10. Measure algebras

As we have seen, given a measure space (X, Σ, μ), we can form the algebra $\mathfrak{A} = \Sigma/\mathcal{N}$, where $\mathcal{N}$ is the ideal of sets of measure zero. Then we can transfer μ to a σ-additive measure on $\mathfrak{A}$ — that will be denoted by the same letter μ. The measure algebra $(\mathfrak{A}, \mu)$ may be seen as a metric space:

Theorem 10.1. *If $(\mathfrak{A}, \mu)$ is a measure algebra then $d_\mu(a, b) = \mu(a \bigtriangleup b)$ defines a metric on $\mathfrak{A}$ and the metric space $(\mathfrak{A}, d_\mu)$ is complete.*

Proof. The axioms of a metric follow by easy calculations. For instance, if $a, b, c \in \mathfrak{A}$ then

$$a \bigtriangleup c \le (a \bigtriangleup b) \vee (b \bigtriangleup c), \text{ so } d_\mu(a, c) \le d_\mu(a, b) + d_\mu(b, c).$$

Take a sequence of $a_n \in \mathfrak{A}$ which is a Cauchy sequence, i.e. $\mu(a_n \bigtriangleup a_k) \to 0$ as $n, k \to \infty$. Then we can find a sequence k_n of natural numbers such that $\mu(a_{k_n} \bigtriangleup a_{k_{n+1}}) \le 1/2^n$. Since algebra $\mathfrak{A}$ is complete we can define (the upper limit)

$$a = \bigwedge_i \bigvee_{k \ge i} a_{n_k} \in \mathfrak{A}.$$

One can check that a is the desired limit, that is $\mu(a_n \bigtriangleup a) \to 0$. This is an exercise!

Alternatively, this argument is a part of the proof of Riesz's theorem (a sequence of functions that is Cauchy in measure converges in measure). Indeed, if we take $A_n \in \Sigma$ such that $A_n/\mathcal{N} = a_n$ then $\mu(A_n \bigtriangleup A_k) \to 0$ (which means that the sequence of χ_{A_n} is Cauchy in measure), so it has a subsequence converging pointwise to a function that is $\{0, 1\}$-almost everywhere etc. □

Every measure algebra $(\mathfrak{A}, \mu)$ has its Boolean structure and is, on the other hand, a metric space. Those two structures are compatible:

Theorem 10.2. *In a measure algebra $(\mathfrak{A}, \mu)$ the Boolean operations are continuous.*

Proof. Suppose that $a_n \to a$, i.e. $\mu(a_n \bigtriangleup a) \to 0$ and suppose that $b_n \to b$. Then $a_n \vee b_n \to a \vee b$ since

$$\mu((a_n \vee b_n) \bigtriangleup (a \vee b)) \le \mu(a_n \bigtriangleup a) + \mu(b_n \bigtriangleup b) \to 0.$$

By a similar argument $a_n \wedge b_n \to a \wedge b$, $a_n^c \to a^c$. □

Recall that the measure space (X, Σ, μ) (or the measure μ itself) is said to be separable if $L_1(\mu)$ is separable, and this is equivalent to saying that there is a countable family $\mathcal{A} \subseteq \Sigma$ such that

$$\inf\{\mu(A \bigtriangleup B) : A \in \mathcal{A}\} = 0,$$

for every $B \in \Sigma$ (see Lemma 4.7 and Theorem 4.8). With this definition we can state the following.

Corollary 10.3. *The measure algebra $\mathfrak{A}$ of a separable measure μ is separable as a metric space $(\mathfrak{A}, d_\mu)$. In particular, the measure algebra of $\mu \in P(X)$, where X is a separable metrizable space, is a Polish space.*

We can now specify the brave statement *there is (essentially) only one measure*. Given a measure μ, we denote by $\mathcal{N}(\mu)$ the ideal of sets of μ-measure zero.

Theorem 10.4. *Suppose that (X, Σ, μ) be a probability measure space that is nonatomic and separable. Then the measure algebra $\mathfrak{A} = \Sigma/\mathcal{N}(\mu)$ is isomorphic to the measure algebra $\mathfrak{B} = Bor[0,1]/\mathcal{N}(\lambda)$ of the Lebesgue measure. In fact, there is an isomorphism $h : \mathfrak{A} \to \mathfrak{B}$ such that $\mu(a) = \lambda(h(a))$ for every $a \in \mathfrak{A}$.*

Here by an isomorphism $h : \mathfrak{A} \to \mathfrak{B}$ of Boolean algebras we mean a bijection preserving Boolean operations in an obvious manner (e.g $h(a \vee b) = h(a) \vee h(b)$). We shall prove Theorem 10.4 below, after some preparations. The theorem is actually an elementary version of the Maharam structure theorem, stating that, in a sense, for every cardinal number κ, there is only one measure algebra of density κ.

