Intro

We write λ for the Lebesgue measure on $\mathbb{R}$ so λ is defined either on $Bor(\mathbb{R})$ (the smallest σ-algebra containing all open sets) or on the algebra of measurable sets. i.e. those $A \subseteq \mathbb{R}$ for which there are Borel sets $B_1 \subseteq A \subseteq B_2$ with $\lambda(B_2 \setminus B_1) = 0$.

The outer measure λ^*, defined for all $A \subseteq \mathbb{R}$, may be written as

$$\lambda^*(A) = \inf\{\lambda(V) : A \subseteq V, V \text{ open}\}.$$

Recall that λ^* is monotone and countably subadditive, that is

if $A_1 \subseteq A_2$ then $\lambda^*(A_1) \leqslant \lambda^*(A_2)$;

$$\lambda^*\left(\bigcup_n A_n\right) \leqslant \sum_n \lambda^*(A_n).$$

Lebesgue Measure and intervals

Lemma. *If* $A \subseteq \mathbb{R}$ *is a measurable set* $0 < \lambda(A) < \infty$ *then for every* $\varepsilon > 0$ *there is a nonempty interval* (a, b) *such that*

$$\frac{\lambda(A \cap (a,b))}{b-a} > 1 - \varepsilon.$$

Sketchy Proof. For every $\delta > 0$ there is (!) a set of the form $E = \cup_{i=1}^{n}(a_i, b_i)$ such that $\lambda(A \triangle E) < \delta$. If δ is really small then one of (a_i, b_i) is as required.

The Steinhaus theorem. $A - A$ *has nonempty interior for every* A *with* $\lambda(A) > 0$.

Note that lemma above says that, given such a set A,

$$(\forall \varepsilon > 0)(\exists x)(\exists \delta)\lambda(A \cap (x - \delta, x + \delta))/(2\delta) > 1 - \varepsilon.$$

Definition. $x \in \mathbb{R}$ *is a density point of (a measurable set)* $A \subseteq \mathbb{R}$ *if*

$$\lim_{\delta \to 0^+} \frac{\lambda(A \cap (x - \delta, x + \delta))}{2\delta} = 1.$$

Lebesgue and Vitali

Lebesgue Density Theorem. *Given measurable $A \subseteq \mathbb{R}$, almost every point $x \in A$ is its density point, i.e.*

$$\lim_{\delta \to 0^+} \frac{\lambda(A \cap (x-\delta, x+\delta))}{2\delta} = 1.$$

Definition. *A family $\mathcal{J}$ of nondegenerate closed intervals is* **a Vitali cover** *of a set A if for every $x \in A$*

$$(\forall \varepsilon > 0)(\exists J \in \mathcal{J}) x \in J \textit{ and } \operatorname{diam}(J) < \varepsilon.$$

Vitali Theorem. *If family $\mathcal{J}$ of nondegenerate closed intervals is a Vitali cover of an arbitrary set $A \subseteq \mathbb{R}$ then there are* **pairwise disjoint** *$J_n \in \mathcal{J}$ such that*

$$\lambda\left(A \setminus \bigcup_n J_n\right) = 0.$$

Vitali yields Lebesgue

Fix **bounded** $A \subseteq \mathbb{R}$ and $k \in \mathbb{N}$. It is sufficient to show that if

$$A_k = \left\{ x \in A : \liminf_{\delta \to 0^+} \frac{\lambda(A \cap [x-\delta, x+\delta])}{2\delta} < 1 - 1/k \right\},$$

then A_k is of measure zero. Fix $\varepsilon > 0$ and open $V \supseteq A_k$ such that $\lambda(V) < \lambda^*(A_k) + \varepsilon$.

Consider a family $\mathcal{J}$ of closed intervals J contained in V such that $\lambda(A \cap J) \leqslant (1 - 1/k)\lambda(J)$. Then $\mathcal{J}$ is a Vitali cover of A_k!

Take pairwise disjoint $J_n \in \mathcal{J}$ almost covering A_k.

$$\lambda^*(A_k) = \lambda^*(A_k \cap \bigcup_n J_n) \leqslant$$
$$\leqslant \sum_n \lambda^*(A_k \cap J_n) \leqslant (1 - 1/k) \sum_n \lambda(J_n) \leqslant$$
$$\leqslant \lambda(V) \leqslant (1 - 1/k)(\lambda^*(A_k) + \varepsilon).$$

This gives $\lambda^*(A_k) \leqslant \varepsilon(k-1)$. Hence $\lambda^*(A_k) = 0$.

Proof of Vitali Theorem

We consider a Vitali cover $\mathcal{J}$ of some bounded $A \subseteq \mathbb{R}$. We may assume that all the intervals from $\mathcal{J}$ are contained in some fixed (a, b).

