21. The Haar measure on a compact group

At the beginning of the course, we discussed the properties of the usual product measure ν on the Cantor set $2^{\mathbb{N}}$. Recall that this measure is translation invariant when $2^{\mathbb{N}}$ is equipped with the natural operation of addition coordinatewise mod 2. The name Haar[1] measure for ν is connected with a very general phenomenon that every compact group admits such a canonical, uniquely defined probability measure. The whole story would make this last lecture to long so we shall concentrate on the basic elements of the construction.

We shall consider an arbitrary, not necessarily abelian, group $(G, \cdot)$. We use the usual convention that for $A, B \subseteq G$,

$$A \cdot B = \{a \cdot b : a \in A, b \in B\} \text{ and } A^{-1} = \{a^{-1} : a \in A\}.$$

By e we denote the neutral element of the group.

A group G is said to be a *topological group* if G is equipped with some topology such that the group operations

$$G \times G \ni (x, y) \to x \cdot y \in G \text{ and } G \ni x \to x^{-1} \in G$$

are continuous (here $G \times G$ is given its product topology). Then for every $x \in G$ the function $f : G \to G$, $f(y) = x^{-1}y$ is a homeomorphism sending x to e. This implies that the topology of G is completely determined by the local base at $e \in G$. We shall write $N(e)$ for the collection of open neighbourhoods of e. The continuity of the group operations means that for every $U \in N(e)$ there is $V \in N(e)$ such that $V \cdot V \subseteq U$ and, likewise, there is $V \in N(e)$ with $V^{-1} \subseteq U$.

Below we deal with a compact (Hausdorff) group, i.e. a topological group which is a compact space in the usual topological sense. One can check that any topological group satisfying T_1 separation axiom (saying that the points are closed) is automatically completely regular — we not discuss it. Recall, however, that every Hausdorff compact space is normal.

For the rest of the section we fix a compact group G. We first note the following general facts on the interplay between the topology and the group structure.

Lemma 21.1. *Suppose that $K \subseteq U \subseteq G$, where K is compact and U is open. Then there is $V \in N(e)$ such that $K \cdot V \subseteq U$.*

Proof. For every $x \in K$ put $W_x = x^{-1} \cdot U$; then $W_x \in N(e)$ so there is $V_x \in N(e)$ such that $V_x \cdot V_x \subseteq W_x$. The family $\{xV_x : x \in K\}$ is an open cover of K so, by compactness, $K \subseteq \bigcup_{i=1}^{k} x_i V_{x_i}$ for some k and $x_i \in K$. Set $V = \bigcap_{i=1}^{k} V_{x_i}$. For any $x \in K$ we have $x \in x_i V_{x_i}$ for some $i \leq k$ so

$$xV \subseteq x_i V_{x_i} \cdot V_{x_i} \subseteq x_i W_{x_i} = U,$$

and hence $K \cdot V \subseteq U$. $\square$

[1] to honour Alfréd Haar

Lemma 21.2. *Suppose that $K_1, K_2 \subseteq G$ are compact and disjoint. Then there is $U \in N(e)$, such that for every $x \in G$ the set $x \cdot U$ cannot meet both K and L.*

Proof. As G is a normal topological space, there are open disjoint $U_1 \supseteq K_1$, $U_2 \supseteq K_2$. By Lemma 21.1, there are $V_1, V_2 \in N(e)$ such that $K_1 \cdot V_1 \subseteq U_1$ and $K_2 \cdot V_2 \subseteq U_2$.

Consider $V' = V_1 \cap V_2$ and $V \in N(e)$ such that $V^{-1} \subseteq V'$. Then V is as required; indeed, suppose $xV \cap K_1 \neq \emptyset$ and $xV \cap K_2 \neq \emptyset$; then

$$x \in (K_1 \cdot V^{-1}) \cap (K_2 \cdot V^{-1}) \subseteq U_1 \cap U_2 = \emptyset.$$

a contradiction. □

The construction of the Haar measure is based on the following concept.