We say that $a \in \mathfrak{A}$ is an atom of $\mathfrak{A}$ if $a \neq 0$ and for every nonzero $b \in \mathfrak{A}$, if $b \leq a$ then $b = a$.

Lemma 10.5. *If $\mathfrak{A}_0$ is a finite Boolean algebra then there are pairwise disjoint atoms $a_1, \ldots, a_k \in \mathfrak{A}_0$ (for some k), such that $a_1 \vee \ldots \vee a_k = 1$. Consequently, every $b \in \mathfrak{A}_0$ is of the form $b = \bigvee_{i \in I} a_i$ for some $I \subseteq \{1, 2, \ldots, k\}$.*

If $h(a_i)$, $i \leq k$, are atoms of another finite Boolean algebra $\mathfrak{B}_0$ and $\bigvee_{i \leq k} h(a_i) = 1$ the h extends to a Boolean homomorphism $\mathfrak{A}_0 \to \mathfrak{B}_0$.

Proof. Exercise. □

Lemma 10.6. *Keeping the notation of 10.4, let $\mathfrak{A}_0$ be a finite subalgebra of $\mathfrak{A}$ and let $h : \mathfrak{A}_0 \to \mathfrak{B}_0 \subseteq \mathfrak{B}$ be an isomorphism such that*

$$(*) \quad \mu(a) = \lambda(h(a) \text{ for every } a \in \mathfrak{A}_0.$$

Then for every $x \in \mathfrak{A}$ there is an extension of h to an isomorphism $h_1 : \mathfrak{A}_1 \to \mathfrak{B}_1$, where $\mathfrak{A}_1$ is a subalgebra of $\mathfrak{A}$ generated by $\mathfrak{A}_0 \cup \{x\}$ such that (*) is still satisfied.

Proof. By Lemma 10.5, every $b \in \mathfrak{A}_0$ is a finite union of atoms $a_1, \ldots, a_k$ of $\mathfrak{A}_0$. Take $x \in \mathfrak{A}$ and note that the algebra $\mathfrak{A}_1$ will have atoms of the form $x \wedge a_i$, $a_i \setminus x$ (some may be excluded as being 0). To define h_1 properly we need to set the values of $h_1(a_i \wedge x)$ and $h_1(a_i \setminus x)$.

In a nontrivial case we have $a_i \wedge x, a_i \setminus x \neq 0$. We use the Darboux property of nonatomic measures. We have $h(a_i) \in \mathfrak{B}$, which is the measure algebra of a nonatomic measure so there is $y_i \in \mathfrak{B}$ such that $y_i \leq h(a_i)$ and $\lambda(y_i) = \mu(a_i \wedge x)$. We define $h_1(a_i \wedge x) = y_i$ and $h_1(a_i \setminus x) = h(a_i) \setminus y_i$. Note that (*) is preserved.

Once h_1 is defined on atoms of $\mathfrak{A}_1$ so that (*) is satisfied for atoms, we let $\mathfrak{B}_1$ be the algebra generated by $\mathfrak{B}_0 \cup \{y_i : i \leq k\}$. We extend h_1 to $h_1 : \mathfrak{A}_1 \to \mathfrak{B}_1$ using Lemma 10.5. □

Lemma 10.7. *In the setting of 10.4, there are countable subalgebras $\widetilde{\mathfrak{A}} \subseteq \mathfrak{A}$ and $\widetilde{\mathfrak{B}} \subseteq \mathfrak{B}$ such that*

(i) $\widetilde{\mathfrak{A}} \subseteq \mathfrak{A}$ and $\widetilde{\mathfrak{B}} \subseteq \mathfrak{B}$ are dense in the metrics d_μ and d_λ, respectively;

(ii) there is an isomorphism $\widetilde{h} : \widetilde{\mathfrak{A}} \to \widetilde{\mathfrak{B}}$ preserving the measure (so that (*) is granted for $a \in \widetilde{\mathfrak{A}}$).

Proof. This is the classical zig-zag argument. Make a list of $x_n \in \mathfrak{A}$ and $y_n \in \mathfrak{B}$ forming countable dense subsets of the corresponding algebras. Start with the trivial isomorphism $h_0 : \mathfrak{A}_0 \to \mathfrak{B}_0$, where $\mathfrak{A}_0 = \{0, 1\}$.

At the odd step n we have $h_{n-1} : \mathfrak{A}_{n-1} \to \mathfrak{B}_{n_1}$; add x_n to $\mathfrak{A}_{n-1}$ and define an extension $h_n : \mathfrak{A}_n \to \mathfrak{B}_n$ by means of Lemma 10.6.

