2.5 Exercises

2.5.1 Check that taking preimages by a given function preserves all the basic set-theoretic operations. Note that

$$f\left[\bigcup_n A_n\right] = \bigcup_n f[A_n],$$

for any subsets A_n of the domain of f. Check that the inclusion

$$f[A_1 \cap A_2] \subseteq f[A_1] \cap f[A_2]$$

may be strict.

2.5.2 Given a sequence of Σ-measurable functions $f_n : X \to \mathbb{R}$, check that the following sets belongs to Σ

(i) the set of those x for which the sequence $f_n(x)$ is increasing;
(ii) the set of those x for which $f_n(x) < 2$ for all n;
(iii) the set of those x for which $f_n(x) < 2$ for almost all n;
(iv) the set of those x for which $f_n(x) < 2$ for infinitely many n;
(v) the set of those x for which $\sup_n f_n(x) < 2$;
(vi) the set of those x for which $\sup_n f_n(x) \leqslant 2$;
(vii) the set of those x for which the sequence $f_n(x)$ converges;
(viii) the set of those x for which $\limsup f_n(x) > \liminf f_n(x)$.

2.5.3 Prove that the sum of a convergent series of measurable functions is measurable.

2.5.4 Let $f : \mathbb{R} \to \mathbb{R}$ be an **arbitrary** function. Put $F_\varepsilon = \{x \in \mathbb{R} : osc_x(f) \geqslant \varepsilon\}$ where $osc_x(f) \geqslant \varepsilon$ denotes that for every $\delta > 0$ there are $x', x'' \in (x - \delta, x + \delta)$ such that $|f(x') - f(x'')| \geqslant \varepsilon$.

Check that the set F_ε is closed; conclude that the set of points of continuity of f is Borel.

2.5.5 Suppose that for every t from some set T we are given a continuous function $f_t : \mathbb{R} \to \mathbb{R}$; consider the function $h = \sup_{t \in T} f_t$. Prove that h is a Borel function (even for an uncountable T). For that purpose consider sets of the form $\{x : h(x) > a\}$.

2.5.6 Check that every simple function which is measurable with respect to a σ–algebra $\Sigma \subseteq \mathcal{P}(X)$ can be written as

(i) $\sum_{i \leqslant n} a_i \chi_{A_i}$, where $A_i \in \Sigma$, $A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n$, and
(ii) $\sum_{i \leqslant n} b_i \chi_{B_i}$ where $B_i \in \Sigma$ are pairwise disjoint.

What conditions guarantee that such representations are unique?

2.5.7 Check that the family of all simple functions is closed under taking linear combinations, absolute value and multiplication.

2.5.8 Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function, that is $|f(x) - f(y)| \leqslant L|x - y|$ for some constant L. Prove that $f[A]$ has Lebeshue measure zero whenever A is a set of measure zero.

2.5.9 Conclude form the previous exercise that the image of a measurable set by a Lipschitz function is measurable.

HINT: $f[F]$ is compact whenever f is continuous and $F \subseteq \mathbb{R}$ is compact; apply Corollary 1.6.3.

2.5.10 Prove that in Exercises 8 and 9 it is sufficient to assume that f is a locally Lipschitz function (satisfies the Lipschitz condition on every interval of the form $[-n, n]$; note that every function having a continuous derivative is locally Lipschitz.

2.5.11 Note that an arbitrary nondecreasing function $f : \mathbb{R} \to \mathbb{R}$ is Borel.

2.5.12 Construct a nondecreasing continuous function $g : [0, 1] \to [0, 1]$ such that $g[C] = [0, 1]$ where $C \subseteq [0, 1]$ is the Cantor set.

HINT: set $g(x) = 1/2$ for $x \in (1/3, 2/3)$; $g(x) = 1/4$ for $x \in (1/9, 2/9)$ and so on.

2.5.13 Use the function g from the previous exercise to demonstrate that the image of a measurable set by a continuous function need not be measurable, and the preimage of a measurable function by such a function also may be nonmeasurable.

2.5.14 Note that if $\mu(X) < \infty$ that for every measurable function $f : X \to \mathbb{R}$ and $\varepsilon > 0$ there is a set A such that $\mu(A) < \varepsilon$ and f is bounded on $X \setminus A$.