Definition 21.3. Given any $U \in N(e)$, for every $A \subseteq K$ define

$$[A : U] = \min\{|I| : I \subseteq G, A \subseteq I \cdot U\}.$$

Put

$$\theta_U(A) = \frac{[A : U]}{[G : U]}.$$

Note that $[A : U]$ is finite whenever U is open since the sets xU cover G so there is a finite subcover by compactness; consequently, $\theta_U(A) \in [0, 1]$. Further properties are listed below.

Lemma 21.4. *For any open $U, V \subseteq G$ and an arbitrary $A, B \subseteq G$ the following hold*

(i) $[A : U] \leq [A \cup B : U] \leq [A : U] + [B : U]$;
(ii) $[A : U] \leq [A : V][V : U]$;
(iii) $[(xA) : U] = [A : U]$ *for any* $x \in G$.

Proof. The first inequality in (i) is clear; for the second note that if $A \subseteq I \cdot U$ and $B \subseteq J \cdot U$ then $A \cup B \subseteq (I \cup J) \cdot U$.

For (ii) take finite $I, J \subseteq G$ such that $A \subseteq I \cdot V$ and $V \subseteq J \cdot U$; then $A \subseteq I \cdot J \cdot U$.

To check (iii) note that if $A \subseteq I \cdot U$ then $xA \subseteq (xI) \cdot U$ and $|xI| = |I|$. This shows that $[(xA) : U] \leq [A : U]$. We then apply this inequality to $A = x^{-1}(xA)$. □

For the next step we might use the concept of a net in a topological space or compactness of some product but it is more transparent to use the limit along an ultrafilter in the following form: If $f : T \to [0, 1]$ is any function from some index set T then for any ultrafilter $\mathcal{F}$ on T there is a unique $a = \lim_{t \to \mathcal{F}} f(t) \in [0, 1]$ such that $\{t \in T : |f(t) - a| < \varepsilon\} \in \mathcal{F}$ for every $\varepsilon > 0$. This can be verified as L4/P6, only now the index set can be uncountable. Note that this operation is additive:

$$\lim_{t \to \mathcal{F}} \Big(f(t) + g(t) \Big) = \lim_{t \to \mathcal{F}} f(t) + \lim_{t \to \mathcal{F}} g(t).$$

Proposition 21.5. *Let $\mathcal{K}$ be a family of all compact subsets of a compact group G. There is a function $\theta : \mathcal{K} \to [0, 1]$ such that*

(i) $\theta(G) = 1$, $\theta(\emptyset) = 0$;

(ii) $\theta(K \cup L) = \theta(K) + \theta(L)$ *whenever* $K, L \in \mathcal{K}$ *are disjoint;*
(iii) $\theta(xK) = \theta(K)$ *for every* $J \in \mathcal{K}$ *and* $x \in G$.

Proof. We shall define θ is the limit of θ_U for $U \in N(e)$. Thus our index set will be $T = N(e)$. The family of all the sets $\{V \in N(e) : V \subseteq U\}$ with $U \in N(e)$ is centred (has the finite intersection property) so there is an ultrafilter $\mathcal{F}$ on $N(e)$ such that $\{V \in N(e) : V \subseteq U\} \in \mathcal{F}$ for every $U \in N(e)$.

For $K \in \mathcal{K}$ we set

$$\theta(K) = \lim_{U \to \mathcal{F}} \theta_U(K).$$

As $\theta_U(K) \in [0,1]$, the limit along $\mathcal{F}$ is well-defined and $\theta_U(K) \in [0,1]$. We have $\theta_U(G) = 1$ by the definition of the index so $\theta(G) = 1$. Likewise, $\theta(\emptyset) = 0$.