At the even step n repeat the above to the inverse isomorphism $h_{n-1}^{-1} : \mathfrak{B}_{n-1} \to \mathfrak{A}_{n-1}$, adding y_n this time.

Finally, put $\widetilde{\mathfrak{A}} = \bigcup_n \mathfrak{A}_n$, $\widetilde{\mathfrak{B}} = \bigcup_n \mathfrak{B}_n$ define $\widetilde{h} : \widetilde{\mathfrak{A}} \to \widetilde{\mathfrak{B}}$ as the unique common extension of h_n's. □

For the final stroke we need to recall the following general fact.

Lemma 10.8. *Suppose that (X, ρ_1) and (Y, ρ_1) are complete metric spaces. If $\widetilde{g} : \widetilde{X} \to \widetilde{Y}$ is an isometry between dense subspaces $\widetilde{X} \subseteq X$, $\widetilde{Y} \subseteq Y$ then $\widetilde{h}$ extends uniquely to an isometry $g : X \to Y$.*

Proof. We simply define $g(x) = \lim_n \widetilde{g}(x_n)$ whenever $x_n \to x$. It is easy to check that the definition is correct. For instance, since $x_n \to x$ then x_n's form a Cauchy sequence. Hence $\widehat{g}(x_n)$ also form a cauchy sequence (as $\widetilde{g}$ is an isometry); Y is complete so the limit exists. □

Proof. (of Theorem 10.4) This follows from Lemma 10.7 and Lemma 10.8. We only need to check that the extension of an isomorphism is again an isomorphism. But this is a consequence of the continuity of Boolean operations, see Theorem 10.2 □

Corollary 10.9. *If (X, Σ, μ) and (Y, Θ, ν) are two nonatomic separable probability measure spaces then the Banach spaces $L_1(\mu)$ and $L_1(\nu)$ are linearly isometric.*

In particular, $L_1(\mu)$ is isometric to $L_1(\nu)$ whenever the measures $\mu \in P(X)$ and $\nu \in P(Y)$ are nonatomic, and the spaces X, Y are separable and metrizable.

Proof. We know that $\mathfrak{A} = \Sigma/\mathcal{N}(\mu)$ and $\mathfrak{B} = \Theta/\mathcal{N}(\nu)$ are isomorphic Boolean algebras so we may fix an isomorphism $h : \mathfrak{A} \to \mathfrak{B}$ preserving the measure.

To define an isometry $I : L_1(\mu) \to L_1(\nu)$ we first consider simple function. If $A \in \Sigma$ then $\chi_A \in L_1(\mu)$ (this is how we think) but formally this is rather 'the characteristic function of' $A/\mathcal{N}(\mu)$, that is an element of $L_1(\mu)$. To follow our former informal custom we can define $I(\chi_A) = \chi_B$, where $B \in \Theta$ is such that $h(A/\mathcal{N}(\mu)) = B/\mathcal{N}(\nu)$. At this level, I preserves the norms since $\|\chi_A\|_1 = \mu(A)$ for $A \in \Sigma$.

This enables us to define I on the subspace of simple functions in $L_1(\mu)$, by the formlua

$$I\left(\sum_{k \le n} t_k \cdot \chi_{A_k}\right) = \sum_{k \le n} t_k \cdot \chi_{B_k},$$

where B_k satisfy $h(A_k/\mathcal{N}(\mu) = B_k/\mathcal{N}(\lambda)$. It requires some work to check that the definition is correct but... let us skip it:-) The proof is very similar to that checking that the integral is well-defined on simple functions.

The fact that I preserves the L_1-norms follow from the fact that if we consider a simple function $f = \sum_{k\le n} t_k \cdot \chi_{A_k}$ where A_k's are pairwise disjoint then $\|f\|_1 = \sum_{k\le n} |t_k|\mu(A_k)$.

Finally we use Lemma 10.8 again, applying it to I defined on a dense subspace of simple functions. □

In particular, $L_1[0,1]$ is the same as $L_1[0,1]^2$ which does not look that obvious. The above Corollary is sometimes useful: For instance, we may use the fact that $L_1[0,1]$ is isometric to $L_1(2^{\mathbb{N}})$ to conclude that in $L_1[0,1]$ there is a sequence of independent random variables of zero mean since such a sequence obviously exists in the other space: take $g_n : 2^{\mathbb{N}} \to \{-1.1\}$ defined by $g_n(x) = (-1)^{x(n)}$.

Let us finally remark that Corollary 10.9 can be proved for L_p-spaces by an analogous argument.