alpha|}, \tag{5.1}$$

and, for every $\varphi \in \mathcal{S}(\mathfrak{g})$, the cancellation condition

$$|\langle K, \varphi\circ\delta_R\rangle| \le CR^m, \qquad R > 0, \tag{5.2}$$

where the constant C does not depend on $R > 0$.

Remark 5.3. Let $K \in \mathcal{F}^m(\mathfrak{g})$. Let

$$\langle K_t, f\rangle = \langle K, f\circ\delta_t\rangle, \qquad t > 0.$$

Then, for every $t > 0$, $t^{-m}K \in \mathcal{F}^m(\mathfrak{g})$ with the same constants.

Remark 5.4. If $K \in \mathcal{F}^m(\mathfrak{g})$, then, for every α,

$$D^\alpha K \in \mathcal{F}^{m+d(\alpha)}(\mathfrak{g}), \qquad x^\alpha K \in \mathcal{F}^{m-d(\alpha)}(\mathfrak{g}).$$

Proposition 5.5. *If $K \in \mathcal{F}^m(\mathfrak{g})$ and $m > 0$, then*

$$|\langle K, \varphi\circ\delta_R\rangle| \le CN(\varphi)R^m, \qquad \varphi \in \mathcal{S}(\boldsymbol{R}^n),\ R > 0,$$

where $N(\varphi) = \max_{|\alpha|\le[m]+1} \|D^\alpha\varphi\|_\infty$.

Proof. Let $\eta \in C_c^\infty(\mathfrak{g})$ be equal to 1 in a neighbourhood of the origin, and keep it fixed. We have

$$\begin{aligned}\int \varphi(Rx)K(x)\,dx &= \int \varphi(x)K_R(x)\,dx = \int \Big(\varphi(x) - \sum_{d(\alpha)\le m} \frac{D^\alpha\varphi(0)}{\alpha!}x^\alpha\Big)\eta(x)K_R(x)\,dx \\ &+ \sum_{d(\alpha)\le m} \frac{D^\alpha\varphi(0)}{\alpha!}\int \eta(x)x^\alpha K_R(x)\,dx + \int \varphi(x)(1-\eta(x))K_R(x)\,dx \\ &= I_1(R) + I_2(R) + I_3(R),\end{aligned}$$

where, by Proposition 2.12,

$$|I_1(R)| \le C_1 N(\varphi) R^m \int_{|x|\le c} |x|^{-n+\widetilde{m}-m}\,dx \le C_2 N(\varphi) R^m,$$

$$|I_2(R)| \le C_1 \sum_{d(\alpha)\le m} \frac{|D^\alpha\varphi(0)|}{\alpha!} R^m \le C_2 N(\varphi) R^m.$$

Finally,

$$I_3 \le C_1 R^m \int |\varphi(x)|(1-\eta(x))|x|^{-n-m}\,dx \le C_2\|\varphi\|_\infty R^m \le C_3 N(\varphi) R^m.$$

□

In a similar way we prove

Proposition 5.6. *Let $K \in \mathcal{F}^0(\mathfrak{g})$. If K has compact support, then*

$$|\langle K, \varphi\circ\delta_R\rangle| \le C N_1(\varphi), \qquad R>0,\ \varphi\in\mathcal{S}(\boldsymbol{R}^n),$$

where $N_1(\varphi) = \max_{|\alpha|\le 1}\|D^\alpha\varphi\|_\infty$. If K is supported away from the origin, then

$$|\langle K, \varphi\circ\delta_R\rangle| \le C N_2(\varphi), \qquad R>0,\ \varphi\in\mathcal{S}(\boldsymbol{R}^n),$$

where $N_2(\varphi) = \|\,|\cdot|\varphi\|_\infty$.

Remark 5.7. It is not hard to see that if $m < 0$, then the size condition implies the cancellation one. In fact,

$$\left|\int \varphi(Rx)K(x)\,dx\right| \le C N(\varphi) R^m, \qquad R>0,\ \varphi\in\mathcal{S}(\boldsymbol{R}^n),$$

where

$$N(\varphi) = \int |x|^{-n+|m|}|\varphi(x)|\,dx.$$

In the vector space $\mathcal{F}^m(\mathfrak{g})$ we introduce seminorms

$$|K|_\alpha = \sup_{x\ne 0} |x|^{|\alpha|-n-m}|D^\alpha K(x)|$$

and

$$|K|_c = \sup_{R>0} R^{-m} \sup_{N(\varphi)\le 1} |\langle K, \varphi\circ\delta_R\rangle|,$$

where in the case $m=0$ we let $N(\varphi) = N_1(\varphi) + N_2(\varphi)$.

