0.4 Exercises

0.4.1 Calculate

(i) $\bigcap_{n=1}^{\infty}(0,1/n)$; $\bigcap_{n=1}^{\infty}(-1/n,1/n)$; $\bigcup_{n=1}^{\infty}[1/n,n)$;
(ii) $\bigcap_{n=1}^{\infty}(n,n+3)$; $\bigcup_{n=1}^{\infty}(n,n+3)$;
(iii) $\bigcap_{n=1}^{\infty}(n,2n)$; $\bigcup_{n=1}^{\infty}(n-n^2,1/n)$.

0.4.2 For a sequence of sets A_n as above, calculate $\limsup_n A_n$ and $\liminf_n A_n$.

0.4.3 Express the interval $[a,b] \subseteq \mathbb{R}$ as an intersection of open intervals. Likewise, write (a,b) as a union of closed intervals.

0.4.4 Check that in the previous exercise one cannot exchange 'open' for 'closed'.

0.4.5 Express the triangle $T = \{(x,y) \in \mathbb{R}^2 : 0 < x < 1, 0 < y < x\}$ as a union of rectangles. Note that in fact T is a countable union of rectangles.

0.4.6 Note that $x \in \limsup_n A_n$ if and only if $x \in A_n$ for infinitely many n; accordingly, $x \in \liminf_n A_n \iff x \in A_n$ for almost all n.

0.4.7 Check the following relations

(i) $\bigcap_{n=1}^{\infty} A_n \subseteq \liminf_n A_n \subseteq \limsup_n A_n \subseteq \bigcup_{n=1}^{\infty} A_n$;
(ii) $(\liminf_n A_n)^c = \limsup_n A_n^c$, $(\limsup_n A_n)^c = \liminf_n A_n^c$;
(iii) $\liminf_n (A_n \cap B_n) = \liminf_n A_n \cap \liminf_n B_n$;
(iv) $\liminf_n (A_n \cup B_n) \supseteq \liminf_n A_n \cup \liminf_n B_n$ i równość na ogół nie zachodzi.

Write properties of lim sup, analogous to (iii)–(iv).

0.4.8 Check that for a given sequence of sets A_n, if we set $B_1 = A_1$, $B_n = A_n \setminus \bigcup_{j<n} A_j$ for $n > 1$, then $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$, and B_n's are pairwise disjoint.

0.4.9 Prove that $\lim_n A_n = A \iff \lim_n (A_n \triangle A) = \emptyset$.

0.4.10 Prove that every family of pairwise disjoint nonempty intervals on the real line is countable.

0.4.11 Let $U \subseteq \mathbb{R}$ be an open set. For $x, y \in U$ define $x \sim y$ if there is an interval (a,b) such that $x, y \in (a,b) \subseteq U$. Check that $\sim$ is an equivalence relation and its equivalence classes are open intervals. Conclude that every open subset of the reals is a countable union of pairwise disjoint open intervals.

0.4.12 Check that a finite intersection of open sets is open.

0.5 Problems

0.5.A Prove the following Cauchy-like condition for a convergence of sets: A sequence of sets A_n is convergent if and only if for any sequences of natural numbers $(n_i)_i$, $(k_i)_i$ with $n_i, k_i \to \infty$ we have $\bigcap_{i=1}^{\infty}(A_{n_i} \triangle A_{k_i}) = \emptyset$.

0.5.B Prove that every sequence of sets $A_n \in \mathcal{P}(\mathbb{N})$ has a converging subsequence.

0.5.C Find an example of a sequence of $A_n \in \mathcal{P}(\mathbb{R})$, without a converging subsequence. REMARK: it is convenient to replace $\mathbb{R}$ by another set of the same cardinality.

0.5.D Prove that if F is a closed and bounded subset of $\mathbb{R}$ then every sequence of $x_n \in F$ has a subsequence converging to some $x \in F$.

HINT: $x \in F$ is a limit of a subsequence of x_n's if and only if for every $\delta > 0$ the interval $(x - \delta, x + \delta)$ contains x_n for infinitely many n's. Assume that no $x \in F$ has such a property and apply Theorem 0.3.5.