14. Universal measures and measurable cardinals

Definition 14.1. Let us say that μ is a universal measure on a set X if μ is a probability measure defined on $\mathcal{P}(X)$ and $\mu(\{x\}) = 0$ for every $x \in X$.

Clearly, a universal measure vanishes on all countable sets (by σ-additivity). Note that if μ is a universal measure on X and $f : X \to Y$ is a bijection then $f[\mu]$ is a universal measure on Y. Hence, every set equipotent with X will have a universal measure (if X does). In other words, carrying a universal measure is a property of the cardinality of X rather than the set itself. The basic, ontological problem is: *Do universal measures exist?* In case they do we introduce the following.

Definition 14.2. The first cardinal number κ for which there is a universal measure on κ is called a *real-valued measurable cardinal* and denoted as $\mathfrak{m}$.

Below we write $\kappa < \mathfrak{m}$ if we want to say that κ is smaller than the first real-valued measurable cardinal (if universal measures exist) and to say that κ is arbitrary (otherwise). Clearly $\omega < \mathfrak{m}$; the theorem below is less obvious.

Theorem 14.3 (Ulam)**.** $\omega_1 < \mathfrak{m}$.

Proof. To be sure, we claim that there is no universal measure on ω_1, the uncountable set well-ordered so that every initial segment is countable. The proof is based on the following device called Ulam's matrix.

Claim. There is a family $\{A_{\alpha,n} : \alpha < \omega_1, n \in \omega\}$ of subsets of ω_1 such that

(i) $A_{\alpha,n} \cap A_{\beta,n} = \emptyset$ whenever $\alpha \neq \beta$;

(ii) $\omega_1 \setminus \bigcup_n A_{\alpha,n}$ is countable for every $\alpha < \omega_1$.

Briefly, the sets in columns are pairwise disjoint, the union of every row is co-countable.

Le us see how to use Ulam's matrix. Suppose that μ is a universal measure on ω_1. Then for every $\alpha < \omega_1$ there is $n(\alpha) \in \omega$ such that $\mu(A_{\alpha,n(\alpha)}) > 0$ by (ii). Then $\alpha \to n(\alpha)$ is a function into a countable set so there is n^* such that the set $I = \{\alpha < \omega_1 : n(\alpha) = n^*\}$ is uncountable. But by (i) we get an uncountable family of pairwise disjoint sets of positive measure, which is impossible.

The construction of Ulam's matrix is short: for every $\alpha < \omega_1$, the initial segment $\{\beta : \beta < \alpha\}$ is countable so there is a 1-1 function $f_\alpha : \{\beta : \beta < \alpha\} \to \omega$. Put $A_{\alpha,n} = \{\beta > \alpha : f_\beta(\alpha) = n\}$, and check that (i) and (ii) hold. $\square$

Lemma 14.4. *If μ is a universal measure on a set X and $\kappa < \mathfrak{m}$ then for every family $\{A_\alpha : \alpha < \kappa\}$ of subsets of X of measure zero we have*

$$\mu\left(\bigcup_{\alpha<\kappa} A_\alpha\right) = 0.$$

Proof. Write $A = \bigcup_{\alpha<\kappa} A_\alpha$ and suppose that $\mu(A) > 0$. Then $A = \bigcup_{\alpha<\kappa} B_\alpha$, where the sets

$$B_\alpha = A_\alpha \setminus \bigcup_{\beta<\alpha} A_\beta,$$

are pairwise disjoint. Define a measure ν on κ by

$$\nu(I) = \mu\left(\bigcup_{\alpha\in I} B_\alpha\right);$$

it is easy to verify that ν is indeed a measure. Then $\nu(\kappa) > 0$, $\nu(\{\xi\}) = 0$ for $\xi \in \kappa$ so, after rescaling, ν becomes a universal measure on κ, a contradiction with $\kappa < \mathfrak{m}$. □

Now, if you feel insecure with the above, then stay with Theorem14.3. Otherwise, you might try to see that the combination of Theorem 14.3 and Lemma 14.4 actually gives the following. Here, given a cardinal number κ, κ^+ denote the next cardinal number.

