INVERTIBILITY OF CONVOLUTION OPERATORS ON HOMOGENEOUS GROUPS

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Abstract

We say that a tempered distribution A belongs to the class $S^m(\mathfrak{g})$ on a homogeneous Lie algebra $\mathfrak{g}$ if its Abelian Fourier transform $a = \widehat{A}$ is a smooth function on the dual $\mathfrak{g}^\star$ and satisfies the estimates

$$|D^\alpha a(\xi)| \le C_\alpha (1+|\xi|)^{m-|\alpha|}.$$

Let $A \in S^0(\mathfrak{g})$. Then the operator $f \mapsto f \star \widetilde{A}(x)$ is bounded on $L^2(\mathfrak{g})$. Suppose that the operator is invertible and denote by B the convolution kernel of its inverse. We show that B belongs to the class $S^0(\mathfrak{g})$ as well. As a corollary we generalize Melin's theorem on the parametrix construction for Rockland operators.

In a former paper [**10**] we describe a calculus of a class of convolution operators on a nilpotent homogeneous group G with the Lie algebra $\mathfrak{g}$. These operators are distinguished by the conditions imposed on the Abelian Fourier transforms of their kernels similar to those required from the L^p-multipliers on $\mathbf{R}^n$. More specifically, a tempered distribution A belongs to the class $S^m(G) = S^m(\mathfrak{g})$ if its Fourier transform $a = \widehat{A}$ is a smooth function on the dual to the Lie algebra $\mathfrak{g}^\star$ and satisfies the estimates

$$|D^\alpha a(\xi)| \le C_\alpha (1+|\xi|)^{m-|\alpha|}, \qquad \xi \in \mathfrak{g}^\star.$$

In [**10**] we follow and extend to the setting of a general homogeneous group the ideas of Melin [**14**] who first introduced such a calculus on the subclass of stratified groups. The classes $S^m(\mathfrak{g})$ of convolution operators have the expected properties of composition and boundedness (see Propositions 1.1 and 1.2 below) which is a generalization of the results of Melin [**14**]. However, a complete calculus should also deal with the problem of invertibility. The aim of the present paper is to fill the gap.

Suppose that $A \in S^0(\mathfrak{g})$. Then, by the boundedness theorem (see Proposition 1.2 below), the operator

$$f \mapsto f \star \widetilde{A}(x) = \int_{\mathfrak{g}} f(xy)A(y)\,dy$$

defined initially on the Schwartz class functions extends uniquely to a bounded operator on $L^2(\mathfrak{g})$. Furthermore, suppose that the operator $f \mapsto f \star \widetilde{A}$ is invertible on $L^2(\mathfrak{g})$ and denote by B the convolution kernel of its inverse. We show here that under these circumstances B belongs to the class $S^0(\mathfrak{g})$ as well. This is done by replacing Melin's techniques of parametrix construction involving the more refined classes $S^{m,s}(\mathfrak{g}) \subset S^m(\mathfrak{g})$ of convolution operators by the calculus of less restrictive classes $S^m_0(\mathfrak{g})$, where no estimates in the central directions are required.

2000 *Mathematics Subject Classification* 22E30 (primary), 47G30 (secondary).

Let us remark that the described result can be also looked upon as a close analogue of the theorem on the inversion of singular integrals, see [**9**] and Christ-Geller [**3**].

By using auxiliary convolution operators, namely accretive homogeneous kernels P^m smooth away from the origin, we construct "elliptic" operators V_1^m of order $m > 0$ and get inversion results for classes $S^m(\mathfrak{g})$ for all $m > 0$, which enables us to generalize Melin's theorem on the parametrix construction for Rockland operators. At the same time, however, we present a direct parametrix construction for Rockland operators which avoids the machinery of Melin and also that of the present paper and depends only on well-known properties of Rockland operators as derived in Folland-Stein [**7**] and the calculus of [**10**].

We believe that the presented symbolic calculus may be a step towards a more comprehensive pseudodifferential calculus on nilpotent Lie groups parallel to that of Christ-Geller-Głowacki-Polin [**4**].

The author is grateful to M. Christ and F. Ricci for their helpful remarks on the subject of the present paper.

1. *Symbolic calculus.*

Let $\mathfrak{g}$ be a nilpotent Lie algebra endowed with a family of diltations $\{\delta_t\}_{t>0}$. We identify $\mathfrak{g}$ with the corresponding nilpotent Lie group by means of the exponential map. Let

$$1 = p_1 < p_2 < \cdots < p_d$$

be the exponents of homogeneity of the dilations. Let $|\cdot|$ be a homogenous norm on V. Let

$$\mathfrak{g}_j = \{x \in \mathfrak{g} : tx = t^{p_j} \cdot x\}, \qquad 1 \leq j \leq d.$$

Denote by $Q = \sum_k \dim \mathfrak{g}_k \cdot p_k$ the homogeneous dimension of $\mathfrak{g}$.

Let $|\cdot|$ be a homogeneous norm on $\mathfrak{g}$. Let

$$\rho(x) = 1 + |x|.$$

A similar notation will be applied for the dual space $\mathfrak{g}^\star$.

