3.5 Exercises

3.5.1 Check that the formula

$$\int_X \sum_{i=1}^n a_i\chi_{A_i}\ \mathrm{d}\mu = \sum_{i=1}^n a_i\mu(A_i)$$

properly defines the integral of simple functions on an arbitrary space (X, Σ, μ).

HINT: If $\sum_{i=1}^n a_i\chi_{A_i} = \sum_{j=1}^k b_j\chi_{B_j}$ then there is a finite partition of X into measurable sets $T_s, 1 \leqslant s \leqslant p$ such that every A_i and B_j is a union of some T_s.

3.5.2 Suppose that $\mu(X) = 1$ and $\mu(A_i) \geqslant 1/2$ for $i = 1, 2, \ldots, n$. Prove that there is $x \in X$ belonging to at least $n/2$ sets A_i. For this purpose evaluate $\int_X \sum_{i\leqslant n} \chi_{A_i}\ \mathrm{d}\mu$ (compare Problem 1.11.E).

3.5.3 Consider $f(x) = -\frac{1}{x^2+1}$ to note that one cannot define the integral $\int_{\mathbb{R}} f\ \mathrm{d}\lambda$ as the supremum of $\int s\ \mathrm{d}\lambda$ for simple functions $s \leqslant f$. Define an analogous function on $[0, 1]$.

3.5.4 Let (X, Σ, μ) be a measure space and let $f, g : X \to \mathbb{R}$ be measurable functions. Check that

(i) if $\int_A f\ \mathrm{d}\mu = 0$ for every $A \in \Sigma$ then $f = 0$ almost everywhere;

(ii) if f is integrable on X then it is integrable on every $X_0 \in \Sigma$;

(iii) if $A, B \in \Sigma$ and $\mu(A \triangle B) = 0$ then $\int_A f\ \mathrm{d}\mu = \int_B f\ \mathrm{d}\mu$ for every f (and that if any of the integrals is finite, so is the other);

(iv) $\int |f - g|\ \mathrm{d}\mu \geqslant |\int |f|\ \mathrm{d}\mu - \int |g|\ \mathrm{d}\mu|$.

3.5.5 Verify whether

(i) the product of two integrable function must be integrable;

(ii) a function f such that $f = 1$ almost everywhere is integrable;

(iii) f is integrable provided it is integrable on every set of finite measure.

3.5.6 Consider the space $(\mathbb{N}, P(\mathbb{N}), \mu)$ where μ is a counting measure, that is $\mu(A) = |A|$ for finite sets and $\mu(A) = \infty$ for infinite $A \subseteq \mathbb{N}$.

Prove that $f : \mathbb{N} \to \mathbb{R}$ is integrable if and only if $\sum_{n=1}^\infty |f(n)| < \infty$. Note that in such a case the integral coincides with the sum of the series.

3.5.7 Can you find a sequence of integrable functions such that

(i) it converges almost everywhere but not in measure;

(ii) converges in measure but not almost everywhere;

(iii) converges almost everywhere but is unbounded;

(iv) uniformly convergent to zero and such that the integrals do not converge;

(v) uniformly convergent to a non-integrable function?

For every question consider two cases: $\mu(X) < \infty$ and $\mu(X) = \infty$.

3.5.8 Let $f : [a,b] \to \mathbb{R}$ be a bounded Borel function. Note that f is Lebesgue integrable.

3.5.9 Prove that if $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue integrable then for every $\varepsilon > 0$ there is an interval $[a,b]$ such that $\int_{[a,b]} |f| \,\mathrm{d}\mu > \int_{\mathbb{R}} |f| \,\mathrm{d}\mu - \varepsilon$.

3.5.10 Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative function with a finite improper Riemann integral $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$. Prove that f is Lebesgue integrable. Check that the assumption $f \geqslant 0$ is essential.

3.5.11 Suppose that $\mu(X) < \infty$. Prove that a measurable function f is integrable if and only if, writing $A_n = \{x : |f(x)| \geqslant n\}$, we have $\sum_{n=1}^{\infty} \mu(A_n) < \infty$.