Theorem 14.5 (Ulam)**.** *If $\kappa < \mathfrak{m}$ then $\kappa^+ < \mathfrak{m}$.*

To understand how it works, consider $\kappa = \omega_2$, construct analogous Ulam's matrix etc.

Theorem 14.5 and Lemma 14.4 mean that $\mathfrak{m}$ (if exists) is a *weakly inaccessible cardinal* (a regular cardinal which is not a successor). Set Theory says: you cannot prove (using the usual axiom) that such creatures exist. You might succeed in proving that inaccessible cardinal do not exists but this is rather unlikely.

Theorem 14.6 (Ulam)**.** *If there are universal measures then either $\mathfrak{m} \le \mathfrak{c}$ or else there is a $\{0,1\}$-valued universal measure on $\mathfrak{m}$.*

Proof. Let μ be a universal measure on $\mathfrak{m}$. There are two cases.

The measure μ is nonatomic. Then there is a function $f : \mathfrak{m} \to [0,1]$ such that $f[\mu] = \lambda$, more precisely, $\lambda(B) = \mu(f^{-1}[B])$ for $B \in Bor[0,1]$. see L2/P10 and P11. But then we can extend the Lebesgue measure λ to a universal measure $\overline{\lambda}$ on $[0,1]$, simply writing $\overline{\lambda}(Z) = \mu(f^{-1}[Z])$ for *arbitrary* $Z \subseteq [0,1]$. It follows that $\mathfrak{m} \le \mathfrak{c}$ which is the cardinality of the unit interval.

The measure μ has an atom. Let $A \subseteq \mathfrak{m}$ be such a set that $\mu(A) > 0$ and for every $B \subseteq A$ we have either $\mu(B) = 0$ or $\mu(B) = \mu(A)$. This clearly enables us to define a $\{0,1\}$-measure on A. We have $|A| \le \mathfrak{m}$ but also $|A| \ge \mathfrak{m}$ by minimality of $\mathfrak{m}$. □

Corollary 14.7. *The Lebesgue measure extends to a universal measure if and only if $\mathfrak{m} \le \mathfrak{c}$.*

Proof. The forward implication is obvious. For the reverse one we use the argument from 14.6. □

Remark 14.8. Some additional comments.

A universal $\{0,1\}$-measure on a set X gives a σ-complete ultrafilter $\{A \subseteq X : \mu(A) = 1\}$ and *vice versa* (an ultrafilter is σ-compete if intersection of countably many of its members is still in the ultrafilter). If $\mathfrak{m} > \mathfrak{c}$ then $\mathfrak{m}$ is called simply a *measurable cardinal.* It is really huge; for instance, if $\kappa < \mathfrak{m}$ then $2^\kappa < \mathfrak{m}$.

We know already from Vitali's construction that the Lebesgue measure cannot be extended to a universal **translation invariant** measure. However, the Lebesgue measure can be extended to translation invariant **finitely additive** measure defined for all subsets of the real line. The same holds for $\mathbb{R}^2$; the resulting set function is invariant with respect to all the isometries of the plane. As Banach and Tarski proved, using their famous paradoxical decomposition of the ball, this is no longer true for $\mathbb{R}^3$.

15. Measures on nonseparable metric spaces

If X is a set admitting a universal measure μ then we may think that X is a metric space, equipped with the discrete metric, and then $Bor(X)$ is the power set of X so μ is a Borel measure on X that vanishes on all separable subspaces. This can be reversed.