In expressions like D^α or x^α we shall use multiindices

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d),$$

where

$$\alpha_k = (\alpha_{k1}, \alpha_{k1}, \ldots, \alpha_{kn_k}),$$

are themselves multiindices with positive integer entries corresponding to the spaces $\mathfrak{g}_k$ or $\mathfrak{g}_k^\star$. The homogeneous length of α is defined by

$$|\alpha| = \sum_{k=1}^{d} |\alpha_k|, \qquad |\alpha_k| = \dim \mathfrak{g}_k \cdot p_k.$$

As usual we denote by $\mathcal{S}(\mathfrak{g})$ or $\mathcal{S}(\mathfrak{g}^\star)$ the Schwartz classes of smooth and rapidly vanishing functions. The Fourier transform

$$\widehat{f}(\xi) = \int_{\mathfrak{g}} f(x)\, e^{-i\langle \xi, x\rangle}\, dx$$

maps $\mathcal{S}(\mathfrak{g})$ onto $\mathcal{S}(\mathfrak{g}^\star)$ and extends to tempered distributions on $\mathfrak{g}$. Let

$$\|f\|^2 = \int_{\mathfrak{g}} |f(x)|^2\, dx, \qquad f \in L^2(\mathfrak{g}).$$

A similar notation will be applied to $f \in L^2(\mathfrak{g}^\star)$, where the Lebesgue measure $d\xi$ on $\mathfrak{g}^\star$ is normalized so that

$$\int_{\mathfrak{g}} |f(x)|^2\, dx = \int_{\mathfrak{g}^\star} |\widehat{f}(x)|^2\, d\xi.$$

The algebra of bounded linear operators on $L^2(\mathfrak{g})$ will be denoted by $\mathcal{B}(L^2(\mathfrak{g}))$.

For a tempered distribution A on $\mathfrak{g}$, we write

$$\mathrm{Op}(A)f(x) = f \star \widetilde{A}(x) = \int_{\mathfrak{g}} f(xy)A(dy), \qquad f \in \mathcal{S}(\mathfrak{g}).$$

Let $m \in \mathbf{R}$. By $S^m(\mathfrak{g}) = S^m(\mathfrak{g}, \rho)$ we denote the class of all distributions $A \in \mathcal{S}'(\mathfrak{g})$ whose Fourier transforms $a = \widehat{A}$ are smooth and satisfy the estimates

$$|D^\alpha a(\xi)| \le C_\alpha \rho(\xi)^{m-|\alpha|}, \tag{1.1}$$

where $|\alpha|$ stands for the homogeneous length of a multiindex. Let us recall that $S^m(\mathfrak{g})$ is a Fréchet space with the family of norms

$$|a|_\alpha = \sup_{\xi \in \mathfrak{g}^\star} \rho(\xi)^{-m+|\alpha|} |D^\alpha a(\xi)|.$$

It is not hard to see that for every $\varphi \in C_c^\infty(\mathfrak{g})$ equal to 1 in a neighbourhood of 0 the distribution $(1-\varphi)A$ is a Schwartz class function. Thus

$$A = A_1 + F, \tag{1.2}$$

where A_1 is compactly supported and $F \in \mathcal{S}(\mathfrak{g})$.

It follows from (1.2) that for every $m \in \mathbf{R}$

$$\mathrm{Op}(A) : \mathcal{S}(\mathfrak{g}) \to \mathcal{S}(\mathfrak{g})$$

is a continuous mapping if $A \in S^m(\mathfrak{g})$. Therefore, it extends to a continuous mapping denoted by the same symbol of $\mathcal{S}'(\mathfrak{g})$. It is also clear that for $A \in S^m(\mathfrak{g})$ and $B \in S^n(\mathfrak{g})$ the convolution $A \star B$ makes sense and $\mathrm{Op}\,(A \star B) = \mathrm{Op}\,(A)\mathrm{Op}\,(B)$.

The following two propositions have been proved in [**10**].

Proposition 1.1. *If $A \in S^m(\mathfrak{g})$ and $B \in S^n(\mathfrak{g})$, then $A \star B \in S^{m+n}(\mathfrak{g})$ and the mapping*

$$S^m(\mathfrak{g}) \times S^n(\mathfrak{g}) \ni (A, B) \mapsto A \star B \in S^{m+n}(\mathfrak{g})$$

is continuous.

Proposition 1.2. *If $A \in S^0(\mathfrak{g})$, then $\mathrm{Op}(A)$ is bounded on $L^2(\mathfrak{g})$ and the mapping*

$$S^0(\mathfrak{g}) \ni A \mapsto \mathrm{Op}(A) \in \mathcal{B}(L^2(\mathfrak{g}))$$

is continuous.

Let $\mathfrak{z}$ be the central subalgebra corresponding to the largest eigenvalue of the dilations. We may assume that

$$\mathfrak{g} = \mathfrak{g}_0 \times \mathfrak{z}, \qquad \mathfrak{g}^\star = \mathfrak{g}_0^\star \times \mathfrak{z}^\star, \tag{1.3}$$

where $\mathfrak{g}_0$ may be identified with the quotient Lie algebra $\mathfrak{g}/\mathfrak{z}$. The multiplication law in $\mathfrak{g}$ can be expressed by

$$(x,t)(y,s) = (x \circ y, t+s+r(x,y)),$$

where $x \circ y$ is mutiplication in $\mathfrak{g}_0$. Here the variable in $\mathfrak{g}$ has been split in accordance with the given decomposition. In a similar way we also split the variable $\xi = (\eta, \lambda)$ in $\mathfrak{g}^\star$.

Let $m \in \mathbf{R}$. By $S_0^m(\mathfrak{g}^\star)$ we denote the class of all distributions $A \in \mathcal{S}'(\mathfrak{g})$ whose Fourier transforms $a = \widehat{A}$ are smooth in the variable η and satisfy the estimates

$$|D_\eta^\alpha a(\eta,\lambda)| \le C_\alpha \rho(\eta,\lambda)^{m-|\alpha|}. \tag{1.4}$$

Again, $S_0^m(\mathfrak{g})$ is a Fréchet space with the family of norms

$$|a|_\alpha = \sup_{(\eta,\lambda)\in\mathfrak{g}^\star} \rho(\eta,\lambda)^{-m+|\alpha|} |D_\eta^\alpha a(\eta,\lambda)|.$$

The following result has not been stated explicitely in [**10**] but follows by the argument given there.