2.5.15 Let $|f_n| \leqslant M$ and suppose that $f_n \xrightarrow{\mu} f$. Check that $|f| \leqslant M$ almost everywhere.

2.5.16 Comsider a nondecreasing sequence of measurable functions f_n such that f_n converge to f in measure. Prove that in such a case $f_n \to f$ almost everywhere.

2.5.17 Prove that if $f_n \xrightarrow{\mu} f$ and $g_n \xrightarrow{\mu} g$ then $f_n + g_n \xrightarrow{\mu} f + g$. Prove that $f_n g_n \xrightarrow{\mu} fg$ under an additional assumption that f_n i g_n are all bounded by the same constant.

2.5.18 Let μ be a finite measure. Prove that if $f_n \xrightarrow{\mu} f$ where $f(x) \neq 0$ for all x then $1/f_n \xrightarrow{\mu} 1/f$.

2.5.19 Suppose that $\mu(X) < \infty$. Prove that if $f_n \xrightarrow{\mu} f$ and $g_n \xrightarrow{\mu} g$ then $f_n g_n \xrightarrow{\mu} fg$ (compare Exercise 17). Show that this may not hold for an infinite measure.

2.6 Problems

2.6.A Let $A \subseteq \mathbb{R}$ be a measurable set of finite Lebesgue measure. Investigate whether the function

$$g : \mathbb{R} \to \mathbb{R}, \quad g(x) = \lambda(A \cap (x + A)),$$

is continuous (here λ denotes the Lebesgue measure, $x + A$ is the translate of the set).

2.6.B Prove that every Lebesgue measurable function $f : \mathbb{R} \to \mathbb{R}$ is a limit of a sequence of continuous functions f_n converging almost everywhere. In fact, one can find such f_n belonging to C^∞.

HINT: Start from the case $f = \chi_A$ where A is a finite union of intervals.

2.6.C Prove that no sequence of continuous function $f_n : \mathbb{R} \to \mathbb{R}$ can converge pointwise to $\chi_\mathbb{Q}$ (the characteristic function of $\mathbb{Q}$).

HINT: I method: argue by contradiction, using the Darboux property of continuous functions. II method: prove that a pointwise limit of a sequence of continuous function must have a point of continuity.

2.6.D Let $f : \mathbb{R} \to \mathbb{R}$ be an **arbitrary** function satisfying the equation $f(x + y) = f(x) + f(y)$. Check that then $f(x) = ax$ for all $x \in \mathbb{Q}$ $(a = f(1))$.

Prove that if the function f is measurable then $f(x) = ax$ for all $x \in \mathbb{R}$.

2.7 Appendix: Upper and lower limits of sequences of reals

Let (a_n) be a sequence of real numbers. We call a a cluster point of the sequence if there is a subsequence of (a_n) converging to a. Likewise, we define when ∞ or $-\infty$ is the cluster point of the sequence.

2.7.1 Prove that every sequence of reals has the least cluster point (which is a real number or one of $-\infty, \infty$). The least cluster point is called the lower limit $\liminf_{n\to\infty} a_n$.

2.7.2 Note that $\liminf_{n\to\infty} a_n = -\infty$ if and only if the sequence (a_n) is not bounded from below.

2.7.3 Prove that $a = \liminf_{n\to\infty} a_n$ (where a is a real number) if and only if for every $\varepsilon > 0$ we have $a_n > a - \varepsilon$ for almost all n and $a_n < a + \varepsilon$ for infinitely many n.

2.7.4 Prove that $\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf_{k\geqslant n} a_k$.

2.7.5 Prove that $\liminf_{n\to\infty}(a_n + b_n) \geqslant \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n$.

2.7.6 Define the upper limit $\limsup$ accordingly and note its analogous properties.

2.7.7 Note that a sequence of reals converges if and only if its upper limit and lower limit coincide (and they are real numbers).

2.7.8 Check that $\liminf_{n\to\infty}(a_n - b_n) = a - \limsup_{n\to\infty} b_n$ whenever $\lim a_n = a$.