Corollary 5.8. *If $m = 0$, then $K \in \mathcal{F}^m(\mathfrak{g})$ if and only if K is a Calderón-Zygmund kernel.*

Proposition 5.9. *Let $m > -Q$. Then the distribution $K \in \mathcal{S}'(\mathfrak{g})$ belongs to $\mathcal{F}^m(\mathfrak{g})$ if and only if its Fourier transform $\widehat{K}$ is a locally integrable function on $\mathfrak{g}^\star$ which is smooth on $\mathfrak{g}^\star \setminus \{0\}$, and satifies the estimates*

$$|D^\alpha \widehat{K}(\xi)| \le C_\alpha |\xi|^{m-|\alpha|}, \qquad \xi \ne 0. \tag{5.10}$$

The set of seminorms

$$|K|^\alpha = \sup_{0 \ne \xi \in \mathfrak{g}^\star} |\xi|^{|\alpha|-m} |D^\alpha \widehat{K}(\xi)|$$

is equivalent to the one defined above.

Proof. By Remark 5.3, the family $R^{-m}K_R$ is bounded in $\mathcal{F}^m(\mathfrak{g})$. Thus, to show that $\widehat{K}$ satisfies (5.10), it is enough to show that, for any α, the distribution $D^\alpha \widehat{K}$ is a continuous function on the annulus $1/2 \le |\xi| \le 2$.

If $m > 0$, then, for every $\varphi \in \mathcal{S}(\boldsymbol{R}^n)$, we have $K \star \varphi \in L^1(\boldsymbol{R}^n)$, hence $\widehat{\varphi}\widehat{K} \in C(\mathfrak{g}^\star)$, which implies that $\widehat{K}$ is continuous on the annulus. Now, if $K \in \mathcal{F}^m(\mathfrak{g})$, where $m \in \boldsymbol{R}$, then, for every α, there exists a finite collection B of β such that $D^\beta x^\alpha K \in \mathcal{F}^{m_1}(\mathfrak{g})$, where $m_1 > 0$, and

$$\sum_{\beta \in B} \xi^\beta \ge c > 0, \qquad 1/2 \le |\xi| \le 2. \tag{5.11}$$

Therefore, for every $\beta \in B$, $\xi^\beta D^\alpha \widehat{K}$ is a continuous function on the annulus. By (5.11), the same holds for $D^\alpha \widehat{K}$. Note that we have not used the condition $m > -Q$ so far.

Now, suppose that (5.10) holds true. By Remark 5.7 and hypothesis $m > -Q$, $\widehat{K} \in \mathcal{F}^{-m-Q}(\mathfrak{g}^\star)$, so, by the first part of the proof, K is smooth away from the origin and satisfies the size condition for $\mathcal{F}^m(\mathfrak{g})$. Furthermore,

$$\begin{aligned} |\langle K, \varphi \circ \delta_R \rangle| &= |\int_{\mathfrak{g}^\star} \widehat{K}(\xi) \widehat{\varphi}_R(\xi)\, d\xi| \\ &\le \int_{\mathfrak{g}^\star} |\widehat{K}(\delta_R \xi) \varphi(\xi)|\, d\xi \le C_1 R^m \int_{\mathfrak{g}^\star} |\xi|^m |\varphi(\xi)|\, d\xi = C_2 R^m, \end{aligned}$$

which shows that K satisfies also the cancellation condition. The equivalence of seminorms follows from the proof. □

Corollary 5.12. *Let $m > -Q$. If $K \in \mathcal{F}^m(\mathfrak{g})$ and $\widehat{K} \in C^\infty(\mathfrak{g}^\star)$, then $K \in S^m(\mathfrak{g})$.*

Remark 5.13. Denote by $\boldsymbol{F}(m)$ the class of all smooth functions f on $\mathfrak{g}$ such that

$$|D^\alpha f(x)| \le C_\alpha (1 + |x|)^{-Q-m-|\alpha|}.$$

Any $Q \in \mathcal{F}^m(\mathfrak{g})$ can be represented as

$$Q = Q_0 + q,$$

where $Q_0 \in S^m(\mathfrak{g})$ and has compact support, and $q \in \boldsymbol{F}(m)$.

Remark 5.14. We say that a pair (m_1, m_2) is *admissible* if

$$m_1, m_2 > -Q, \quad m_1 + m_2 > -Q.$$

If the pair (m_1, m_2) is admissible, then the convolution $K_1 \star K_2$ is well defined for $K_1 \in \mathcal{F}^{m_1}(\mathfrak{g})$, $K_2 \in \mathcal{F}^{m_2}(\mathfrak{g})$. In fact,

$$K_1 \star K_2 = ((K_1)_0 + k_1) \star ((K_2)_0 + k_2),$$

where $(K_1)_0$, $(K_2)_0$ have compact support and $k_1 \in \boldsymbol{F}(m_1)$, $k_2 \in \boldsymbol{F}(m_2)$. Thus, the only problem is to justify $k_1 \star k_2$. This can be done by observing that there exist $1 < p, q < \infty$ such that $1/p + 1/q = 1$ and $k_1 \in L^p(\mathfrak{g})$, $k_2 \in L^q(\mathfrak{g})$, which implies that $k_1 \star k_2$ is a continuous function vanishing at infinity.