Proposition 1.3. *If $A \in S_0^m(\mathfrak{g}^\star)$ and $B \in S_0^n(\mathfrak{g}^\star)$, then $A \star B \in S_0^{m+n}(\mathfrak{g}^\star)$ and the mapping*

$$S_0^m(\mathfrak{g}^\star) \times S_0^n(\mathfrak{g}^\star) \ni (A,B) \mapsto A \star B \in S_0^{m+n}(\mathfrak{g}^\star)$$

is continuous.

Let us introduce the following notation:

$$\widehat{f} \# \widehat{g}(\xi) = \widehat{f \star g}(\xi), \qquad \xi \in \mathfrak{g}^\star,$$

for $f, g \in \mathcal{S}(\mathfrak{g})$. Then, for every fixed $\lambda \in \mathfrak{z}^\star$,

$$a \# b(\eta,\lambda) = a(\cdot,\lambda) \#_\lambda b(\cdot,\lambda)(\eta), \tag{1.5}$$

where

$$\widehat{f} \#_\lambda \widehat{g}(\eta) = (f \star_\lambda g)\widehat{\ }(\eta), \qquad f \star_\lambda g(x) = \int_{\mathfrak{g}_0} f(x \circ y^{-1}) g(y)\, e^{i\langle r(x,y^{-1}),\lambda\rangle} dy$$

for $f, g \in \mathcal{S}(\mathfrak{g}_0)$. In particular, $f \star_0 g$ is the usual convolution on the quotient group $\mathfrak{g}_0$.

Let

$$T_{k_i} F(x) = x_{k_i} F(x), \qquad T_\alpha F(x) = x^\alpha F(x).$$

For a given multiindex γ, let

$$k(\gamma) = \max_{1\le k\le d} \{k : \gamma_k \ne 0\},$$

and

$$\mathcal{P}(\gamma) = \{\alpha : \alpha_k = 0,\ k \ge k(\gamma)\}.$$

Lemma 1.4. *Let $f, g \in \mathcal{S}(\mathfrak{g})$. Then for every γ,*

$$T_\gamma(f \star g) = T_\gamma f \star g + f \star T_\gamma g + \sum_{\alpha,\beta\in\mathcal{P}(\gamma), |\alpha|+|\beta|=|\gamma|} c_{\alpha\beta}^\gamma T_\alpha f \star T_\beta g.$$

By applying the Fourier transform, we obtain

$$D^\gamma(f\#g) = D^\gamma f\#g + f\#D^\gamma g + \sum_{\alpha,\beta\in\mathcal{P}(\gamma),|\alpha|+|\beta|=|\gamma|} c^\gamma_{\alpha\beta}\, D^\alpha f\#D^\beta g \tag{1.6}$$

for $f, g \in \mathcal{S}(\mathfrak{g}^\star)$.

LEMMA 1.5. *Let $A \in S^m(\mathfrak{g})$. If $B \in S_0^{-m}(\mathfrak{g})$ is the inverse of A, that is,*

$$A \star B = B \star A = \delta_0,$$

then $B \in S^m(\mathfrak{g})$.

Proof. Let $a = \widehat{A}$, $b = \widehat{B}$. By (1.6),

$$0 = D^{\gamma_d}(a\#b) = D^{\gamma_d}a\#b + a\#D^{\gamma_d}b + \sum c^\gamma_{\alpha\beta} D^\alpha a\#D^\beta b,$$

where the summation extends over α, β such that

$$|\alpha| + |\beta| = |\gamma_d|,\ \ |\alpha_d|, |\beta_d| < |\gamma_d|$$

and every multiindex is split as $\alpha = (\alpha', \alpha_d)$, α_d being the part correspodning to $\mathfrak{g}_d^\star$. Therefore,

$$D^{\gamma_d}b = -b\#D^{\gamma_d}a\#b + \sum c^\gamma_{\alpha\beta} b\#D^\alpha a\#D^\beta b,$$

where the sybol on the right-hand side belongs to $\widehat{S}_0^{-m-|\gamma_d|}$ provided that $b \in \widehat{S}_0^{-m-\kappa}$ for $\kappa < |\gamma_d|$. By induction, $D^{\gamma_d}b \in \widehat{S}_0^{-m-|\gamma_d|}(\mathfrak{g})$, which is our assertion. □

Let $A_j \in S_0^{m_j}(\mathfrak{g}^\star)$, where $m_j \searrow -\infty$. Then there exists a distribution $A \in S_0^{m_1}(\mathfrak{g}^\star)$ such that

$$A - \sum_{j=1}^N A_j \in S_0^{m_{N+1}}(\mathfrak{g}^\star)$$

for every $N \in \mathbf{N}$. The distribution A is unique modulo the class

$$S_0^{-\infty}(\mathfrak{g}^\star) = \bigcap_{n<0} S_0^n(\mathfrak{g}^\star).$$

We shall write

$$A \approx \sum_{j=1}^\infty A_j, \tag{1.7}$$

and call the distribution A the asymptotic sum of the series $\sum A_j$ (cf., e.g., Hörmander [**13**], Proposition 18.1.3).

We say that $A \in S^m(\mathfrak{g})$, where $m \geq 0$, has a parametrix $B \in S^{-m}(\mathfrak{g})$ if

$$B \star A - \delta_0 \in \mathcal{S}(\mathfrak{g}), \qquad A \star B - \delta_0 \in \mathcal{S}(\mathfrak{g}),$$

where δ_0 stands for the Dirac delta at 0. If B_1 is a left-parametrix and B_2 a right one, then it is easy to see that $B_1 = B_2$ modulo the Schwartz class functions so both B_1 and B_2 are parametrices. In particular, if A is symmetric, then either of the conditions implies the other one.

