11. Measures and set theory

Some natural, even quite simple, questions about measures lead to set-theoretic considerations. Below we cover the following.

(1) We construct $X \subseteq [0,1]$ such that no continuous measure in $P(X)$ is tight (but there are continuous Borel measures on X).
(2) We prove that there may be an uncountable separable metric space X carrying no continuous Borel measures.
(3) We try to answer the question if every Borel measure on a metrizable space must be concentrated on a separable subspace.
(4) We consider the master problem: Can one define 'the Lebesgue measure' on all subsets of the real line.

The verb 'construct' in (1) may be misleading; actually, here it means 'we prove by transfinite induction', relying heavily on the axiom of choice; this is fine because the axiom of choice is true, isn't it? The phrase 'there may be' means 'it is relatively consistent'. Do not worry if it does not explain anything.

The problems related to (4) were investigated soon after the Lebesgue measure had been defined. There are classical results on that due to Banach, Ulam, Marczewski and others. We discuss them here because they are interesting, they belong to the core of mathematics, and, actually, they can be explained within very basic understanding of set theory. Set-theoretic aspects of the measure have been intensively studied since then. We shall not discuss here more recent results because they require quite sophisticated methods.

The next sections should make you understand that a *countably* additive measure is a rare bird. We have the Lebesgue measure and we use is extensively in numerous parts of mathematics, but we can check that it is countably additive only using compactness. There is no abstract method of constructing a measure in spaces without any handy topology available. Actually, on the real line we can use directly Dedekind's axiom of completeness — the compactness of intervals is an emanation of the axiom. The condition of **finite** additivity is quite different; this was partially demonstrated by the last three problems on List no 7.

12. Ordinal and cardinal numbers

Nothing is proved here or even formally defined; this section is an attempt to give you some intuition if you have never considered ordinals.

A set X is well-ordered by the relation $\prec$ if $\prec$ is a linear order on X and every nonempty set $A \subseteq X$ has the least element. Zermelo's axiom *every set can be well-ordered* is an equivalent form of the axiom of choice.

Suppose that $(X, \prec)$, $(Y, \prec)$ are well-ordered by some relations (formally, we should write $\prec_X$ and $\prec_Y$ but nobody does it). Then X and Y can be compared and there are three possibilities:

(i) they are the same (isomorphic): there is an order-preserving bijection $f : X \to Y$;
(ii) X is shorter than Y, it is isomorphic to some initial segment $\{y \in Y : y \prec y_0\}$;
(iii) Y is shorter than X isomorphic to some initial segment $\{x \in X : x \prec x_0\}$.

This is nearly obvious; try to define $f : X \to Y$ so that once you have defined $f(x)$ for $x \prec x'$ then you want to say that $f(x')$ is the least element in $Y \setminus \{f(x) : x \prec x'\}$. You will either define the whole bijection $X \to Y$, or the will be no room in the domain or else no room in the codomain.

Ordinal numbers may be seen as **types** of well-ordered set (informally, equivalence classes of the isomorphic relation as in (i)). It is now quite common to see ordinal numbers as concrete well-ordered sets, providing models of all possible well-orderings. The modern definition of an ordinal numbers says: it is a transitive set well-ordered by the relation $\in$. Ordinal numbers are usually denoted by Greek letters $\alpha, \beta, \ldots$; ω is the symbol reserved for special ordinals. We use the usual symbol $<$ to compare ordinal numbers, so $\alpha < \beta$ is the same as $\alpha \in \beta$.

The natural numbers $0, 1, 2, \ldots$ are ordinals. The first infinite ordinal is denoted by ω, it is the order type of $\mathbb{N}$, in which every initial segment is finite. As said, $n < \omega$ is the same as $n \in \omega$ so we put $\omega = \{0, 1, 2, \ldots\}$.

If we consider $X = \{1/2, 2/3, 3/4, \ldots, 1\}$ (with the usual order on the real line) then it is not of type ω — it has the greatest element. Give a name $\omega + 1$ to such a well-order. Can you guess how $\omega + 7$ or $\omega + \omega$ look like?

The ordinal number ω_1 is the least uncountable ordinal,

$$\omega_1 = \{\alpha : \alpha \text{ is a countable ordinal}\}.$$

In other words, ω_1 is an uncountable set that is well-ordered so that every initial segment is countable. We define ω_2 in a similar manner. can you guess the definition of ω_ω?

We say that an ordinal number α is a **cardinal number** if there is no bijection $\alpha \to \beta$ for $\beta < \alpha$. Note that ω is a cardinal number while $\omega + 1$ is not (when you add one element to a countable set it remains countable).

We have a scale of ordinals

$$0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2, \ldots, \omega + \omega, \ldots, \omega_1 \ldots \omega_2 \ldots,$$

and, from time to time, some new cardinal number pops up. Sometimes we write $\aleph_0 = \omega$, $\aleph_1 = \omega_1$ etc.; this is to distinguish the ordinal arithmetic from the cardinal one.

The scale above contains all the cardinal numbers. Indeed, given X of cardinality κ, well order X somehow, then it is isomorphic to some ordinal number α on our scale. If α is not a cardinal number itself, find the first $\beta < \alpha$ for which there is a bijection $\alpha \to \beta$. Then $\kappa = \beta$.

The main puzzle was: Where is $\mathfrak{c}$, the cardinality of $\mathbb{R}$, on the scale of ordinal numbers? For the first 50-60 years (since Georg Cantor invented set theory) people tended to think that $\mathfrak{c} = \omega_1$ and this was called **the continuum hypothesis** (CH for short). Later it became clear that we can have $\mathfrak{c} = \omega_2$ or even $\mathfrak{c} = \omega_{2020}$ — the usual axioms of set theory cannot decide the value of the continuum; $\mathfrak{c}$ can be nearly any $\aleph$ (except fo some, such as $\aleph_\omega$).