Proposition 5.15. *Let K be a distribution such that $\widehat{K}$ is locally integrable on $\mathfrak{g}^\star$, smooth away from $\lambda = 0$, and satisfies, for all α, β,*

$$|D_\eta^\alpha D_\lambda^\beta \widehat{K}(\eta, \lambda)| \le C_{\alpha\beta}(|\eta| + |\lambda|)^{m-|\alpha|-|\beta|}, \qquad \eta \in \mathfrak{g}_0^\star,\ \lambda \in \mathfrak{z}^\star \setminus \{0\}.$$

Then, $K \in \mathcal{F}^m(\mathfrak{g})$.

Proof. This follows by Sobolev's lemma. □

The following is Theorem B of Coré-Geller [7].

Theorem 5.16. *Let $(m_1, m_2) \in \boldsymbol{R}^2$ be admissible. Let $K_1 \in \mathcal{F}^{m_1}(\mathfrak{g})$, $K_2 \in \mathcal{F}^{m_2}(\mathfrak{g})$. Then, $K = K_1 \star K_2 \in \mathcal{F}^{m_1+m_2}(\mathfrak{g})$ and each of the seminorms of K depends on a seminorm of K_1 and a seminorm of K_2.*

An important subclass of $\mathcal{F}^m(\mathfrak{g})$ is the class of all $T \in \mathcal{S}'(\mathfrak{g})$ which are smooth away from the origin and homogeneous of degree $-m - Q$. The last property means that

$$\langle T, f \circ \delta_R \rangle = R^m \langle T, f \rangle, \qquad f \in \mathcal{S}'(\mathfrak{g}), \quad R > 0.$$

A model homogeneous kernel of class $\mathcal{F}^m(\mathfrak{g})$, where $0 < m < 1$, is

$$\langle P, f \rangle = \int_{\mathfrak{g}} \Big(f(x) - f(0) \Big) \frac{dx}{|x|^{Q+m}}, \qquad f \in \mathcal{S}(\mathfrak{g}).$$

(As a matter of fact, one could consider analogous kernels for $0 < m < 2$, but we do not need this.) The distribution P is a *generalised laplacian* (see Duflo [9], Section 2), that is, satisfies the maximum principle

$$\langle P, f \rangle \le 0$$

if $f \in C_c^\infty$ is real and attains its maximal value at 0. Therefore, P is a generating functional of a continuous semigroup of subprobability measures μ_t (Hunt [17]). The measures μ_t have densities h_t, because the Lévy measure of P

$$\nu(dx) = \frac{dx}{|x|^{Q+m}}$$

is absolutely continuous with respect to Haar measure and unbounded on $\mathfrak{g} \setminus \{0\}$ (see Janssen [18]). In other words,

$$\mu_t \star \mu_s = \mu_{t+s}, \qquad t, s > 0,$$

and

$$\lim_{t \to 0} \langle \mu_t, f \rangle = f(0), \qquad f \in \mathcal{S}(\mathfrak{g}),$$

as well as

$$\frac{d}{dt}\Big|_{t=0} \langle \mu_t, f \rangle = \langle P, f \rangle, \qquad f \in \mathcal{S}(\mathfrak{g}).$$

(See Duflo [9], Proposition 4 or Hunt [17]) The operator $\boldsymbol{P} f = f \star P$ is nonpositive and essentially selfadjoint with $\mathcal{S}(\mathfrak{g})$ for its core domain. $\boldsymbol{P}$ is also an infinitesimal generator of a strongly continuous semigroup of contractions

$$T_t = f \star \mu_t, \qquad t > 0,$$

on the Hilbert space $L^2(\mathfrak{g})$ (see Duflo [9], Example 4, p 247).

By Theorem 2.3 of [15], the densities h_t are smooth functions, and

$$|D^\alpha h_t(x)| \le C_\alpha \frac{t}{(t^{1/m} + |x|)^{Q+m}}.$$

(Actually, [15] considers only the case $m = 1$. The case $0 < m < 1$ is proved in the same way by just changing exponents in the right places.) It follows that the fundamental solution for P

$$R(x) = \int_0^\infty h_t(x)\, dt$$

is integrable and smooth. R is also homogeneous of degree $-Q + m$; therefore belongs to $\mathcal{F}^{-m}(\mathfrak{g})$.