2. *Sobolev spaces*

We shall say that a tempered distribution T is a *regular kernel of order* $r \in \mathbf{R}$, if it is homogeneous of degree $-Q-r$ and smooth away from the origin. A symmetric distribution T is said to be *accretive*, if

$$\langle T, f\rangle \geq 0$$

for real $f \in C_c^\infty(\mathfrak{g})$ which attain their maximal value at 0. Such a T is an infinitesimal generator of a continuous semigroup of subprobability measures μ_t. By the Hunt theory (see, eg., Duflo [**5**]), $T = \mathrm{Op}\,(T)$ is a positive selfadjoint operator on $L^2(\mathfrak{g})$ with $\mathcal{S}(\mathfrak{g})$ as its core domain and for every $0 < m < 1$

$$\mathrm{Op}\,(T)^m = \mathrm{Op}\,(T^m), \qquad \langle T^m, f\rangle = \frac{1}{\Gamma(-m)}\int_0^\infty t^{-1-m}\langle \delta_0 - \mu_t, f\rangle\, dt,$$

where the distribution T^m is also accretive.

Let T be a fixed symmetric accretive regular kernel of order $0 < m \leq 1$. Then there exists a symmetric nonnegative function $\Omega \in C^\infty(\mathfrak{g} \setminus \{0\})$ which is homogeneous of degree 0 such that

$$\langle T, f\rangle = cf(0) + \lim_{\varepsilon\to}\int_{|x|\geq\varepsilon}\Big(f(0) - f(x)\Big)\frac{\Omega(x)\,dx}{|x|^{Q+m}},$$

where $c \geq 0$. If $c = 0$, T is an infinitesimal generator of a continuous semigroup of probability measures with smooth densities. For every $0 < a < 1$, T^a is also a symmetric regular kernel of order am.

Let

$$\langle P, f\rangle = \lim_{\varepsilon\to}\int_{|x|\geq\varepsilon}\frac{f(0) - f(x))\,dx}{|x|^{Q+1}}$$

be a fixed symmetric accretive distribution of order 1. Let us warn the reader that the distributions P^m do not belong to any of the classes $S^m(\mathfrak{g})$ as they do not vanish rapidly at infinity which is a certain technical complication. That is why we introduce the truncated kernels

$$V_0 = I, \qquad V_m = \varphi P^m, \qquad m > 0,$$

where φ is a symmetric nonnegative $[0,1]$-valued smooth function with compact support and equal to 1 on the unit ball. Thus defined $V_m \in S^m(\mathfrak{g})$ is also accretive and it differs from P^m by a finite measure. Therefore, for every $0 < m \leq 1$, there exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$C_1\|(I + \mathrm{Op}\,(P))^m f\| \leq \|(I + \mathrm{Op}\,(V_m))f\| \leq C_2\|(I + \mathrm{Op}\,(P))^m f\|, \tag{2.1}$$

for $f \in \mathcal{S}(\mathfrak{g})$.

Proposition 2.1. *For every* $0 < m \leq 1$, *there exists a constant* $C_m > 0$ *such that*

$$\|f \star V_m\| \geq C_m\|f\|, \qquad f \in \mathcal{S}(\mathfrak{g}).$$

Proof. In fact, let $f \in \mathcal{S}(\mathfrak{g})$ and $F = \widetilde{f} \star f$. Then

$$\begin{aligned}\langle f \star V_m, f\rangle &= \langle T, F\rangle \\ &= \lim_{\varepsilon\to} \int_{\varepsilon\le|x|\le 1} \Big(F(0) - \varphi(x)F(x)\Big)\frac{\Omega_m(x)\,dx}{|x|^{Q+1}} + F(0)\int_{|x|\ge 1}\frac{\Omega_m(x)\,dx}{|x|^{Q+1}} \\ &\ge C_m^2 F(0) = C_m^2\|f\|^2\end{aligned}$$

since the first integral is nonnegative. □

It follows from (2.1) and Proposition 2.1 that there exist new constants $C_1 > 0$ and $C_2 > 0$ such that

$$C_1\|(I + \mathrm{Op}\,(P))^m f\| \le \|\mathrm{Op}\,(V_m)f\| \le C_2\|(I + \mathrm{Op}\,(P))^m f\|, \tag{2.2}$$

for $f \in \mathcal{S}(\mathfrak{g})$ and $0 \le m \le 1$.

Recall from [**8**] that P is *maximal*, that is, for every regular symmetric kernel T of arbitrary order $m > 0$ there exists a constant $C > 0$ such that

$$\|f \star \widetilde{T}\| \le C\|f \star P^m f\|, \qquad f \in \mathcal{S}(\mathfrak{g}). \tag{2.3}$$

We introduce a scale of Sobolev spaces. For every $m \in \mathbf{R}$

$$H(m) = \{f \in L^2(\mathfrak{g}) : (I + \mathrm{Op}\,(P))^m f \in L^2(\mathfrak{g})\}$$

with the usual norm $\|f\|_{(m)} = \|(I + \|\mathrm{Op}\,(P))^m f\|_2$. The dual space to $H(m)$ can be identified with $H(-m)$. By (2.2), the norms defined by V_m for $0 < m \le 1$ are equivalent. It follows that for every $0 \le m \le 1$

$$V_m : H(m) \to H(0)$$

is an isomorphism.

3. *Main step*

Here comes a preliminary version of our theorem.

Proposition 3.1. *Let $0 \le m \le 1$. Let $A = A^\star \in S^m(\mathfrak{g})$ and let $\mathrm{Op}\,(A)$: $H(m) \to H(0)$ be an isomorphism. If $A\star V_m = V_m\star A$, then there exists $B \in S^{-m}(\mathfrak{g})$ such that*

$$A \star B = B \star A = \delta_0.$$

In particular $\mathrm{Op}\,(B) = \mathrm{Op}\,(A)^{-1}$.