By Lemma 21.4(i), we have $\theta_U(K \cup L) \leq \theta_U(K) + \theta_U(L)$ so, passing to a limit, we have $\theta(K \cup L) \leq \theta(K) \cup \theta(L)$ for any $K, L \in \mathcal{K}$. More subtle is to check that

$$\theta(K \cup L) \geq \theta(K) + \theta(L) \text{ whenever } K \cap L = \emptyset.$$

By Lemma 21.2 there is $U \in N(e)$ such that every translate xU can meet K or L but never both of them. This clearly holds also for every $V \in N(e)$ with $V \subseteq U$. For such V take a finite $I \subseteq G$ such that the value of $[K \cup L : V]$ is witnessed by $K \cup L \subseteq I \cdot V$. We can now divide I into disjoint parts $I(K)$ and $I(L)$ by the rule that xV meets K whenever $x \in I(K)$ and $I(L) = I \setminus I(K)$. Then $|I| = |I(K)| + |I(L)|$ and $L \subseteq I(L) \cdot V$ so

$$\theta_V(K) + \theta_V(L) \leq |I(K)| + |I(L)| = |I| = \theta_V(K \cup L).$$

From the point of view of $\mathcal{F}$ those V which are not contained in U are inessential so, passing to the limit, we get $\theta(K) + \theta(L) \leq \theta(K \cup L)$, as required.

The fact that $\theta(xK) = \theta(K)$ follows from Lemma 21.4(iii). $\square$

Theorem 21.6. *On every compact group* G *there is a probability Borel measure* ν *such that (denoting by* $\mathcal{K}$ *the family o all compact subsets of* G*)*
(i) ν *is translation invariant, that is* $\nu(xB) = \nu(B)$ *for* $B \in Bor(G)$ *and* $x \in G$*;*
(ii) ν *is regular, i.e.* $\nu(B) = \sup\{\nu(K) : K \subseteq B, K \in \mathcal{K}\}$ *for* $B \in Bor(G)$.

Proof. We take θ from Proposition 21.5 and perform a two-step regularization procedure:

$$\theta_*(U) = \sup\{\theta(K) : K \in \mathcal{K}, K \subseteq L\} \text{ for open } U \subseteq G.$$

$$\eta(K) = \inf\{\theta_*(U) : K \subseteq U, U \text{ open}\} \text{ for } K \in \mathcal{K}.$$

It is somewhat technical but quite routine, see below, to check that this defines η on $\mathcal{K}$ in such a way that

$$\eta(K) + \sup\{\eta(L') : L' \subseteq L \setminus K, L' \in \mathcal{K}\} = \eta(L),$$

for every $K \subseteq L$, $K, L \in \mathcal{K}$.

Then we extend ν to the algebra $\mathcal{A}$ generated by $\mathcal{K}$ by the formula

$$\nu(A) = \sup\{\eta(K) : K \subseteq A, K \in \mathcal{K}\},$$

and check that this is $\mathcal{K}$-regular finitely additive measure on $\mathcal{K}$. Finally by Lemma 19.3 we conclude that ν extends to a regular Borel measure; see appendix for details.

It is not difficult to verify that the invariance to left translations is preserved at each step of the construction. □

There are several ways to derive Theorem 21.6 from Proposition 21.5, see e.g. this handy text of Jonathan Gleason The approach sketched above is based on David's Fremlin monograph (where everything is treated in a more general context).

Theorem 21.7. *If ν is a left-invariant regular Borel measure on a compact group G then $\nu(U^{-1}) = \nu(U)$ for every open set $U \subseteq G$.*

Consequently, ν is also right-invariant and $\nu' = \nu$ for every left-invariant regular probability measure ν'

Proof. Consider any open $U \subseteq G$ and

$$\Gamma = \{(x, y) \in G \times G : y^{-1}x \in U\}.$$

Recall that the product measure $\nu \otimes \nu$ (defined on $Bor(G) \otimes Bor(G)$) extends to a Borel measure μ on $Bor(G \times G)$ so that μ satisifes the usual Fubini formula

$$\mu(\Gamma) = \int_G \nu(G_x)\, \mathrm{d}\nu(x) = \int_G \nu(G^y)\, \mathrm{d}\nu(y),$$

see P11/L9 (note that checking the formula for open $\Gamma \subseteq G \times G$ is simpler).