Define inductively $\gamma_n > 0$ and $J_n \in \mathcal{J}$ as follows:

$$\mathcal{J}_n = \{J \in \mathcal{J} : J \cap J_i = \emptyset \text{ for every } i < n\},$$
$$\gamma_n = \sup\{\operatorname{diam}(J) : J \in \mathcal{J}_n\}.$$

Choose $J_n \in \mathcal{J}_n$ such that $\operatorname{diam}(J_n) > 1/2\gamma_n$. Note that $\sum_n \lambda(J_n) \leqslant b - a < \infty$. Hence $\gamma_n \leqslant 2\lambda(J_n) \to 0$. Let J_n' denotes an interval 5 times bigger than J_n (if $J = (z - \delta, z + \delta)$ then $J' = (x - 5\delta, x + 5\delta)$).

Claim. For every n

$$A \subseteq \bigcup_{i<n} J_i \cup \bigcup_{i \geqslant n} J_i'.$$

Note that Claim immediately implies $\lambda^*(A \setminus \cup_n J_n) = 0$:

$$\lambda^*\left(A \setminus \bigcup_{i<n} J_i\right) \leqslant \lambda\left(\bigcup_{i \geqslant n} J_i'\right) \leqslant \sum_{i \geqslant n} \lambda(J_i') = 5 \cdot \sum_{i \geqslant n} \lambda(J_i) \to 0.$$

Claim

CLAIM. For every n

$$A \subseteq \bigcup_{i<n} J_i \cup \bigcup_{i \geqslant n} J_i'.$$

Proof. Take $x \in A \setminus \cup_{i<n} J_i$. There is (!) $I \in \mathcal{J}$ such that $x \in I$ and I is disjoint from $\cup_{i<n} J_i$. Then $\operatorname{diam}(I) > 0$ and $\gamma_k \to 0$ so $I \notin \mathcal{J}_k$ for almost all k. But $\mathcal{J}_1 = \mathcal{J}$ so there is a maximal m such that $I \in \mathcal{J}_m$.This implies (!) that $I \cap J_{m+1} \neq \emptyset$ which gives $x \in J_{m+1}'$.

(Can we use 4 rather than 5?)

Integral form of the Lebesgue density theorem

Theorem. For every $f \in L_1(\mathbb{R})$ we have

$$\lim_{\delta \to 0^+} (1/2\delta) \cdot \int_{x-\delta}^{x+\delta} f \; \mathrm{d}\lambda = f(x),$$

for almost all $x \in \mathbb{R}$.

Proof. Consider the family $\mathcal{F}$ of those integrable functions f for which the assertion holds.

Then $\mathcal{F}$ contains all χ_A for measurable A of finite measure (directly by the Lebsgue density theorem).

The family $\mathcal{F}$ is closed under addition and contains every f which is a uniform limit of a sequence $f_n \in \mathcal{F}$.

Hence $\mathcal{F}$ contains all bounded integrable functions etc.

Differentiating integrals

Remark. If

$$\lim_{\delta\to 0^+} \frac{\lambda(A\cap(x-\delta,x+\delta))}{2\delta} = 1,$$

then

$$\lim_{\delta\to 0^+} \frac{\lambda(A\cap(x,x+\delta))}{\delta} = 1,$$

This follows from

$$\frac{\lambda(A\cap(x-\delta,x+\delta))}{2\delta} =$$

$$= 1/2\left(\frac{\lambda(A\cap(x,x+\delta))}{\delta} + \frac{\lambda(A\cap(x-\delta,x))}{\delta}\right).$$

Hence for almost all x and an integrable f we have (by analogous argument)

$$\lim_{\delta\to 0^+} (1/\delta)\cdot \int_x^{x+\delta} f \,\mathrm{d}\lambda = f(x).$$

Corollary. Given $f \in L_1([a,b])$, the function

$$F(x) = \int_a^x f \,\mathrm{d}\lambda,$$

satisifes $F' = f$ for almost all $x \in [a,b]$.

Remarks

(1) In Vitali's theorem the set A does not have to be measurable itself.
(2) The Lebesgue density folds for nonmeasurable A but we need to replace λ by λ^* in the definition of a point of density.
(3) Vitali's theorem (and so Lebesgue's theorem) holds for some measures μ on a metric space (X, ρ). The only essential assumption is that we control the measure of balls that are 5 times bigger. For instance, it is enough to assume that there a constant C such that

$$\mu(B_{2r}(x))) \leqslant C \cdot \mu(B_r(x))$$

for every $x \in X$ and $r > 0$ (here $B_r(x)$ is the closed ball around x of radius r.).