By hypothesis, A is invertible in $\mathcal{B}(L^2(\mathfrak{g}))$. There exists a symmetric distribution B such that

$$\mathrm{Op}\,(A)^{-1}f = f \star B, \qquad f \in \mathcal{S}(\mathfrak{g}).$$

We have to show that $B \in S^{-m}(\mathfrak{g})$.

Let $\mathcal{S}_1(\mathfrak{g})$ denote the subspace of $\mathcal{S}(\mathfrak{g})$ consisting of those functions whose Fourier transform is supported where $1 \le |\lambda| \le 2$. Note that this subspace is invariant under convolutions.

Lemma 3.2. $\mathrm{Op}\,(B)$ *maps continuously $\mathcal{S}(\mathfrak{g})$ into $\mathcal{S}(\mathfrak{g})$. The same applies to the invariant space $\mathcal{S}_1(\mathfrak{g})$.*

Proof. Being a convolution operator bounded on $L^2(\mathfrak{g})$, $\mathrm{Op}\,(B)$ commutes with right-invariant vector fields Y and hence maps $\mathcal{S}(\mathfrak{g})$ into $L^2(\mathfrak{g})\cap C^\infty(\mathfrak{g})$. Therefore, by Lemma 1.4,

$$\begin{aligned} T_\gamma \mathrm{Op}\,(B) &= \mathrm{Op}\,(B)T_\gamma + \mathrm{Op}\,(B)[T_\gamma, \mathrm{Op}\,(A)]\mathrm{Op}\,(B) \\ &= \mathrm{Op}\,(B)T_\gamma + \mathrm{Op}\,(B)\mathrm{Op}\,(A_\gamma)\mathrm{Op}\,(B) \\ &+ \sum_{\alpha,\beta\in\mathcal{P}(\gamma),|\alpha|+|\beta|=|\gamma|} c_{\alpha\beta}\cdot \mathrm{Op}\,(B)\mathrm{Op}\,(A_\alpha)T_\beta \mathrm{Op}\,(B), \end{aligned} \tag{3.1}$$

where $A_\alpha = T_\alpha A$. Note that $A_\alpha \in S^{m-|\alpha|}\subset S^0$ so, by Proposition 1.2, $\mathrm{Op}\,(A_\alpha)$ is bounded on $L^2(\mathfrak{g})$. By induction it follows that $\mathrm{Op}\,(B)$ maps $\mathcal{S}(\mathfrak{g})$ into the space of functions vanishing rapidly at infinity. Since $\mathcal{S}(\mathfrak{g})$ is invariant under $\mathrm{Op}\,(B)$, the operators $\mathrm{Op}\,(A)$ and $\mathrm{Op}\,(B) = \mathrm{Op}\,(A)^{-1}$ are isomorphisms of $\mathcal{S}(\mathfrak{g})$ and $\mathcal{S}_1(\mathfrak{g})$. □

For $n\in\mathbf{Z}$, let

$$\langle A_n, f\rangle = 2^{-nm}\int_{\mathfrak{g}} f(2^n x)\,A(dx), \qquad \langle B_n, f\rangle = 2^{nm}\int_{\mathfrak{g}} f(2^n x)\,B(dx).$$

Corollary 3.3. *The operators* $\mathrm{Op}\,(B_n)$ *are equicontinuous on* $\mathcal{S}_1(\mathfrak{g})$.

Proof. By Proposition 1.2, the mapping

$$S^m(\mathfrak{g})\ni A\to \mathrm{Op}\,(B)\in \mathcal{B}(L^2(\mathfrak{g}))$$

is continuous. Since the family $\{A_n\}$ is bounded in $S^m(\mathfrak{g})$ so is $\{\mathrm{Op}\,(B_n)\}$ in $\mathcal{B}(L^2(\mathfrak{g}))$. Hence our assertion follows by induction using (3.1). □

Let $a=\widehat{A}$, and let

$$\widehat{A_\lambda}(\eta) = a_\lambda(\eta) = a(\eta,\lambda), \qquad \lambda\in\mathfrak{z}^\star.$$

Lemma 3.4. *For every* $f\in\mathcal{S}(\mathfrak{g}_0^\star)$ *the function*

$$\lambda\to \|f\#_\lambda a_\lambda\|^2$$

is continuous.

Proof. Let $0<h\in\mathcal{S}(\mathfrak{z}^\star)$ and $h(0)=1$. Then $F=(f\otimes h)\#a\in\mathcal{S}(\mathfrak{g}^\star)$ and

$$\lambda\to\int_{\mathfrak{g}_0^\star}|F(\eta,\lambda)|^2 d\eta = |h(\lambda)|^2\|f\#_\lambda a_\lambda\|^2$$

is continuous, which implies our claim. □

From now on we shall proceed by induction. The assertion is obviously true in the Abelian case. Let us assume that it holds for $\mathfrak{g}_0=\mathfrak{g}/\mathfrak{z}$.

Lemma 3.5. *The distribution* A_0 *satisfies the hypothesis of the theorem on* $\mathfrak{g}_0$.

Proof. Observe that under the remaining assumptions of Proposition 3.1 the condition that $\mathrm{Op}\,(A): H(m)\to H(0)$ is an isomorphism is equivalent to the estimate

$$\|f\star A\|\geq C\|f\star V_m\|, \qquad f\in\mathcal{S}(\mathfrak{g}).$$

Now, since $A \star V_m = V_m \star A$, we also have

$$A_0 \star (V_m)_0 = (V_m)_0 \star A,$$

where $(V_m)_0$ is the counterpart of V_m on $\mathfrak{g}_0$. Furthermore, we have

$$\|f \star A\| \geq C\|f \star V_m\|$$

so, by Lemma 3.4,

$$\|f_0 \star A_0\| \geq C\|f_0 \star (V_m)_0\|, \qquad f \in \mathcal{S}(\mathfrak{g}),$$

which implies

$$\|f \star A_0\| \geq C\|f \star (V_m)_0\|, \qquad f \in \mathcal{S}(\mathfrak{g}_0).$$

□

Let $b = \widehat{B}$ and $b_n = \widehat{B_n}$. Of course, $b_n \in \mathcal{S}'(\mathfrak{g}^\star)$.