13. Bernstein and Lusin sets

Theorem 13.1. *There is a set $Z \subseteq [0,1]$, called a Bernstein set, having the property that*

$$K \cap Z \neq \emptyset \neq K \setminus Z,$$

for every uncountable closed set $K \subseteq [0,1]$.

Proof. We need to know that there are only $\mathfrak{c}$ many closed uncountable subsets of $[0,1]$, and every uncoutable closed such K is of cardinality $\mathfrak{c}$. The latter holds since K contains the Cantor set which has the cardinality $\mathfrak{c}$ (see L2/PA).

Let us make a list $\{K_\alpha : \alpha < \mathfrak{c}\}$ of all uncountable closed sets to consider. The very construction is straightforward: for every $\alpha < \mathfrak{c}$ pick distinct x_α, y_α so that

$$x_\alpha, y_\alpha \in K_\alpha \setminus \{x_\beta, y_\beta : \beta < \alpha\};$$

then $Z = \{x_\alpha : \alpha < \mathfrak{c}\}$ is the required Bernstein set.

Why this works? We can carry on the construction at step α since K_α is of size $\mathfrak{c}$ while the set $\{x_\beta, y_\beta : \beta < \alpha\}$ has smaller cardinality, the same as cardinality of α which is $< \mathfrak{c}$. The difference of those sets is hence quite big. If we take any closed uncountable K then it was given some number α on our list, so $K = K_\alpha$, and we have picked two elements to witness non-emptiness: $x_\alpha \in Z \cap K$, $y_\alpha \in K \setminus Z$. $\square$

Theorem 13.2. *Let $Z \subseteq [0,1]$ be a Bernstein set.*

(a) For every continuous measure $\mu \in P([0,1]$ we have $\mu^*(Z) = 1$ and $\mu^*([0,1] \setminus Z) = 1$ (so Z is veeery nonmeasurable).

(b) There is a continuous measure $\mu \in P(Z)$ and such a measure cannot be tight.

Proof. (a). For a continuous measure $\mu \in P([0,1])$ and $B \in Bor[0,1]$, if $\mu(B) > 0$ then there is a closed $K \subseteq B$ with $\mu(K) > 0$. Then K is uncountable (as μ is continuous). Hence $K \cap Z \neq \emptyset \neq K \setminus Z$. This shows that every Borel set of positive measure intersects both Z and its complement, and this is what we needed to check.

(b) Define $\mu \in P(Z)$ by the formula $\mu(B \cap Z) = \lambda(B)$, $B \in Bor[0,1]$; this is a particular case of the construction of L3/P1.

Clearly μ is a continuous measure on Z. If $K \subseteq Z$ is compact then K is countable (by the Bernstein property) so $\mu(K) = 0$. $\square$

The inductive contruction below is also not very involved; however, this time we need to assume that CH holds. (In fact, one cannot construct a Lusin set working only with the usual axioms of set theory.)

Theorem 13.3. *Assume CH (i.e. assume that $\mathfrak{c} = \omega_1$). There is an uncountable set $L \subseteq [0,1]$, called a Lusin set, such that $L \cap F$ is countable for every closed nowhere dense set F.*

Proof. Using $\mathfrak{c} = \omega_1$, make a list $\{F_\alpha : \alpha < \omega_1\}$ of all closed subsets of $[0,1]$ with empty interior. For every $\alpha < \omega_1$ pick

$$x_\alpha \in [0,1] \setminus \left(\bigcup_{\beta<\alpha} F_\beta \cup \{x_\beta : \beta < \alpha\} \right),$$

and put $L = \{x_\alpha : \alpha < \omega_1\}$. This works!

The only delicate point: if $\alpha < \omega_1$ then $\bigcup_{\beta<\alpha} F_\beta$ is a **countable** union of closed nowhere dense sets so $[0,1] \setminus \bigcup_{\beta<\alpha} F_\beta \neq \emptyset$ by the Baire category theorem.

Note that if F is closed set with empty interior then $F = F_\alpha$ for some $\alpha < \omega_1$ and hence $L \cap F \subseteq \{x_\beta : \beta < \alpha\}$ is countable. □

Theorem 13.4. *Suppose that $L \subseteq [0,1]$ is a Lusin set.*

(a) For every continuous measure $\mu \in P([0,1])$ we have $\mu^*(L) = 0$ (one can say that L is universally measurable).

(b) There is no continuous measure $\mu \in P(L)$.

Proof. For the first part note the following: Every continuous $\mu \in P([0,1])$ is concentrated on a set of the form $\bigcup_k F_k$, where F_n's are closed nowhere dense.

Indeed, take an enumeration $\{q_n : n < \omega\}$ of rational numbers in $[0,1]$; for every k and n, using continuity of the measure, find an interval $I_n^k \ni q_n$ such that $\mu(I_n^k) < 2^{-n-k}$, Then $V_k = \bigcup_n I_n^k$ is an open dense set of measure $\mu(V_k) \leq 2^{-k+1}$ so $F_k = [0,1] \setminus V_k$ are as required.

Now (a) is clear: if μ is continuous then $\mu(\bigcup_k F_k) = 1$ for F_k's as above; at the same time $L \cap \bigcup_n F_n$ is countable so $\mu^*(L) = 0$.

For (b) suppose that $\mu \in P(L)$ is continuous. Consider $\nu \in P([0,1])$ given by $\nu(B) = \mu(B \cap L)$, $B \in Bor[0,1]$. Then $\nu^*(L) = 1$, a contradiction with (a). □