Theorem 15.1 (Marczewski & Sikorski)**.** *Suppose that universal measures do not exist. Then every Borel measure on a metrizable space is concentrated on a separable subspace.*

For the proof we need some concepts from general topology. If X is a topological space then a family $\mathcal{A}$ of its subsets is said to be *locally finite* if every $x \in X$ has a neighbourhood V such that $V \cap A \neq \emptyset$ only for finitely many $A \in \mathcal{A}$. A space X is *paracompact* if every open cover of X has a refinement which is locally finite. The following fact is the classical Stone theorem; it can be found in Engelking's book, we take it for granted.

Theorem 15.2 (Stone)**.** *Every metrizable space is paracompact.*

We shall also use the following variant of Lemma 14.4.

Lemma 15.3. *Let μ be a Borel measure on a metrizable space X. Assume that there is a pairwise disjoint family $\{W_\alpha : \alpha < \kappa\}$ of open sets of measure zero such that its union has positive measure. Then κ carries a universal measure.*

Proof. We simply repeat the trick from 14.4: Define ν on κ by

$$\nu(I) = \mu\left(\bigcup_{\alpha \in I} W_\alpha\right);$$

for $I \subseteq \kappa$. Every union is open so the measure ν is well-defined for all subsets of κ. $\square$

Proof. (of Theorem 15.1). Let X be a metrizable space and let μ be a Borel measure on X. Consider the family $\mathcal{V}$ of all open sets $V \subseteq X$ such that $\mu(V) = 0$; let $X_0 = \bigcup \mathcal{V}$. We are going to prove that $\mu(X_0) = 0$. Once it is done, we take $Y = X \setminus X_0$. Then for every open $U \subseteq X$, if $U \cap Y \neq \emptyset$ then $\mu(U \cap Y) > 0$. In other words, the measure μ, treated as a measure on Y, is positive on every nonempty open set in Y. By L8/P2 this means that Y is separable and we are done.

Assume that $\mu(X_0) > 0$; we shall define a universal measure and this will be a contradiction. By the Stone theorem applied to the metrizable space X_0, the family $\mathcal{V}$ has an open locally finite refinement: There is a family of open sets $\mathcal{W}$ such that $\bigcup \mathcal{W} = X_0$ and $\mathcal{W}$ is a locally finite refinement of $\mathcal{V}$, that is every $W \in \mathcal{W}$ is contained in some $V \in \mathcal{V}$. This implies that $\mu(W) = 0$ for $W \in \mathcal{W}$.

Let X_n be the set of those $x \in X_0$ which belong to at most n sets from $\mathcal{W}$. Note that $\bigcup_n X_n = X_0$ because $\mathcal{W}$ is locally finite. Moreover, X_n is a closed set in X_0: if $x \notin X_n$ then we have $n+1$ sets from $\mathcal{W}$ witnessing that fact and their intersection is disjoint from X_n.

The main point: the sets $Z_n = X_n \setminus X_{n-1}$ are Borel and $\mu(\bigcup_n Z_n) = \mu(X_0) > 0$ so there is n such that $\mu(Z_n) > 0$. Every $x \in Z_n$ belongs to exactly n sets $W_1^x, \ldots, W_n^x$ from $\mathcal{W}$. Put $U^x = \bigcap_{i=1}^n W_i^x$; then $\mu(W^x) = 0$ and $W^x \cap W^{x'} = \emptyset$ whenever $x \neq x'$. This means that we can apply Lemma 15.3 to Z_n and get a universal measure. $\square$

The Stone theorem on paracompactness is quite nontrivial. Oxtoby in his *Measure and category* reproduces a different self-contained argument of 15.1. However the proof above can be generalized to the following version. A topological space X is *metacompact* if every open cover has a point-finite refinement (every point belongs to finitely many sets for that refinement). What we have proved is: if μ is a Borel measure on a metacompact space X and for every $x \in X$, $\mu(U) = 0$ for some open set $U \ni x$ then $\mu(X) = 0$.

Another remark is that for the Marczewski-Sikorski theorem it is enough to assume that a metrizable space X in question has weight $< \mathfrak{m}$ (i.e. has a base of cardinality $< \mathfrak{m}$).