Lemma 3.6. *There exist $p \in \widehat{S}_0^{-m}(\mathfrak{g}^\star)$ and $q \in \mathcal{S}(\mathfrak{g}^\star)$ such that*

$$p\#a(\eta,\lambda) = 1 - q(\eta,\lambda), \qquad 1 \leq \lambda \leq 2. \tag{3.2}$$

Proof. Let $u \in C_c^\infty([0,\infty)$ be equal to 1 in a neighbourhood of $[0,1]$ and supported in $[0,2)$. Then

$$\psi(\eta,\lambda) = u\Big(\frac{\rho(0,\lambda)}{\rho(\eta,0)}\Big)$$

is an element of $\widehat{S}^0(\mathfrak{g}^\star)$. By Lemma 3.5 and the induction hypothesis, there exists $b_0 \in \widehat{S}^{-m}(\mathfrak{g}^\star)$ on a such that

$$b_0\#_0 a_0 = 1.$$

Let

$$p(\eta,\lambda) = \psi(\eta,\lambda)b_0(\eta).$$

Then $p \in \widehat{S}^{-m}(\mathfrak{g}^\star)$ and

$$\begin{aligned} p\#a(\eta,\lambda) &= p\#(a-a_0)(\eta,\lambda) + b_0\#_0 a_0(\eta) + (1-\psi)(\cdot,\lambda)b_0\#_0 a_0(\eta) \\ &= 1 - q_0(\eta,\lambda), \end{aligned}$$

where for every $\varphi \in C_c^\infty(\mathfrak{z}^\star)$, $\varphi(\lambda)q_0(\eta,\lambda)$ is in $\widehat{S}_0^{-1}(\mathfrak{g}^\star)$. Therefore we take $\varphi \in C_c^\infty(\mathfrak{z}^\star)$ which equals 1 where $1 \leq |\lambda| \leq 2$ and modify p_0 and q_0 by letting

$$p_1(\eta,\lambda) = p_0(\eta,\lambda)\varphi(\lambda), \qquad q_1(\eta,\lambda) = q_0(\eta,\lambda)\varphi(\lambda).$$

Now, $p_1 \in \widehat{S}_0^{-m}(\mathfrak{g}^\star)$, $q_1 \in \widehat{S}_0^{-1}(\mathfrak{g}^\star)$, and

$$p_1\#a = 1 - q_1, \qquad 1 \leq |\lambda| \leq 2.$$

Let

$$p \approx \sum_{k=1}^\infty q_1^k \# p_1,$$

where the infinite sum is understood as in (1.7). Then $p \in S_0^{-m}$ and

$$p\#a = 1 - q, \qquad 1 \leq \lambda \leq 2,$$

where $q \in \mathcal{S}(\mathfrak{g}^\star)$. □

Now we are in a position to conclude the proof of Proposition 3.1. By acting with b on the right on both sides of (3.2), we get

$$b = p + q\#b, \qquad 1 \le |\lambda| \le 2,$$

where $q\#b \in \mathcal{S}(\mathfrak{g})$. Consequently,

$$|D^\alpha_\eta b(\eta, \lambda)| \le C_\alpha \rho(\eta, \lambda)^{-m-|\alpha|}, \qquad 1 \le |\lambda| \le 2.$$

However, the same applies to b_n for every $n \in \mathbf{Z}$ with the same constants C_α. Therefore, $B \in S^{-m}_0(\mathfrak{g})$. Finally, by Lemma 1.5, we conclude that $B \in S^{-m}(\mathfrak{g})$.

Corollary 3.7. *Let $A \in S^0(\mathfrak{g})$ and let*

$$\|f \star A\| \ge C\|f\|, \qquad f \in \mathcal{S}(\mathfrak{g}).$$

There exists $B \in S^0(\mathfrak{g})$ such that

$$B \star A = \delta_0.$$

Proof. It is not hard to see that

$$\|\mathrm{Op}\,(A^\star \star A)f\| \ge C\|f\|, \qquad f \in \mathcal{S}(\mathfrak{g}),$$

so $\mathrm{Op}\,(A^\star \star A) : L^2(\mathfrak{g}) \to L^2(\mathfrak{g})$ is an isomorphism. By Proposition 3.1 there exists $B_1 \in S^0(\mathfrak{g})$ such that $B_1 \star A^\star \star A = \delta_0$. Therefore $B_1 \star A^\star$ is the left-inverse for A. □

Corollary 3.8. *For every $0 \le m \le 1$, there exists $V_{-m} \in S^{-m}(\mathfrak{g})$ such that*

$$V_m \star V_{-m} = V_{-m} \star V_m = \delta_0.$$

4. *The operator* $\mathrm{Op}\,(V_1)$

In this section we show that the role of the family of distributions $V_m \in S^m(\mathfrak{g})$ in defining the Sobolev spaces can be taken over by the family of fractional powers of one single distribution V_1. This will enable the final step towards our theorem.

Recall that if a positive selfadjoint operator $A : L^2(\mathfrak{g}) \to L^2(\mathfrak{g})$ is invertible, then

$$A^{-k} f = \frac{\sin k\pi}{\pi} \int_0^\infty t^{-k}(tI + A)^{-1} f\, dt \tag{4.1}$$

for $0 < k < 1$ (see, e.g, Yosida [**18**], IX.11).