Every horizontal section Γ^y is equal to yU so it satisfies $\nu(\Gamma^y) = \nu(U)$ by left-invariance. For a vertical section Γ_x we have

$$y \in \Gamma_x \iff y^{-1} \in Ux^{-1} \iff y \in xU^{-1},$$

so $\nu(\Gamma_x) = \nu(U^{-1})$ and Fubini says $\nu(U) = \nu(U^{-1})$.

By outer regularity we have $\nu(B^{-1}) = \nu(B)$ for every Borel $B \subseteq G$. Note that, formally speaking, $B \mapsto \nu(B^{-1})$ defines a right-invariant measure which is equal to ν.

For uniqueness, apply the above Fubini-like argument to $\nu \otimes \nu'$. □

Remark 21.8. (1) There is a nontrivial theorem stating that every Haar measure satisfies the assertion of Theorem 18.2, that $Bor(G)$ is contained in the completion of $Baire(G)$ with respect to the Haar measure.

(2) The construction above can be adapted to prove that the invariant Borel measure exists on every group which is locally compact; we start by choosing $G_0 \in N(e)$ having compact closure and in the definition of θ_U replace G by G_0. This gives ν with $\nu(G_0) = 1$. If the group is locally compact but not compact then the Haar measure is necessarily infinite (note that the Lebesgue measure is the Haar measure on a locally compact group $(\mathbb{R}, +)$). In a non-compact case, a left-invariant measure is still unique up to a constant but may be different from a right-invariant measure.

(3) If a infinite group G is equipped with the discrete topology then its Haar measure is simply the counting measure (assigning ∞ to infinite sets and $|A|$ for finite $A \subseteq G$).

(4) If G is infinite discrete group then the existence of an invariant finitely additive probability on (the power set of) G is another interesting story. A group is said to be *amenable* if it admits such a function. We have already seen (L7/P2) that $(\mathbb{Z}, +)$ is amenable (actually, every abelian group is amenable).

(5) The classical non-amenable group is F_2, the free group of two generators. If you cannot find a reason why F_2 does not admit an invariant probability (and you cannot fall asleep for that reason) then check it in WIKIPEDIA or look into this introductory text by Alejandra Garrido

Appendix A. How to construct a measure in 21.6

This part is fairly general (does not refer to the group structure). We are given a (monotone, additive and sub-additive) set function θ defined on the lattice of compact subsets of G. We define

$$\theta_*(U) = \sup\{\theta(K) : K \in \mathcal{K}, K \subseteq L\} \text{ for open } U \subseteq G.$$

$$\eta(K) = \inf\{\theta_*(U) : K \subseteq U, U \text{ open}\} \text{ for } K \in \mathcal{K}.$$

Then we want to prove that there is a Borel measure ν such that $\nu(B) = \sup\{\eta(K) : K \in \mathcal{K}, K \subseteq B\}$ for $B \in Bor(G)$. Let us agree that below U, V (possibly with indices) stand for open sets while K, L are always compact.

STEP 1. $\theta_*(U_1 \cup U_2) \le \theta_*(U_1) + \theta_*(U_2)$.

Indeed if $K \subseteq U_1 \cup U_2$ then, by normality of G, we have $K = K_1 \cup K_2$ where $K_i \subseteq U_i$; this immediately implies the formula.

STEP 2. $\theta_*(U_1 \cup U_2) = \theta_*(U_1) + \theta_*(U_2)$ whenever $U_1 \cap U_2 = \emptyset$. Likewise, $\eta(K_1 \cup K_2) = \eta(K_1) + \eta(K_2)$ whenever $K_1 \cap K_2 = \emptyset$.

Call it an exercise!