We associate with P another kernel $V \in S^m(\mathfrak{g})$ in the following way: We let $\eta \in C_c^\infty(\mathfrak{g})$ be nonnegative, less than 1, and equal to 1 for $|x| \le 1$. Then,

$$\langle V, f\rangle = \langle P, \eta f\rangle, \qquad f \in \mathcal{S}(\mathfrak{g}), \tag{5.17}$$

is a compactly supported distribution in $S^m(\mathfrak{g})$. If $f \in C_c^\infty(\mathfrak{g})$ is real and $f(x) \le f(0)$, then

$$\langle V, f\rangle = \int_{\mathfrak{g}} \frac{\eta(x) f(x) - f(0)}{|x|^{Q+m}}\, dx \le -f(0) \int_{\mathfrak{g}} \frac{1 - \eta(x)}{|x|^{Q+m}}\, dx \le -C f(0),$$

where $C > 0$, which shows that not only V, but also $V + C\delta_0$ is a generalized laplacian. By δ_0 we denote the Dirac measure at 0. It follows that

$$\|f\| \le C\|\mathrm{Op}(V) f\|, \qquad f \in \mathcal{S}(\mathfrak{g}). \tag{5.18}$$

Denote by v_t the densities of the semigroup generated by V. Then, $u_t = e^{Ct/2} v_t$ are the densities of that generated by $V + C\delta_0$. Since $\|u_t\|_{L^1} \le 1$, the fundamental solution for V

$$W(x) = \int_0^\infty v_t(x)\, dt = \int_0^\infty e^{-Ct} u_t(x)\, dt$$

is an integrable function.

Proposition 5.19. $\widehat{W} \in C^\infty(\mathfrak{g}^\star)$.

Proof. It is sufficient to show that $T_\alpha W \in L^1(\mathfrak{g})$, for every α. This is true for $\alpha = 0$. Assume that it is true for $d(\beta) < k$, and let $d(\alpha) = k$. Since $W \star V = \delta_0$, we have $T_\alpha(W \star V) = 0$. Therefore, by (2.11),

$$T_\alpha W = - \sum_{\substack{d(\beta)+d(\gamma)=k \\ d(\beta)<k}} c_{\beta\gamma} T_\beta W \star T_\gamma V \star W,$$

where, by induction hypothesis, $T_\beta W \in L^1(\mathfrak{g})$, for all $d(\beta) < k$. Recall that $0 < m < 1$, so $T_\gamma V \in L^1(\mathfrak{g})$, for all $\gamma \neq 0$. This completes the proof.

□

Let $\lambda \in \mathfrak{z}^\star$. We have the following Plancherel formula

$$\|f\|^2 = \int_{\mathfrak{z}^\star} \|f^\lambda\|^2\, d\lambda, \qquad f \in \mathcal{S}(\mathfrak{g}), \tag{5.20}$$

where

$$f^\lambda(x) = \int_{\mathfrak{z}} f(x, u) e^{-i\langle u, \lambda\rangle}\, du, \qquad f \in \mathcal{S}(\mathfrak{g}),\ x \in \mathfrak{g}_0.$$

Here and below, by $\|\cdot\|$ we denote the L^2-norm on $\mathfrak{g}$ or $\mathfrak{g}_0$.

Recall that $\circ$ denotes the group multiplication in $\mathfrak{g}_0$ (see (2.4)). Denote by $\star_0$ the convolution on $\mathfrak{g}_0$ so that

$$f \star_0 \widetilde{g}(x) = \int_{\mathfrak{g}_0} f(x \circ y) g(y)\, dy, \qquad f, g \in \mathcal{S}(\mathfrak{g}_0).$$

Let $K \in \mathcal{F}^m(\mathfrak{g})$, where $m \geq 0$. For every $\lambda \in \mathfrak{z}^\star$, we define a new distribution K^λ on $\mathfrak{g}_0$ by

$$\widehat{K^\lambda}(\eta) = \widehat{K}(\eta, \lambda), \qquad \eta \neq 0.$$

Lemma 5.21. *For every $\lambda \neq 0$, $K^\lambda \in S^m(\mathfrak{g}_0)$, and $K^0 \in \mathcal{F}^m(\mathfrak{g}_0)$. Each seminorm of K_0 in $\mathcal{F}^m(\mathfrak{g}_0)$ depends on a seminorm of K in $\mathcal{F}^m(\mathfrak{g})$. We have*

$$(f \star \widetilde{K})^\lambda(x) = \int_{\mathfrak{g}_0} e^{-i\langle (x,0)(z,0), \tilde{\lambda}\rangle} f(x \circ z) K^\lambda(z)\, dz,$$

where $\langle (x,u), \tilde{\lambda}\rangle = \langle u, \lambda\rangle$. In particular, for $\lambda = 0$,

$$(f \star \widetilde{K})^0 = f^0 \star_0 \widetilde{K^0} = \mathrm{Op}(K^0) f^0, \qquad f \in \mathcal{S}(\mathfrak{g}).$$

Finally, for every $f \in \mathcal{S}(\mathfrak{g})$, the mapping

$$\mathfrak{z}^\star \ni \lambda \mapsto (f \star \widetilde{K})^\lambda \in L^2(\mathfrak{g})$$

is continuous.