The operator $\mathrm{Op}\,(V_1)$ is positive selfadjoint and invertible. In the proof of the next proposition we follow Beals [**2**], Theorem 4.9.

Proposition 4.1. *For every $m \in \mathbf{R}$, $\mathrm{Op}\,(V_1)^m = \mathrm{Op}\,(V_1^m)$, where $V_1^m \in S^m(\mathfrak{g})$.*

Proof. It is sufficient to prove the proposition for $-1 < m < 0$. For $t \ge 0$ let

$$R_t = (V_1 + t\delta_0)^{-1}, \qquad r_t = \widehat{R}_t.$$

The operators $\mathrm{Op}\,(V_1) + tI$ satisfy the hypothesis of Proposition 3.1 with the exponent $m = 1$ uniformly so there exist constants C'_α independent of t such that

$$|D^\alpha r_t| \le C'_\alpha \rho^{-1-|\alpha|}. \tag{4.2}$$

On the other hand

$$tR_t = \delta_0 - R_t \star V_1 \in S^0(\mathfrak{g})$$

uniformly in t so that

$$t|D^\alpha r_t| \le C''_\alpha \rho^{-\alpha}. \tag{4.3}$$

Combining (4.2) with (4.3) we get

$$|D^\alpha r_t| \le C_\alpha (t+\rho)^{-1}\rho^{-\alpha}$$

with C_α independent of $t \ge 0$.

Now, the operator $\mathrm{Op}\,(V_1)$ is positive and invertible so, by (4.1), $\mathrm{Op}\,(V_1)^m = \mathrm{Op}\,(V_1^m)$, where

$$(V_1^m)^\wedge = -\frac{\sin m\pi}{\pi}\int_0^\infty t^m r_t\,dt,$$

where $-1 < m < 0$. Therefore

$$\begin{aligned}|D^\alpha (V_1^m)^\wedge| &\le \frac{C_\alpha}{\pi}\int_0^\infty t^m (t+\rho)^{-1}\,dt \cdot \rho^{-|\alpha|}\\ &\le C'_\alpha \rho^{m-|\alpha|},\end{aligned}$$

which proves our case. □

Lemma 4.2. *Let K be a distribution on $\mathfrak{g}$ smooth away from the origin and satisfying the estimates*

$$|D^\alpha K(x)| \le C_\alpha |x|^{m-Q-|\alpha|}, \qquad x \ne 0, \tag{4.4}$$

for some $m > 0$. Then,

$$K = R + \nu,$$

where $R \in S^{-m}(\mathfrak{g})$ and $\partial\mu \in L^1(\mathfrak{g})$ for every left-invariant differential operator on $\mathfrak{g}$.

Proof. It is sufficient to observe that (4.4) implies that $\widehat{K}$ is smooth away from the origin and

$$|D^\alpha \widehat{K}(\xi)| \le C_\alpha |x|^{-m-|\alpha|}, \qquad \xi \ne 0,$$

and let $R = \varphi K$, $\nu = K - R$, where $\varphi \in C_c^\infty(\mathfrak{g})$ is equal to 1 in a neighbourhood of 0. □

Recall that

$$P^m = V_m + \mu,$$

where $V_m \in S^m(\mathfrak{g})$ and $\partial\mu \in L^1(\mathfrak{g})$ for every invariant differential operator ∂ on $\mathfrak{g}$.

Proposition 4.3. *Let $m > 0$. Then*

$$(P^m + \delta_0)^{-1} = R + \nu,$$

where $R \in S^{-m}(\mathfrak{g})$ and $\partial\nu \in L^1(\mathfrak{g})$ for every invariant differential operator ∂ on $\mathfrak{g}$.

Proof. Since the kernel P^m is maximal (see (2.3) above), it follows (see Dziubański [**6**], Theorem 1.13) that the semigroup generated by P^m consists of operators with the convolution kernels

$$h_t(x) = t^{-Q/m} h_1(t^{-1/m}x), \qquad t > 0,$$

which are smooth functions satisfying the estimates

$$|D^\alpha h_t(x)| \le \frac{C_\alpha t}{(t^{1/m} + |x|)^{Q+m+|\alpha|}}, \qquad x \in \mathfrak{g}.$$

Therefore,

$$(P^m + \delta_0)^{-1}(x) = \int_0^\infty e^{-t} h_t(x)\, dt,$$

and consequently satisfies the estimates (4.4). □

We know that there exists a constant $C > 0$ such that

$$C^{-1}\|f \star V_1\| \le \|f \star P\| + \|f\| \le C\|f \star V_1\|,$$

whence

$$\|f \star V_1^m\| \ge C_m \|f\|, \qquad f \in \mathcal{S}(\mathfrak{g}), \tag{4.5}$$

for $m > 0$.

Now we have much more.

Corollary 4.4. *For every $m > 0$ there exists a constant $C > 0$ such that*

$$C^{-1}\|f \star V_1^m\| \le \|f \star P^m\| + \|f\| \le C\|f \star V_1^m\|. \tag{4.6}$$

Proof. In fact, we have

$$V_1^m = V_1^m \star (P^m + \delta_0)^{-1} \star (P^m + \delta_0) = \Big(V_1^m \star R + V_1^m \star \nu\Big) \star (P^m + \delta_0),$$

where R and ν are as in Proposition 4.3. Then $V_1^m \star R \in S^0(\mathfrak{g})$ and $V_1^m \star \nu \in L^1(\mathfrak{g})$ so

$$\|f \star V_1^m\| \le C_1(\|f \star P^m\| + \|f\|).$$

The proof of the opposite inequality uses the identity

$$f \star P^m = f \star V_m \star V_1^{-m} \star V_1^m + f \star \mu$$

and (4.5). □

5. *Main theorem*

Here comes our main theorem and the conclusion of its proof.