STEP 3. Given $K_1 \subseteq K_2$, we have

$$\eta(K_1) + \sup\{\eta(K) : K \subseteq K_2 \setminus K_1, L' \in \mathcal{K}\} = \eta(K_2).$$

Fix $\varepsilon > 0$; choose $U_i \supseteq K_i$ such that $\theta_*(U_i) < \eta(K_i) + \varepsilon$. Then $K_2 \setminus U_1$ is a compact set contained in U_2 and disjoint from K_1 so there is V such that $K_2 \setminus U_1 \subseteq V \subseteq U_2$ and $V \cap K_1 = \emptyset$. We can moreover assume that $\theta_*(V) < \eta(K_2 \setminus U_1) + \varepsilon$. Now, using Step 1,

$$\eta(K_2) \le \theta_*(U_1 \cup V) \le \theta_*(U_1) + \theta_*(V) \le \eta(K_1) + \varepsilon + \eta(K_2 \setminus U_1) + \varepsilon,$$

which verifies the harder inequality in the formula. The reverse one follows by Step 2.

STEP 4. For any K_1, K_2 we have

$$\eta(K_1 \cup K_2) + \eta(K_1 \cap K_2) \ge \eta(K_1) + \eta(K_2).$$

By Step 3, for any $\varepsilon > 0$ there are $L_i \subseteq K_i \setminus (K_1 \cap K_2)$ such that $\eta(L_i) + \eta(K_1 \cap K_2^) > \eta(K_i) - \varepsilon$. Then

$$-2\varepsilon + \eta(K_1) + \eta(K_2) \le \eta(L_1) + \eta(K_1 \cap K_2) + \eta(L_2) + \eta(K_1 \cap K_2) \le$$

$$\eta(L_1 \cup L_2 \cup (K_1 \cap K_2)) + \eta(K_1 \cap K_2) \le \eta(K_1 \cup K_2) + \eta(K_1 \cap K_2).$$

Step 5. The family $\mathcal{A}$ of those Borel sets B such that for every $\varepsilon > 0$ there are $K \subseteq B$ and $L \subseteq G \setminus B$ with $\eta(K) + \eta(L) > 1 - \varepsilon$ is an algebra of sets containing all open sets (and all closed sets).

Indeed, every open set is in $\mathcal{A}$ by Step 3 (where we put $K_2 = G$. Clearly $\mathcal{A}$ is closed under taking complements. It remains to check that if $A_1, A_2 \in \mathcal{A}$ then $A_1 \cap A_2 \in \mathcal{A}$. For this fix $\varepsilon > 0$ and choose $K_i \subseteq A_i$ and $L_i \subseteq G \setminus A_i$ witnessing that $A_i \in \mathcal{A}$. Then, using Step 4 twice,

$$\eta(K_1 \cap K_2) + \eta(L_1 \cup L_2) \geq \eta(K_1) + \eta(K_2) - \eta(K_1 \cup K_2) + \eta(L_1) + \eta(L_2) - \eta(L_1 \cap L_2) \geq$$
$$2 - 2\varepsilon - \eta((K_1 \cup K_2) \cup (L_1 \cap L_2)) \geq 1 - 2\varepsilon,$$

so $K_1 \cap K_2$, $L_1 \cup L_2$ are witnesses for $A_1 \cap A_2$ with a constant 2ε.

Step 6. We can now define ν on $\mathcal{A}$ by $\nu(A) = \sup\{\eta(K) : K \subseteq A\}$; such ν is finitely additive $\mathcal{K}$-regular measure on $\mathcal{A}$. Indeed, if $A_1 \cap A_2 = \emptyset$ then $\nu(A_1 \cup A_2) \geq \nu(A_1) + \nu(A_2)$ follows from Step 2 while for the reverse inequality take any $K \subseteq A_1 \cup A_2$, fix $\varepsilon > 0$ and choose K_1, L_1 witnessing that $A_1 \in \mathcal{A}$ and use Step 3.

Step 7. We can now apply (anstract) Lemma 19.3 saying (in particular) that if ν is an additive set function on an algebra and ν is regular with respect to compact sets then ν extends uniquely to a regular measure on the generated σ-algebra (recall that countable additivity on $\mathcal{A}$ is equivalent to being continuous from above at $\emptyset$ which follows from compactness).