Proof. This is an exercise in Fourier transform. Note that the case $m > 0$ is simpler. □

Corollary 5.22. *Let $K \in \mathcal{F}^0(\mathfrak{g})$. If $\mathrm{Op}(K)$ is invertible on $L^2(\mathfrak{g})$, then $\mathrm{Op}(K^0)$ is invertible on $L^2(\mathfrak{g}_0)$, and $\|\mathrm{Op}(K^0)^{-1}\| \leq \|\mathrm{Op}(K)^{-1}\|$.*

Proof. Let $C = \|\mathrm{Op}(K)\|$. By hypothesis,

$$\|f\| \leq C \|\mathrm{Op}(K) f\|, \qquad \|f\| \leq C \|\mathrm{Op}(K^\star)\|,$$

for $f \in \mathcal{S}(\mathfrak{g})$. Therefore, by Plancherel's formula,

$$\int_{\mathfrak{z}^\star} \|f^\lambda\|^2\, d\lambda \leq C \int_{\mathfrak{z}^\star} \|(f \star K)^\lambda\|^2\, d\lambda.$$

Since both integrands are continuous and f is arbitrary, we get $\|f^0\| \leq C\|\mathrm{Op}(K^0) f^0\|$. Similarly, $\|f^0\| \leq C\|\mathrm{Op}(K^0)^\star f^0\|$. Every element of $\mathcal{S}(\mathfrak{g}_0)$ is of the form f^0, where $f \in \mathcal{S}(\mathfrak{g})$, so the above implies that $\mathrm{Op}(K^0)$ is invertible and $\|\mathrm{Op}(K^0)^{-1}\|$ does not exceed C. □

Corollary 5.23. *There exists a constant C such that*

$$\|f\| \leq C \|\mathrm{Op}(V^0) f\|, \qquad f \in \mathcal{S}(\mathfrak{g}_0).$$

Proof. This follows from (5.18) and Corollary 5.22. □

6. Proof of the main theorem

Let us recall that if $K \in \mathcal{F}^0$, then the operator $\mathrm{Op}(K)$ is bounded on $L^2(\mathfrak{g})$ with

$$\|\mathrm{Op}(K)\| \leq C \max_{|\alpha|\leq m} \sup_{\xi\in\mathfrak{g}^\star\setminus\{0\}} |\xi|^{d(\alpha)}|\widehat{K}(\xi)|, \tag{6.1}$$

for some $m \in \boldsymbol{N}$. This follows from Ricci [21].

Theorem 6.2. *Let $K \in \mathcal{F}^0(\mathfrak{g})$. If the bounded operator $\mathrm{Op}(K)$ is invertible on $L^2(\mathfrak{g})$, then there exists $L \in \mathcal{F}^0(\mathfrak{g})$ such that*

$$L \star K = K \star L = \delta_0,$$

and each seminorm of L in $\mathcal{F}^0(\mathfrak{g})$ depends on a seminorm of K in $\mathcal{F}^0(\mathfrak{g})$ and the operator norm $\|\mathrm{Op}(L)\|$. If $K \in S^0(\mathfrak{g})$, then $L \in S^0(\mathfrak{g})$, and each seminorm of L in $S^0(\mathfrak{g})$ depends on a seminorm of K in $S^0(\mathfrak{g})$ and the operator norm $\|\mathrm{Op}(L)\|$.

Proof. We proceed by induction on the step d. If $d = 1$, the goup $\mathfrak{g}$ is Abelian and our hypothesis implies

$$|\widehat{K}(\xi)| \geq c > 0, \qquad \xi \in \mathfrak{g}^\star \setminus \{0\},$$

and it is easily checked that L defined by $\widehat{L} = 1/\widehat{K}$ satisfies the required properties.

Let $d > 0$ and assume that our claim holds for homogeneous groups of step strictly less than d. Let $\mathfrak{g}$ be a homogeneous group of step d. Then $\mathfrak{g}_0$ is a homogeneous group of step $d-1$. Let $K \in \mathcal{F}^0(\mathfrak{g})$ satisfy the hypothesis of the theorem. Then, by Lemma 5.21 and Corollary 5.22, $K^0 \in \mathcal{F}^0(\mathfrak{g}_0)$ and $\mathrm{Op}(K^0)$ is invertible. Let $R = K \star V$, where $V \in S^{1/2}(\mathfrak{g})$ has been defined by (5.17). By Theorem 5.16, $R \in \mathcal{F}^{1/2}(\mathfrak{g})$. Then,

$$\mathrm{Op}(R^\lambda) - \mathrm{Op}(R^0) = \mathrm{Op}(M_\lambda),$$

where $\widehat{M_\lambda}(\eta) = \widehat{R}(\eta, \lambda) - \widehat{R}(\eta, 0)$. Since $R \in \mathcal{F}^{1/2}(\mathfrak{g})$, and

$$D^\alpha_\eta \widehat{M_\lambda}(\eta) = \int_0^1 D_\lambda D^\alpha_\eta \widehat{R}(\eta, t\lambda)\, dt,$$