Theorem 5.1. *Let $A \in S^m(\mathfrak{g})$, where $m \ge 0$. If A satisfies the estimate*

$$\|f \star A\| \ge C(\|f \star P^m\| + \|f\|), \qquad f \in \mathcal{S}(\mathfrak{g}),$$

then there exists $B \in S^{-m}(\mathfrak{g})$ such that

$$B \star A = \delta_0$$

Proof. Let $A \in S^m(\mathfrak{g})$ satisfy the hypothesis of our theorem. Then $A \star V_1^{-m}$ satisfies the hypothesis of Corollary 3.7 so there exists $B_1 \in S^0(\mathfrak{g})$ such that

$$B_1 \star A \star V_1^{-m} = \delta_0.$$

By acting by convolution with V_1^m on the right and with V_1^{-m} on the left, we see that $B = V_1^{-m} \star B_1$ is the left-inverse for A. □

COROLLARY 5.2. *Let $A = A^\star \in S^m(\mathfrak{g})$ for some $m \geq 0$. The following conditions are equivalent:*

(i) *There exists $B \in S^{-m}$ such that $B \star A = A \star B = \delta_0$,*
(ii) *For every $k \in \mathbf{R}$, $\mathrm{Op}\,(A) : H(k+m) \to H(k)$ is an isomorphism,*
(iii) *$\mathrm{Op}\,(A) : H(m) \to H(0)$ is an isomorphism,*
(iv) *There exists $C > 0$ such that*

$$\|f \star A\| \geq C(\|f \star P^m\| + \|f\|), \qquad f \in \mathcal{S}(\mathfrak{g}).$$

COROLLARY 5.3. *Let $A \in S^m(\mathfrak{g})$, where $m > 0$, and let $\mathrm{Op}\,(A)$ be positive in $L^2(\mathfrak{g})$. Then A has a parametrix if and only if there exists $C > 0$ such that*

$$||f \star A\| + \|f\| \geq C\|f \star P^m\|. \tag{5.1}$$

Proof. Let $B \in S^{-m}(\mathfrak{g})$ be a parametrix for A. Then

$$B \star A = \delta_0 + h,$$

where $h \in \mathcal{S}(\mathfrak{g})$. Consequently,

$$P^m = V_1^m \star B \star A + g$$

where $g \in L^1(\mathfrak{g})$. Now, $V_1^m \star B \in S^0(\mathfrak{g})$ so it is easy to see that the estimate (5.1) holds.

Suppose now that (5.1) holds true. Then

$$\|f \star P^m\| \leq C_1 \|f \star (A + \delta_0)\|,$$

which, by Corollary 5.2, implies that $A + \delta_0 \in S^m(\mathfrak{g})$ has an inverse $B_1 \in S^{-m}$. Thus

$$B_1 \star A = \delta_0 - B_1,$$

and the parametrix B can be found as an asymptotic series

$$B \approx \sum_{k=1}^{\infty} B_1^k.$$

□

6. *Rockland operators*

A left-invariant homogeneous differential operator R is said to be a *Rockland operator* if for every nontrivial irreducible unitary representation π of $\mathfrak{g}$, π_R is injective on the space of C^∞-vectors of π.

Let R be a left-invariant differential operator homogeneous of degree $-Q-m$, that is,

$$R(f\circ\delta_t)=t^mRf,\qquad f\in\mathcal{S}(\mathfrak{g}),\quad t>0.$$

It is well-known that the following conditions are equivalent:

(1) R is a Rockland operator,
(2) R is hypoelliptic,
(3) For every regular kernel T of order m, there exists a constant $C>0$ such that

$$\|\mathrm{Op}\,(T)f\|\le C\|Rf\|,\qquad f\in\mathcal{S}(\mathfrak{g}).$$

That (1) is equivalent to (2) was proved by Helffer-Nourrigat [**12**] with a contribution from Beals [**1**] and Rockland [**16**]. Helffer-Nourrigat [**12**] also contains the proof of equivalence of (1)-(3) for $\mathrm{Op}\,(T)$ being a differential operator. The remaining part was obtained by the present author in [**8**] and [**11**].

It has been proved by Melin [**14**] that a Rockland operator on a *stratified* homogenenous group has a parametrix. We are going to show that in fact this is so on any homogeneous group.

COROLLARY 6.1. *A Rockland operator on* $\mathfrak{g}$ *has a parametrix.*

Proof. Without any loss of generality we may assume that R is positive. Then the assertion follows from (3) and Corollary 5.3. □

Thus we have one more condition equivalent to (1)-(3). However, the techniques of the present paper can be applied directly to Rockland operators rending unnecessary any reference to Theorem 5.1 or Corollary 5.3. What is needed are well-known properties of Rockland operators and the symbolic calculus of Proposition 1.1. Here is a brief sketch of a direct parametrix construction for a Rockland operator R.

We may assume that R is positive. By Folland-Stein [**7**], Chapter 4.B, R is essentially selfadjoint on $L^2(\mathfrak{g})$ with $\mathcal{S}(\mathfrak{g})$ for its core domain. Moreover, the semigroup generated by it consists of convolution operators with kernels

$$p_t(x)=t^{-Q/m}p_1(t^{-1/m}x),$$

where p_1 is a Schwartz class function. Note that $R=\mathrm{Op}\,(R\delta_0)$. Let $S=(\delta_0+R\delta_0)^{-1}$. It follows that

$$\widehat{S}(\xi)=\int_0^\infty e^{-t}\widehat{p_1}(t^{1/m}\xi)\,dt$$

is a smooth function satisfying the estimates which show that $S\in S^{-m}(\mathfrak{g})$. Moreover,

$$S\star R\delta_0=\delta_0-S,$$

and by the usual argument the asymptotic series

$$S_1\approx\sum_{k=1}^{\infty}S^k$$

defines a parametrix for R (cf. Melin [**14**]).

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