3.5.12 Prove so called Chebyshev's inequality: for an integrable function f it holds

$$\int |f| \,\mathrm{d}\mu \geqslant \varepsilon \mu(\{x : \ |f(x)| \geqslant \varepsilon\}).$$

3.5.13 Conclude from Chebyshev's inequality that

$$\text{jeżeli} \quad \int |f - f_n| \,\mathrm{d}\mu \to 0 \quad \text{to} \quad f_n \xrightarrow{\mu} f.$$

3.5.14 Let A_n be a sequence of measurable sets such that $\mu(A_n \triangle A_k) \to 0$ for $n,k \to \infty$. Prove that there is a measurable set A such that $\mu(A \triangle A_n) \to 0$.

3.5.15 Define continuous integrable functions $f_n : [0,1] \to [0,\infty)$ such that $f_n \to 0$ almost everywhere but the function $\sup_n f_n$ is not integrable.

3.5.16 Let $f : \mathbb{R} \to \mathbb{R}$ be an integrable function. Check that the function $F(x) = \int_{[0,x]} f(t)\,\mathrm{d}\lambda(t)$ is continuous. Give examples showing that F need not be differentiable.

3.5.17 Note that the Fatou lemma does not hold without the assumption that the functions in question are nonnegative. Investigate when the following is true:

$$\limsup_n \int_X f_n \,\mathrm{d}\mu \leqslant \int_X \limsup_n f_n \,\mathrm{d}\mu.$$

3.5.18 Suppose that (f_n) is such a sequence of integrable functions that $\sum_{n=1}^{\infty} \int |f_n| \,\mathrm{d}\mu < \infty$. Prove that the series $\sum_n f_n$ converges almost everywhere and

$$\int \sum_{n=1}^{\infty} f_n \,\mathrm{d}\mu = \sum_{n=1}^{\infty} \int f_n \,\mathrm{d}\mu.$$

3.5.19 Analyze the formula from the previous exercise for $f_n(x) = x^{n-1} - 2x^{2n-1}$ on the interval $(0,1)$.

3.5.20 Check whether

$$\int_0^1 \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+x}}\, \mathrm{d}x = \sum_{n=1}^{\infty} \int_0^1 \frac{(-1)^n}{\sqrt{n+x}}\, \mathrm{d}x.$$

Any generalisations?

3.5.21 Let μ be a finite measure on X and let $f_n, f : X \to \mathbb{R}$ be measurable functions such that $f_n \xrightarrow{\mu} f$. Prove that of $h : \mathbb{R} \to \mathbb{R}$ is bounded and uniformly continuous then

$$\lim_{n\to\infty} \int_X h(f_n)\, \mathrm{d}\mu = \int_X h(f)\, \mathrm{d}\mu.$$

3.5.22 Let f_n be a sequence of integrable functions converging to an integrable function f almost everywhere. Prove that $\lim_{n\to\infty} \int |f_n - f|\, \mathrm{d}\lambda \to 0$ if and only if $\lim_{n\to\infty} \int |f_n|\, \mathrm{d}\lambda = \int |f|\, \mathrm{d}\lambda$.

HINT: The Fatou lemma.

3.6 Problems

3.6.A We say that a measure space (X, Σ, μ) is *semi-finite* if for every $A \in \Sigma$

$$\mu(A) = \sup\{\mu(B) : B \in \Sigma,\ B \subseteq A,\ \mu(B) < \infty\}.$$

Note that every σ-finite measure is semi-finite (but not vice versa).

3.6.B Note that in the definition of the integral of a nonnegative function on a semi-finite measure space one can take the supremum over integrable simple functions. Check that the basic limit theorems remain true for semi-finite measures.

3.6.C Prove that a measure space (X, Σ, μ) which is not semi-finite contains an atom of infinite measure, i.e. there is $A \in \Sigma$ such that $\mu(A) = \infty$ and $\mu(B) \in \{0, \infty\}$ for every set $B \subseteq A$ from σ-algebra Σ.