we see that

$$\begin{aligned} |D^\alpha_\eta \widehat{M_\lambda}(\eta)| &\leq C_\alpha |\lambda|^{p_d} \int_0^1 (|\eta| + t^{\frac{1}{p_d}}|\lambda|)^{1/2-p_d-d(\alpha)}\, dt, \\ &\leq C_\alpha |\lambda|^{1/2}|\eta|^{-d(\alpha)} \int_0^1 t^{\frac{1}{2p_d}-1}\, dt = C'_\alpha |\lambda|^{1/2}|\eta|^{-d(\alpha)}, \end{aligned}$$

which shows that M_λ is a Calderón-Zygmund kernel and, by (6.1), $\|\mathrm{Op}(M^\lambda)\| \leq C|\lambda|^{1/2}$. By Corollary 5.23, the subspace of functions of the form $g = \mathrm{Op}(V^0)f$, where $f \in \mathcal{S}(\mathfrak{g}_0)$, is dense in $L^2(\mathfrak{g}_0)$, and

$$\|\mathrm{Op}(K^\lambda)g - \mathrm{Op}(K^0)g\| = \|\mathrm{Op}(M_\lambda)f\| \leq C|\lambda|^{1/2}\|f\| \leq C_1|\lambda|^{1/2}\|g\|,$$

which implies $\|\mathrm{Op}(K^\lambda) - \mathrm{Op}(K^0)\| \leq C_1|\lambda|^{1/2}$. Hence $\mathrm{Op}(K^\lambda)$ is also invertible if $|\lambda|$ is small enough, say $|\lambda| < 4\varepsilon$. Now, by Corollary refproj, $K^\lambda \in S^0(\mathfrak{g})$, so by induction hypothesis, for every $|\lambda| < 4\varepsilon$, there exists $S_\lambda \in S^0(\mathfrak{g}_0)$ such that

$$K^\lambda \star_0 S_\lambda = S_\lambda \star_0 K^\lambda = \delta_0.$$

Let

$$U = \{\lambda \in \mathfrak{z}^\star : \varepsilon/4 < |\lambda| < 4\varepsilon\}, \qquad W = \{\lambda \in \mathfrak{z}^\star : \varepsilon/3 < |\lambda| < 3\varepsilon\}.$$

Let $\varphi \in C_c^\infty(U)$ be equal to 1 on W. Let $a(\eta,\lambda) = \widehat{K}(\eta,\lambda)$, $s(\eta,\lambda) = \widehat{S_\lambda}(\eta)$, $p(\eta,\lambda) = \varphi(\lambda)^2 s(\eta,\lambda)$. The family $\{K^\lambda\}_{\lambda\in U}$ is smooth in λ, hence, by Lemma 4.5, so is the family $\{S_\lambda\}_{\lambda\in U}$. Therefore, $p \in \widehat{S}_0^0(\mathfrak{g}^\star)$, and

$$p\#_0 a(\eta,\lambda) = \varphi(\lambda)^2, \qquad \eta \in \mathfrak{g}_0^\star,\ \lambda \in \mathfrak{z}^\star.$$

Consequently, by Proposition 3.9,

$$p\# a(\eta,\lambda) = \varphi(\lambda)^2 - q(\eta,\lambda), \qquad \eta \in \mathfrak{g}_0^\star,\ \lambda \in \mathfrak{z}^\star,$$

where $q \in \widehat{S}_0^{-p_d}(\mathfrak{g}^\star)$. By Lemma 4.2, for every positive integer N, there exists $p_N \in \widehat{S}_0^0(\mathfrak{g}^\star)$ and $q_N \in \widehat{S}_0^{-2^N p_d}(\mathfrak{g}^\star)$ such that

$$p_N\# a(\eta,\lambda) = \varphi(\lambda)^{2^{N+1}} - q_N(\eta,\lambda). \tag{6.3}$$

Let $\psi \in C_c^\infty(\mathfrak{z}^\star \setminus \{0\})$ be equal to 1 on U. Let $\widehat{K_1} = \widehat{K}\psi$. Then $K_1 \in \widehat{S}_0^0(\mathfrak{g}^\star)$, and $\mathrm{Op}(K) = \mathrm{Op}(K_1)$ on $\mathcal{S}_U(\mathfrak{g})$ which is invariant under $\mathrm{Op}(K_1)$. Let $b = \widehat{L}$. By Lemma 4.6, the linear mapping

$$\mathrm{Op}(L) : \mathcal{S}_U(\mathfrak{g}) \to \mathcal{S}_U(\mathfrak{g}) \tag{6.4}$$

is continuous so, for every $N_1 \in \boldsymbol{N}$, there exists $N_2 \in \boldsymbol{N}$ such that, for $N \geq N_2$,

$$|D_\eta^\alpha D_\lambda^\beta \Big(q_N\# b\Big)(\eta,\lambda)| \leq C_{\alpha\beta}(1+|\eta|+|\lambda|)^{-N_1}. \qquad d(\alpha), d(\beta) \leq N_1,$$

Since $\varphi(\lambda) = 1$, for $\lambda \in W$, (6.3) implies

$$b(\eta,\lambda) = p_N(\eta,\lambda) + q_N\# b(\eta,\lambda), \qquad (\eta,\lambda) \in \mathfrak{g}_0^\star \times W, \tag{6.5}$$

where N can be taken arbitrarily large. Since N_1 can also be taken as large as we please and $p_N \in \widehat{S}_0^0(\mathfrak{g}^\star)$, it follows that b coincides with a smooth function on $\mathfrak{g}^\star \times W$, and

$$|D_\eta^\alpha D_\lambda^\beta b(\eta,\lambda)| \leq C_{\alpha\beta}(1+|\eta|+|\lambda|)^{-d(\alpha)}, \qquad (\eta,\lambda) \in \mathfrak{g}_0^\star \times W. \tag{6.6}$$

For every $\psi \in C_c^\infty(W)$ equal to 1 on a neighbourhood of V, where

$$V = \{\lambda \in \mathfrak{z}^\star : \varepsilon/2 < |\lambda| < 2\varepsilon\},$$

$b\psi \in \widehat{S}_0^0(\mathfrak{g}^\star)$, which, by Lemma 4.3, yields an improvement in estimates (6.6), albeit on a smaller set:

$$|D_\eta^\alpha D_\lambda^\beta b(\eta,\lambda)| \leq C_{\alpha\beta}(1+|\eta|+|\lambda|)^{-d(\alpha)-d(\beta)}, \qquad (\eta,\lambda) \in \mathfrak{g}_0^\star \times V. \tag{6.7}$$

We claim that each of the constants $C_{\alpha\beta}$ depends on a Calderón-Zygmund seminorm of K and the operator norm of $\mathrm{Op}(L)$. In fact, by induction hypothesis, if λ stays in a compact set, each of the $S^0(\mathfrak{g}_0)$-seminorms of K_λ^{-1} depends on a Calderón-Zygmund seminorm of K and the operator norm of $\mathrm{Op}(L)$ uniformly in λ. Consequently, the $\widehat{S}_0^0(\mathfrak{g})$-seminorms of p and, by Propositions 3.9 and 3.4, the seminorms of q_N have similar dependence. Finally, our claim is completed by Lemma 4.3 which takes care of the seminorms of b.

The Calderón-Zygmund seminorms do not change if we replace K with

$$\langle K_n, f\rangle = \langle K, f\circ \delta_{2^n}\rangle, \qquad n \in \boldsymbol{Z}.$$

Furthermore, for every $n \in \boldsymbol{Z}$,

$$\mathrm{Op}(L_n)\,\mathrm{Op}(K_n) = \mathrm{Op}(K_n)\,\mathrm{Op}(L_n) = I,$$

and the operator norms of $\mathrm{Op}(L_n)$ are all equal to that of $\mathrm{Op}(L)$. Therefore, $b_n = \widehat{L_n}$ are smooth functions on $\mathfrak{g}_0^\star \times V$, and satisfy (6.7) uniformly, which easily translates into $\widehat{L} \in C^\infty(\{(\eta,\lambda) \in \mathfrak{g}^\star : \lambda \neq 0\})$ and the estimates

$$|D_\eta^\alpha D_\lambda^\beta \widehat{L}(\eta,\lambda)| \le C_{\alpha\beta}(1+|\eta|+|\lambda|)^{-d(\alpha)-d(\beta)}, \qquad 2^{n-1}\varepsilon < |\lambda| < 2^{n+1}\varepsilon, \tag{6.8}$$

for every α,β, with the same constants $C_{\alpha\beta}$ as in (6.7). By Proposition 5.15, L is a Calderón-Zygmund kernel, which is our assertion. The dependence of the constants has already been discussed.

To complete the proof we need to consider also $K \in S^0(\mathfrak{g})$ in which case we have to prove that $L \in S^0(\mathfrak{g})$. We already know that $L \in \mathcal{F}^0(\mathfrak{g})$, so it will suffice to show that

$$|D_\eta^\alpha D_\lambda^\beta \widehat{L}(\eta,\lambda)| \le C_{\alpha\beta}(1+|\eta|+|\lambda|)^{-d(\alpha)-d(\beta)}, \qquad |\lambda| < 1. \tag{6.9}$$

Recall that $a = \widehat{K}$. Let $\varphi \in C_c^\infty(\mathfrak{z}^\star)$ be equal to 1, for $|\lambda| < 1$. By Lemma 4.1, there exists $p \in \widehat{S}_0^0(\mathfrak{g}^\star)$ and $q \in \widehat{S}_0^{-p_d}(\mathfrak{g}^\star)$ such that

$$p \# a = \varphi^2 - q.$$

Now, the argument leading to the estimate (6.7) can be repeated to yield (6.9) and the expected dependence of seminorms. □

Remark 6.10. The seminorms in $S^0(\mathfrak{g})$ are not invariant under dilations. Accordingly, in the last part of the proof such invariance is neither required nor used.

Corollary 6.11. *For every $m \in \boldsymbol{R}$, there exist kernels $V \in S^m(\mathfrak{g})$ and $W \in S^{-m}(\mathfrak{g})$ such that*

$$V \star W = W \star V = \delta_0.$$

Proof. It is enough to consider the case $0 < m < 1$. Let $P, R = P^{-1}$ and $V, W = V^{-1}$ be as defined in Section 5. Then,

$$P = V + k,$$

where $k \in \boldsymbol{F}(m)$. Thus,

$$R \star V = \delta_0 - R \star k,$$

where $R \star k \in \mathcal{F}^0(\mathfrak{g})$, and

$$W \star P = \delta_0 + W \star k,$$

where $W \star k \in L^1(\mathfrak{g})$. Since $R \in \mathcal{F}(-m)$, we see that $K = R \star V \in \mathcal{F}^0(\mathfrak{g})$ and $\mathrm{Op}(K)$ has an inverse on $L^2(\mathfrak{g})$, namely $\mathrm{Op}(W \star P)$. By Theorem 6.2, $W \star P \in \mathcal{F}^0(\mathfrak{g})$, so

$$W = (W \star P) \star R \in \mathcal{F}^{-m}(\mathfrak{g}).$$

However, we know that $\widehat{W} \in C^\infty(\mathfrak{g}^\star)$ (Proposition 5.19), which, by Corollary 5.12, is enough to conclude that $W \in S^{-m}(\mathfrak{g})$. □

References

[1] A. P. Calderón, A. Zygmund, On singular integral operators, *Amer. J. Math.*, 78 (1956), 289-309,

[2] A. P. Calderón, A. Zygmund, Algebras of certain singular integral operators, *Amer. J. Math.*, 78 (1956), 310-320,

[3] M. Christ, Inversion in some algebras of singular integral operators, *Rev. Mat. Iberoam.* 4 (1988), 219–225,

[4] M. Christ, On the regularity of inverses of singular integral operators, *Rev. Mat. Iberoam.* 4 (1988), 219-225,

[5] M. Christ, Regularity properties of $\bar{\partial}_b$ equation on weakly pseudoconvex CR manifolds of dimension 3, *J. Amer. Math. Soc.* 1 (1988), 587-646,

[6] M. Christ, D. Geller, Singular integral characterization of Hardy spaces on homogeneous groups, *Duke Math. J.* 51 (1984), 547-598,

[7] S. Coré, D. Geller, Convolution on homogeneous groups, *J. Math. Anal. Appl.* 370 (2010), 472-485

[8] L. Corwin, F.P. Greenleaf, *Representations of nilpotent Lie groups and their applications. Part 1: Basic theorey and examples*, Cambridge University Press, Cambridge 1990,

[9] M. Duflo, Représentations de semi-groupes de mesures sur un groupe localement compact, *Ann. Inst. Fourier, Grenoble*, 28.2 (1978), 225-249,

[10] J. Dziubański, P. Głowacki, Sobolev spaces related to Schrödinger operators with polynomial potentials, *Math. Z.* 262 (2009), 881-894,

[11] G.B. Folland and E.M. Stein, *Hardy spaces on homogeneous groups,* Princeton University Press, Princeton NJ 1982,

[12] P. Głowacki, The Melin calculus for general homogeneous groups, *Arkiv mat.*, 45 (2007), 31-48;

[13] P. Głowacki, An inversion problem for singular integral operators on homogeneous groups, *Studia Math.* 87 (1987), 53-69,

[14] P. Głowacki, Invertibility of convolution operators on homogeneous groups, *Revista Mat. Iberoam.*, 28 (2012), 141-156,

[15] P. Głowacki, Stable semigroups of measures as commutative approximate identities on non-graded homogeneous groups, *Inv. Math.*, **83** (1986), 557–582,

[16] L. Hörmander, The analysis of linear partial differential operators *vol. III, Berlin - Heidelberg - New York - Tokyo* 1983,

[17] G. A. Hunt, Semigroups of measures on Lie groups, *Trans. Amer. Math. Soc.* 81 (1956), 264-293,

[18] A. Janssen, Charakterisierung stetiger Faltungshalbgruppen durch das Lévy-Mass, *Math. Ann.* 246 (1980), 233-240,

[19] D. Manchon, Formule de Weyl pour les groupes de Lie nilpotents, J. reine angew. Math. 418 (1991), 77-129,

[20] A. Melin, Parametrix constructions for right-invariant differential operators on nilpotent Lie groups, Ann. Glob. Anal. Geom. 1 (1983), 79-130;

[21] F. Ricci, Calderón-Zygmund kernels on nilpotent Lie groups, in *Harmonic analysis* (1981 Proceedings, ed. F. Ricci and G. Weiss), Lecture Notes in Math. #908, Springer-Verlag,

[22] E. M. Stein, Harmonic analysis, *Princeton University Press*, Princeton NJ 1993,

Institute of Mathematics, University of Wrocław, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland

E-mail address: glowacki@math.uni.wroc.pl